LINEAR MODEL $y = X\beta + \varepsilon$ ASSUME $E(\varepsilon) = 0$

X: n rows, p COLUMNS p = number of model parameters

β: p row, 1 COLUMN ε: n rows, 1 column

$X_{n\times p}$, $\beta_{p\times 1} \rightarrow X\beta$ is $n\times 1$

y: n rows, 1 column VECTOR OF RESPONSES

CASE 1: ONE SAMPLE (ONE MEAN μ) $\hat{\mu} = \bar{y}$, $s.e(\bar{y}) = s/\sqrt{n}$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1} \quad X = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1} \quad \beta = [\mu]_{1\times 1} \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n\times 1}$$

$$y = X\beta + \varepsilon \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} \Rightarrow \begin{array}{l} y_1 = \mu + \varepsilon_1 \\ y_2 = \mu + \varepsilon_2 \\ \vdots \\ y_n = \mu + \varepsilon_n \end{array}$$

$$(X'X) = [n] \Rightarrow (X'X)^{-1} = \left[\tfrac{1}{n}\right] \qquad X'y = \left[\sum_{i=1}^{n} y_i\right]$$

THEN $\hat{\beta} = \hat{\mu} = (X'X)^{-1}X'y = \left[\tfrac{1}{n}\right]\left[\sum_{1}^{n} y_i\right] = [\bar{y}]$ IS THE LEAST SQUARES ESTIMATE OF μ

$$VAR(\hat{\beta}) = VAR(\hat{\mu}) = \sigma^2 (X'X)^{-1} = \sigma^2\left[\tfrac{1}{n}\right] = \left[\tfrac{\sigma^2}{n}\right]$$

$$\widehat{VAR}(\hat{\beta}) = \widehat{VAR}(\hat{\mu}) = \left[\tfrac{s^2}{n}\right] \Rightarrow se(\hat{\beta}) = \tfrac{s}{\sqrt{n}}$$

CASE 2 TWO SAMPLES (TWO MEANS μ_1, μ_2): $\hat{\mu}_1 = \bar{y}_1$, $\hat{\mu}_2 = \bar{y}_2$
SAMPLE SIZES: m, n

$$y = \begin{bmatrix} y_{11} \\ \vdots \\ y_{1m} \\ \hline y_{21} \\ \vdots \\ y_{2n} \end{bmatrix}_{(m+n)\times 1} \quad X = \underbrace{\begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ \hline 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix}}_{(m+n)\times 2} \begin{array}{l} \} m\times 2 \\ \\ \} n\times 2 \end{array} \quad \beta = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \varepsilon = \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1m} \\ \hline \varepsilon_{21} \\ \vdots \\ \varepsilon_{2n} \end{bmatrix}$$

$\boxed{\text{MEANS MODEL}}$

$$y = X\beta + \varepsilon \Rightarrow \begin{bmatrix} y_{11} \\ \vdots \\ y_{1m} \\ \hline y_{21} \\ \vdots \\ y_{2n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ \hline 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1m} \\ \hline \varepsilon_{21} \\ \vdots \\ \varepsilon_{2n} \end{bmatrix} \Rightarrow \begin{array}{l} y_{11} = \mu_1 + \varepsilon_{11} \\ \vdots \\ y_{1m} = \mu_1 + \varepsilon_{1m} \\ \hline y_{21} = \mu_2 + \varepsilon_{21} \\ \vdots \\ y_{2n} = \mu_2 + \varepsilon_{2n} \end{array}$$

$$(X'X) = \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix} \to (X'X)^{-1} = \begin{bmatrix} 1/m & 0 \\ 0 & 1/n \end{bmatrix} \qquad X'y = \begin{bmatrix} \sum_{j=1}^{m} y_{1j} \\ \sum_{j=1}^{n} y_{2j} \end{bmatrix} = \begin{bmatrix} y_{1\cdot} \\ y_{2\cdot} \end{bmatrix}$$

$$\Rightarrow \hat{\beta} = \begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{bmatrix} = (X'X)^{-1} X'y = \begin{bmatrix} 1/m & 0 \\ 0 & 1/n \end{bmatrix} \begin{bmatrix} y_{1\cdot} \\ y_{2\cdot} \end{bmatrix} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix}$$

$$\mathrm{VAR}(\hat{\beta}) = \sigma^2 (X'X)^{-1} = \sigma^2 \begin{bmatrix} 1/m & 0 \\ 0 & 1/n \end{bmatrix} = \begin{bmatrix} \sigma^2/m & 0 \\ 0 & \sigma^2/n \end{bmatrix}$$

$$\widehat{\mathrm{VAR}}(\hat{\beta}) = S_p^2 \begin{bmatrix} 1/m & 0 \\ 0 & 1/n \end{bmatrix} \qquad S_p^2 = \frac{(m-1)S_1^2 + (n-1)S_2^2}{m+n-2} = MS_E$$

$$\mathrm{VAR}(\hat{\mu}_1 - \hat{\mu}_2) = \mathrm{VAR}(\hat{\mu}_1) + \mathrm{VAR}(\hat{\mu}_2) = \frac{\sigma^2}{m} + \frac{\sigma^2}{n} = \sigma^2\left(\frac{1}{m} + \frac{1}{n}\right)$$

$$\widehat{\mathrm{VAR}}(\hat{\mu}_1 - \hat{\mu}_2) = \widehat{\mathrm{VAR}}(\hat{\mu}_1) + \widehat{\mathrm{VAR}}(\hat{\mu}_2) = S_p^2\left(\frac{1}{m} + \frac{1}{n}\right)$$

$$\Rightarrow \mathrm{s.e.}(\hat{\mu}_1 - \hat{\mu}_2) = \mathrm{s.e.}(\bar{y}_1 - \bar{y}_2) = \sqrt{S_p^2\left(\frac{1}{m} + \frac{1}{n}\right)} = S_p\sqrt{\frac{1}{m} + \frac{1}{n}}$$

WHAT ABOUT THE EFFECTS MODEL?

$y_{ij} = \mu + \tau_i + \varepsilon_{ij} \quad i = 1, 2$ THAT IS, $\left\{ \begin{array}{l} \mu_1 = \mu + \tau_1 \\ \mu_2 = \mu + \tau_2 \end{array} \right\}$

THERE ARE THREE PARAMETERS, BUT ~~ONLY TWO MEANS~~.

→ THIS IS AN "OVER-PARAMETERIZED" MODEL. WHY?

$$y = \begin{bmatrix} y_{11} \\ \vdots \\ y_{1m} \\ \hline y_{21} \\ \vdots \\ y_{2n} \end{bmatrix}_{(m+n)\times 1} \quad X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{bmatrix}_{(m+n)\times 3} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} m\times 3 \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n\times 3 \end{array} \quad \beta = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \end{bmatrix}_{3\times 1} \quad \varepsilon = \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1m} \\ \hline \varepsilon_{21} \\ \vdots \\ \varepsilon_{2n} \end{bmatrix}_{(m+n)\times 1}$$

NOTE

$$X\beta = \begin{bmatrix} \mu+\tau_1 \\ \mu+\tau_1 \\ \vdots \\ \mu+\tau_1 \\ \hline \mu+\tau_2 \\ \mu+\tau_2 \\ \vdots \\ \mu+\tau_2 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m\times 1 \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n\times 1 \end{matrix}$$

THEN $y = X\beta + \varepsilon \rightarrow$

$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1m} \\ \hline y_{21} \\ \vdots \\ y_{2n} \end{bmatrix} = \begin{bmatrix} \mu+\tau_1+\varepsilon_{11} \\ \mu+\tau_1+\varepsilon_{12} \\ \vdots \\ \mu+\tau_1+\varepsilon_{1m} \\ \hline \mu+\tau_2+\varepsilon_{21} \\ \mu+\tau_2+\varepsilon_{22} \\ \vdots \\ \mu+\tau_2+\varepsilon_{2n} \end{bmatrix}$$

OR, $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$ (THE EFFECTS MODEL)

WHAT ABOUT ESTIMATION OF μ, τ_1, τ_2?

$$X'X = \begin{bmatrix} m+n & m & n \\ m & m & 0 \\ n & 0 & n \end{bmatrix}$$

BUT $(X'X)^{-1}$ DOES NOT EXIST, THAT IS, $X'X$ IS SINGULAR.

[NOTE: COLUMN 1 = COLUMN 2 + COLUMN 3]

RANK $(X'X) = 2$ NUMBER OF PARAMETERS $= 3$

SO WHAT CAN WE DO? IMPOSE A CONSTRAINT

FOR EXAMPLE: $m\tau_1 + n\tau_2 = 0$ OR, EQUIVALENTLY, $\tau_2 = -\frac{m}{n}\cdot\tau_1$

WE CAN NOW REWRITE X SO THAT IT ONLY HAS TWO COLUMNS

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ \hline 1 & -m/n \\ 1 & -m/n \\ \vdots & \vdots \\ 1 & -m/n \end{bmatrix} \begin{matrix} \\ \\ \\ m \\ \\ \\ \\ n \end{matrix}$$

AND REMOVE τ_2 FROM β: $\beta = \begin{bmatrix} \mu \\ \tau_1 \end{bmatrix}$

NOTE

$$X\beta = \begin{bmatrix} \mu+\tau_1 \\ \mu+\tau_1 \\ \vdots \\ \mu+\tau_1 \\ \hline \mu-\frac{m}{n}\tau_1 \\ \mu-\frac{m}{n}\tau_1 \\ \vdots \\ \mu-\frac{m}{n}\tau_1 \end{bmatrix} = \begin{bmatrix} \mu+\tau_1 \\ \mu+\tau_1 \\ \vdots \\ \mu+\tau_1 \\ \hline \mu+\tau_2 \\ \mu+\tau_2 \\ \vdots \\ \mu+\tau_2 \end{bmatrix}$$

AFTER APPLYING THE CONSTRAINT

NOW USE THE LEAST SQUARES ESTIMATION FORMULAS:

$$X'X = \begin{bmatrix} m+n & m+n(-\frac{m}{n}) \\ m+n(-\frac{m}{n}) & m+n(-\frac{m}{n})^2 \end{bmatrix} = \begin{bmatrix} m+n & 0 \\ 0 & m+\frac{m^2}{n} \end{bmatrix} = \begin{bmatrix} m+n & 0 \\ 0 & \frac{mn+m^2}{n} \end{bmatrix} = \begin{bmatrix} m+n & 0 \\ 0 & \frac{m(m+n)}{n} \end{bmatrix}$$

$$\Rightarrow (X'X)^{-1} = \begin{bmatrix} \frac{1}{m+n} & 0 \\ 0 & \frac{n}{m(m+n)} \end{bmatrix} = \frac{1}{m+n}\begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{m} \end{bmatrix} \qquad X'y = \begin{bmatrix} y_{..} \\ y_{1.} - \frac{m}{n} y_{2.} \end{bmatrix}$$

$$\Rightarrow \hat{\beta} = \begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \end{bmatrix} = (X'X)^{-1}X'y = \frac{1}{m+n}\begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{m} \end{bmatrix} \begin{bmatrix} y_{..} \\ y_{1.} - \frac{m}{n} y_{2.} \end{bmatrix}$$

$$= \frac{1}{m+n}\begin{bmatrix} y_{..} \\ \frac{n}{m} y_{1.} - y_{2.} \end{bmatrix} = \frac{1}{m+n}\begin{bmatrix} y_{..} \\ \frac{n}{m} y_{1.} + y_{1.} - \underbrace{y_{1.} - y_{2.}} \end{bmatrix}$$

$$= \frac{1}{m+n}\begin{bmatrix} y_{..} \\ \frac{m+n}{m} y_{1.} - y_{..} \end{bmatrix} = \begin{bmatrix} \overline{y}_{..} \\ \overline{y}_{1.} - \overline{y}_{..} \end{bmatrix} \qquad \begin{matrix} \hat{\mu} = \overline{y}_{..} \\ \hat{\tau}_1 = \overline{y}_{1.} - \overline{y}_{..} \end{matrix}$$

$$\Rightarrow \hat{\tau}_2 = -\frac{m}{n}\hat{\tau}_1 = -\frac{m}{n}\left(\overline{y}_{1.} - \overline{y}_{..}\right) = -\frac{m}{n}\left(\frac{y_{1.}}{m} - \overline{y}_{..}\right)$$

$$= -\frac{m}{n}\left(\frac{y_{..} - y_{2.}}{m} - \overline{y}_{..}\right) = -\frac{y_{..}}{n} + \frac{y_{2.}}{n} + \frac{m}{n}\overline{y}_{..}$$

$$= -\frac{(m+n)}{n}\overline{y}_{..} + \overline{y}_{2.} + \frac{m}{n}\overline{y}_{..} = \overline{y}_{2.} - \overline{y}_{..} \qquad \hat{\tau}_2 = \overline{y}_{2.} - \overline{y}_{..}$$

NOTE: IF $m=n$, THE CONSTRAINT $m\tau_1 + n\tau_2 = 0$ REDUCES TO $\tau_1 + \tau_2 = 0$ OR, $\tau_2 = -\tau_1$.

NOTE ALSO THAT WE CAN IMPOSE MANY REASONABLE FORMS FOR THE CONSTRAINT. FOR EXAMPLE, SUPPOSE TREATMENT 2 IS A CONTROL TREATMENT. LET'S SET $\tau_2 = 0$. THEN

$$X = \begin{array}{c}\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ \hline 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{array}\right]\begin{array}{l} \\ \\ \\ m \\ \\ \\ \\ n \end{array}\end{array} \quad \beta = \begin{bmatrix} \mu \\ \tau_1 \end{bmatrix} \quad \text{AND} \quad (X'X) = \begin{bmatrix} m+n & m \\ m & m \end{bmatrix} \Rightarrow (X'X)^{-1} = \frac{1}{n}\begin{bmatrix} 1 & -1 \\ -1 & \frac{m+n}{m} \end{bmatrix}$$

$$X'y = \begin{bmatrix} y_{..} \\ y_{1.} \end{bmatrix}$$

$$\text{THEN } \hat{\beta} = \begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \end{bmatrix} = (X'X)^{-1}X'y = \frac{1}{n}\begin{bmatrix} 1 & -1 \\ -1 & \frac{m+n}{m} \end{bmatrix}\begin{bmatrix} y_{..} \\ y_{1.} \end{bmatrix} = \begin{bmatrix} \frac{1}{n}(y_{..} - y_{1.}) \\ \frac{1}{n}\left(-y_{..} + \frac{m+n}{m} y_{1.}\right) \end{bmatrix}$$

$$= \begin{bmatrix} y_{2.}/n \\ \frac{1}{n}(-y_{2.} - y_{1.}) + \frac{m+n}{mn} y_{1.} \end{bmatrix} = \begin{bmatrix} \overline{y}_{2.} \\ -\overline{y}_{2.} + y_{1.}\left(\frac{m+n}{mn} - \frac{1}{n}\right) \end{bmatrix} = \begin{bmatrix} \overline{y}_{2.} \\ -\overline{y}_{2.} + y_{1.}\left(\frac{m+n-m}{mn}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \overline{y}_{2.} \\ -\overline{y}_{2.} + y_{1.}\left(\frac{1}{m}\right) \end{bmatrix} = \begin{bmatrix} \overline{y}_{2.} \\ \overline{y}_{1.} - \overline{y}_{2.} \end{bmatrix}$$

$\hat{\mu} = \overline{y}_{2.}$ (CONTROL TREATMENT MEAN)

$\hat{\tau}_1 = \overline{y}_{2.} - \overline{y}_{1.}$ (DIFFERENCE FROM THE CONTROL MEAN)

AND $\hat{\tau}_2 = 0$

FOR CONSTRAINT $m\tau_1 + n\tau_2 = 0$:

$$\begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \end{bmatrix} = \begin{bmatrix} \bar{y}_{..} \\ \bar{y}_{1.} - \bar{y}_{..} \\ \bar{y}_{2.} - \bar{y}_{..} \end{bmatrix} \Rightarrow \begin{aligned} \hat{\mu}_1 &= \hat{\mu} + \hat{\tau}_1 = \bar{y}_{..} + (\bar{y}_{1.} - \bar{y}_{..}) = \boxed{\bar{y}_{1.}} \quad \hat{\mu}_1 \\ \hat{\mu}_2 &= \hat{\mu} + \hat{\tau}_2 = \bar{y}_{..} + (\bar{y}_{2.} - \bar{y}_{..}) = \boxed{\bar{y}_{2.}} \quad \hat{\mu}_2 \end{aligned}$$

FOR CONSTRAINT $\tau_2 = 0$

$$\begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \end{bmatrix} = \begin{bmatrix} \bar{y}_{2.} \\ \bar{y}_{1.} - \bar{y}_{2.} \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} \hat{\mu}_1 &= \hat{\mu} + \hat{\tau}_1 = \bar{y}_{2.} + (\bar{y}_{1.} - \bar{y}_{2.}) = \boxed{\bar{y}_{1.}} \quad \hat{\mu}_1 \\ \hat{\mu}_2 &= \hat{\mu} + \hat{\tau}_2 = \bar{y}_{2.} + 0 = \boxed{\bar{y}_{2.}} \quad \hat{\mu}_2 \end{aligned}$$

NOTE: $\hat{\mu}_i$ $i=1,2$ ARE THE SAME, DESPITE DIFFERENT CONSTRAINTS.

BECAUSE $\hat{\mu}_1$, $\hat{\mu}_2$ ARE THE SAME FOR BOTH CASES, $VAR(\hat{\mu}_1 - \hat{\mu}_2) = VAR(\bar{y}_1 - \bar{y}_2)$ SHOULD ALSO BE THE SAME IN MATRIX FORM.

BY DEFINITION $\quad VAR(\hat{\mu}_i) = VAR(\hat{\mu} + \hat{\tau}_i) = VAR(\hat{\mu}) + VAR(\hat{\tau}_i) + 2COV(\hat{\mu}, \hat{\tau}_i)$ Ⓐ

Ⓐ FOR CONSTRAINT $m\tau_1 + n\tau_2 = 0$ (OR $\tau_2 = -\frac{m}{n}\tau_1$)

$$VAR(\hat{\beta}) = VAR\left(\begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \end{bmatrix}\right) = \sigma^2 (X'X)^{-1} = \sigma^2 \frac{1}{m+n}\begin{bmatrix} 1 & 0 \\ 0 & n/m \end{bmatrix} = \begin{bmatrix} \sigma^2/m+n & 0 \\ 0 & \sigma^2 \frac{n}{m}\left(\frac{1}{m+n}\right) \end{bmatrix}$$

SUBSTITUTING INTO (*), WE GET

$$VAR(\hat{\mu}_1) = VAR(\hat{\mu}) + VAR(\hat{\tau}_1) + 2COV(\hat{\mu}, \hat{\tau}_1)$$

$$= \frac{\sigma^2}{m+n} + \frac{\sigma^2}{m+n}\cdot\frac{n}{m} + 2(0) = \frac{\sigma^2}{m+n}\left(1 + \frac{n}{m}\right) = \frac{\sigma^2}{m+n}\left(\frac{m+n}{m}\right) = \boxed{\frac{\sigma^2}{m}}$$

$$= VAR(\bar{y}_{1.})$$

$$VAR(\hat{\mu}_2) = VAR(\hat{\mu}) + VAR(\hat{\tau}_2) + 2COV(\hat{\mu}, \hat{\tau}_2)$$

$$= VAR(\hat{\mu}) + VAR\left(-\frac{m}{n}\hat{\tau}_1\right) + 2COV\left(\hat{\mu}, -\frac{m}{n}\hat{\tau}_1\right)$$

$$= \frac{\sigma^2}{m+n} + \frac{m^2}{n^2} VAR(\hat{\tau}_1) - \frac{2m}{n} COV(\hat{\mu}, \hat{\tau}_1)$$

$$= \frac{\sigma^2}{m+n} + \frac{m^2}{n^2}\sigma^2 \frac{n}{m}\left(\frac{1}{m+n}\right) - \frac{2m}{n}(0) = \frac{\sigma^2}{m+n}\left(1 + \frac{m}{n}\right)$$

$$= \frac{\sigma^2}{m+n}\left(\frac{n+m}{n}\right) = \boxed{\frac{\sigma^2}{n}} = VAR(\bar{y}_{2.})$$

(B) FOR CONSTRAINT $\tau_2 = 0$

$$\text{VAR}(\hat\beta) = \text{VAR}\left(\begin{bmatrix}\hat\mu \\ \hat\tau_1\end{bmatrix}\right) = \sigma^2(X'X)^{-1} = \sigma^2\left(\frac{1}{n}\right)\begin{bmatrix}1 & -1 \\ -1 & \frac{m+n}{m}\end{bmatrix} = \begin{bmatrix}\sigma^2/n & -\sigma^2/n \\ -\sigma^2/n & \sigma^2\frac{m+n}{mn}\end{bmatrix}$$

SUBSTITUTING INTO (A), WE GET

$$\text{VAR}(\hat\mu_1) = \text{VAR}(\hat\mu) + \text{VAR}(\hat\tau_1) + 2\,\text{COV}(\hat\mu, \hat\tau_1)$$

$$= \frac{\sigma^2}{n} + \sigma^2\frac{m+n}{mn} - 2\frac{\sigma^2}{n} = \frac{\sigma^2}{n}\left(\frac{m+n}{m} - 1\right) = \frac{\sigma^2}{n}\left(\frac{m+n-m}{m}\right)$$

$$= \frac{\sigma^2}{n}\left(\frac{n}{m}\right) = \boxed{\frac{\sigma^2}{m}} = \text{VAR}(\bar y_{1\cdot})$$

$$\text{VAR}(\hat\mu_2) = \text{VAR}(\hat\mu) + \text{VAR}(\hat\tau_2) + 2\,\text{COV}(\hat\mu, \hat\tau_2)$$

$$= \text{VAR}(\hat\mu) + \text{VAR}(0) + 2\,\text{COV}(\hat\mu, 0) = \text{VAR}(\hat\mu) + 0 + 0$$

$$= \text{VAR}(\hat\mu) = \boxed{\frac{\sigma^2}{n}} = \text{VAR}(\bar y_{1\cdot})$$

WHAT ABOUT $\text{VAR}(\hat\mu_1 - \hat\mu_2) = \text{VAR}(\bar y_{1\cdot} - \bar y_{2\cdot}) = \frac{\sigma^2}{m} + \frac{\sigma^2}{n} = \sigma^2\left(\frac{1}{m} + \frac{1}{n}\right)$? YOU SHOULD GET ~~THIS~~ FROM THE MATRIX FORMS FOR BOTH CONSTRAINTS.

BY DEFINITION,

$$\text{VAR}(\hat\mu_1 - \hat\mu_2) = \text{VAR}(\hat\mu_1) + \text{VAR}(\hat\mu_2) + 2\,\text{COV}(\hat\mu_1, \hat\mu_2)$$

WHERE $\text{COV}(\hat\mu_1, \hat\mu_2) = \text{COV}(\hat\mu + \hat\tau_1, \hat\mu + \hat\tau_2)$

$$= E\left[(\hat\mu + \hat\tau_1 - \mu - \tau_1)(\hat\mu + \hat\tau_2 - \mu - \tau_2)\right] = E\left\{[(\hat\mu - \mu) + (\hat\tau_1 - \tau)][(\hat\mu - \mu) + (\hat\tau_2 - \tau_2)]\right\}$$

$$= E\left[(\hat\mu - \mu)^2 + (\hat\tau_1 - \tau_1)(\hat\mu - \mu) + (\hat\tau_2 - \tau_2)(\hat\mu - \mu) + (\hat\tau_1 - \tau_1)(\hat\tau_2 - \tau_2)\right]$$

$$= \text{VAR}(\hat\mu) + \text{COV}(\hat\tau_1, \hat\mu) + \text{COV}(\hat\tau_2, \hat\mu) + \text{COV}(\hat\tau_1, \hat\tau_2) \quad (**)$$

CLAIM 1. THE EXPRESSION IN (**) EQUALS 0

THUS, $\text{VAR}(\hat\mu_1 - \hat\mu_2) = \text{VAR}(\hat\mu_1) + \text{VAR}(\hat\mu_2) = \frac{\sigma^2}{m} + \frac{\sigma^2}{n} = \sigma^2\left(\frac{1}{m} + \frac{1}{n}\right)$

FOR BOTH CASES.

CLAIM 2: YOU CAN PROVE CLAIM 1 WITHOUT INVOKING INDEPENDENCE OF THE SAMPLES BY USING $\sigma^2(X'X)^{-1}$.

ONEWAY ANOVA MEANS MODEL $y_{ij} = \mu_i + \varepsilon_{ij}$

$$y = \begin{array}{c} \begin{bmatrix} y_{11} \\ \vdots \\ y_{1n_1} \\ \hline y_{21} \\ \vdots \\ y_{2n_2} \\ \hline \vdots \\ \hline y_{a1} \\ \vdots \\ y_{an_a} \end{bmatrix} \\ N\times 1 \end{array} \begin{array}{l} n_1 \\ \\ n_2 \\ \\ \vdots \\ \\ n_a \end{array} \qquad X = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & & & a \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & & 0 \\ \hline 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & & 0 \\ \hline & & 0 & \\ \hline 0 & 0 & & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix} \\ N\times a \end{array} \qquad \beta = \begin{array}{c} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_a \end{bmatrix} \\ a\times 1 \end{array} \qquad \varepsilon = \begin{array}{c} \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1n_1} \\ \hline \varepsilon_{21} \\ \vdots \\ \varepsilon_{2n_2} \\ \hline \vdots \\ \hline \varepsilon_{a1} \\ \vdots \\ \varepsilon_{an_a} \end{bmatrix} \\ N\times 1 \end{array}$$

NOTE: The i^{th} COLUMN OF X CONSISTS OF n_i ONES PRECEDED BY $n_1 + \cdots + n_{i-1}$ ZEROS AND FOLLOWED BY $n_{i+1} + \cdots + n_a$ ZEROS.

THEN $X'X = \begin{bmatrix} n_1 & & & 0 \\ & n_2 & & \\ & & \ddots & \\ 0 & & & n_a \end{bmatrix} = \text{DIAG}(n_1, n_2, \ldots, n_a)$

$$(X'X)^{-1} = \begin{bmatrix} 1/n_1 & & & 0 \\ & 1/n_2 & & \\ & & \ddots & \\ 0 & & & 1/n_a \end{bmatrix} = \text{DIAG}\left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_a}\right)$$

$$X'y = \begin{bmatrix} y_{1\cdot} \\ y_{2\cdot} \\ \vdots \\ y_{a\cdot} \end{bmatrix} \Rightarrow (X'X)^{-1}X'y = \begin{bmatrix} y_{1\cdot}/n_1 \\ y_{2\cdot}/n_2 \\ \vdots \\ y_{a\cdot}/n_a \end{bmatrix} = \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \\ \vdots \\ \bar{y}_{a\cdot} \end{bmatrix} = \begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \\ \vdots \\ \hat{\mu}_a \end{bmatrix}$$

$$\text{VAR}(\hat{\beta}) = (X'X)^{-1}\sigma^2 = \begin{bmatrix} \sigma^2/n_1 & & & 0 \\ & \sigma^2/n_2 & & \\ & & \ddots & \\ 0 & & & \sigma^2/n_3 \end{bmatrix} = \text{DIAG}\left(\frac{\sigma^2}{n_1}, \frac{\sigma^2}{n_2}, \ldots, \frac{\sigma^2}{n_a}\right)$$

THUS, $\text{VAR}(\hat{\mu}_i) = \dfrac{\sigma^2}{n_i}$ (i^{th} DIAGONAL ENTRY OF $(X'X)^{-1}\sigma^2$)

AND $\widehat{\text{VAR}}(\hat{\mu}_i) = \dfrac{MSE}{n_i} \Rightarrow s.e.(\hat{\mu}_i) = \sqrt{\dfrac{MSE}{n_i}}$

ONEWAY ANOVA EFFECTS MODEL $\quad y_{ij} = \mu + \tau_i + \varepsilon_{ij}$

WITH ALL PARAMETERS

$$X = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 0 & & 0 \\ \hline 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 1 & & 0 \\ \hline \vdots & \vdots & \vdots & & \vdots \\ \hline 1 & 0 & 0 & & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & & 1 \end{bmatrix}_{N\times(a+1)}$$

ALL ONES

$$X'X = \left[\begin{array}{c|cccc} N & n_1 & n_2 & \cdots & n_a \\ \hline n_1 & n_1 & 0 & \cdots & 0 \\ n_2 & 0 & n_2 & & 0 \\ \vdots & \vdots & & \ddots & \\ n_a & 0 & \cdots & 0 & n_a \end{array}\right] \quad \text{DIAG}(n_1, n_2, \ldots, n_a)$$

NOTE: COLUMN 1 = SUM OF COLUMNS 2 TO a+1

$\Rightarrow$ $X'X$ IS NOT FULL RANK

$\Rightarrow$ $(X'X)^{-1}$ DOES NOT EXIST

SUPPOSE WE IMPOSE THE CONSTRAINT $\sum_{i=1}^{a} n_i \tau_i = 0$.

THEN $\tau_a = \dfrac{-\sum_{i=1}^{a-1} n_i \tau_i}{n_a}$. WE CAN REWRITE X TO BE FULL RANK:

$$X = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 0 & & 0 \\ \hline 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 1 & & 0 \\ \hline & & \vdots & & \\ \hline 1 & 0 & 0 & & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & & 1 \\ \hline 1 & \frac{-n_1}{n_a} & \frac{-n_2}{n_a} & \cdots & \frac{-n_{a-1}}{n_a} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \frac{-n_1}{n_a} & \frac{-n_2}{n_a} & \cdots & \frac{-n_{a-1}}{n_a} \end{bmatrix} \\ N\times a \end{array} \begin{array}{l} n_1 \\ \\ n_2 \\ \\ \vdots \\ \\ n_{a-1} \\ \\ n_a \end{array} \qquad \beta = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \vdots \\ \tau_{a-1} \end{bmatrix}_{a\times 1} \qquad y = \begin{bmatrix} y_{11} \\ \vdots \\ y_{1n_1} \\ \hline y_{21} \\ \vdots \\ y_{2n_2} \\ \hline \vdots \\ \hline y_{a-1,1} \\ \vdots \\ y_{a-1,n_{a-1}} \\ \hline y_{a1} \\ \vdots \\ y_{an_a} \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1n_1} \\ \hline \varepsilon_{21} \\ \vdots \\ \varepsilon_{2n_2} \\ \hline \vdots \\ \hline \varepsilon_{a-1,1} \\ \vdots \\ \varepsilon_{a-1,n_{a-1}} \\ \hline \varepsilon_{a1} \\ \vdots \\ \varepsilon_{an_a} \end{bmatrix}$$

$$X'X = \left[\begin{array}{c|cccc} N & 0 & 0 & \cdots & 0 \\ \hline 0 & n_1+\frac{n_1^2}{n_a} & \frac{n_1n_2}{n_a} & \cdots & \frac{n_1n_{a-1}}{n_a} \\ 0 & \frac{n_1n_2}{n_a} & n_2+\frac{n_2^2}{n_a} & \cdots & \frac{n_2n_{a-1}}{n_a} \\ \vdots & & & \ddots & \\ 0 & \frac{n_1n_{a-1}}{n_a} & \frac{n_2n_{a-1}}{n_a} & & n_{a-1}+\frac{n_{a-1}^2}{n_a} \end{array}\right] = \left[\begin{array}{c|cccc} N & 0 & 0 & \cdots & 0 \\ \hline 0 & \frac{n_1}{n_a}(n_1+n_a) & \frac{n_1n_2}{n_a} & \cdots & \frac{n_1n_{a-1}}{n_a} \\ 0 & \frac{n_1n_2}{n_a} & \frac{n_2}{n_a}(n_2+n_a) & \cdots & \frac{n_2n_{a-1}}{n_a} \\ \vdots & \vdots & & & \vdots \\ 0 & \frac{n_1n_{a-1}}{n_a} & \frac{n_2n_{a-1}}{n_a} & \cdots & \frac{n_{a-1}}{n_a}(n_{a-1}+n_a) \end{array}\right]$$

$$(X'X)^{-1} = \left[\begin{array}{c|cccc} \frac{1}{N} & 0 & 0 & \cdots & 0 \\ \hline 0 & \frac{N-n_1}{n_1N} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ 0 & -\frac{1}{N} & \frac{N-n_2}{n_2N} & & -\frac{1}{N} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & -\frac{1}{N} & -\frac{1}{N} & \cdots & \frac{N-n_{a-1}}{n_{a-1}N} \end{array}\right]_{a\times a} \qquad X'y = \begin{bmatrix} y_{..} \\ y_{1.} - \frac{n_1}{n_a}y_{a.} \\ y_{2.} - \frac{n_2}{n_a}y_{a.} \\ \vdots \\ y_{a-1.} - \frac{n_{a-1}}{n_a}y_{a.} \end{bmatrix}_{a\times 1}$$

$$\Rightarrow \hat{\beta} = (X'X)^{-1}X'y = \begin{bmatrix} y_{..}/N \\ \frac{N-n_1}{n_1N}\left(y_{1.} - \frac{n_1}{n_a}y_{a.}\right) - \frac{1}{N}\overbrace{\sum_{\substack{i=1\\ i\neq 1}}^{a-1}\left(y_{i.} - \frac{n_i}{n_a}y_{a.}\right)}^{\star} \\ \frac{N-n_2}{n_2N}\left(y_{2.} - \frac{n_2}{n_a}y_{a.}\right) - \frac{1}{N}\sum_{\substack{i=1\\ i\neq 2}}^{a-1}\left(y_{i.} - \frac{n_i}{n_a}y_{a.}\right) \\ \vdots \qquad\qquad \vdots \\ \frac{N-n_{a-1}}{n_{a-1}N}\left(y_{a-1.} - \frac{n_{a-1}}{n_a}y_{a.}\right) - \frac{1}{N}\sum_{\substack{i=1\\ i\neq a-1}}^{a-1}\left(y_{i.} - \frac{n_i}{n_a}y_{a.}\right) \end{bmatrix}_{a\times 1}$$

$$= \frac{1}{N}\begin{bmatrix} y_{..} \\ \frac{N-n_1}{n_1}y_{1.} - \frac{N-n_1}{n_a}y_{a.} - \overbrace{\sum_{i=1}^{a} y_{i.} + y_{1.} + y_{a.} + \frac{N-n_1-n_a}{n_a}y_{a.}}^{\star} \\ \vdots \\ \frac{N-n_{a-1}}{n_{a-1}}y_{a-1.} - \frac{N-n_{a-1}}{n_a}y_{a.} - \underbrace{\sum_{i=1}^{a} y_{i.}}_{y_{..}} + y_{a-1.} + y_{a.} + \frac{N-n_{a-1}-n_a}{n_a}y_{a.} \end{bmatrix}$$

$$= \frac{1}{N}\begin{bmatrix} y_{..} \\ \frac{N}{n_1}y_{1.} + \emptyset y_{a.} - y_{..} \\ \vdots \\ \frac{N}{n_{a-1}}y_{a-1.} + \emptyset y_{a} - y_{..} \end{bmatrix} = \begin{bmatrix} y_{..}/N \\ y_{1.}/n_1 - y_{..}/N \\ \vdots \\ y_{a-1.}/n_{a-1} - y_{..}/N \end{bmatrix} = \begin{bmatrix} \bar{y}_{..} \\ \bar{y}_{1.} - \bar{y}_{..} \\ \vdots \\ \bar{y}_{a-1.} - \bar{y}_{..} \end{bmatrix} = \begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \vdots \\ \hat{\tau}_{a-1} \end{bmatrix}$$

AND USING $\hat{\tau}_a = \dfrac{-\sum_{i=1}^{a-1} n_i \hat{\tau}_i}{n_a} = \dfrac{-\sum_{i=1}^{a-1} n_i (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})}{n_a}$

$= \bar{y}_{a\cdot} - \bar{y}_{\cdot\cdot}$ (VERIFY IN HOMEWORK)

FINALLY, $\text{VAR}(\hat{\beta}) = \sigma^2 (X'X)^{-1}$ WITH

$\text{VAR}(\hat{\tau}_i) = \sigma^2 \dfrac{N-n_i}{n_i N}$ $\left(\dfrac{N-n_i}{n_i N} \text{ IS ALONG THE DIAGONAL OF } (X'X)^{-1} \right)$

AND $\text{VAR}(\hat{\mu}_i) = \text{VAR}(\hat{\mu} + \hat{\tau}_i) = \text{VAR}(\hat{\mu}) + \text{VAR}(\hat{\tau}_i) + 2 \underbrace{\text{COV}(\hat{\mu}, \hat{\tau}_i)}_{0 \text{ IN } (X'X)^{-1}}$

$$= \frac{\sigma^2}{N} + \frac{N-n_i}{n_i N}\sigma^2 + 0 = \sigma^2\left(\frac{n_i + (N-n_i)}{n_i N}\right) = \frac{\sigma^2}{n_i}$$

$$= \text{VAR}(\bar{y}_{i\cdot})$$

IF WE USED CONSTRAINT $\tau_a = 0$ THEN

$$X = \begin{array}{c} \begin{array}{cccc} \;\; & 1 & 2 \cdots & a-1 \end{array} \\ \left[\begin{array}{c|cccc} 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 0 & & 0 \\ \hline 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 1 & & 0 \\ \hline & & \vdots & \vdots & \\ \hline 1 & 0 & 0 & & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & & 1 \\ \hline 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{array}\right] \begin{array}{l} n_1 \\ \\ \\ n_2 \\ \\ \\ \\ n_{a-1} \\ \\ \\ n_a \\ \\ \\ \end{array} \\ N \times a \end{array}$$

$$(X'X) = \left[\begin{array}{c|cccc} N & n_1 & n_2 & \cdots & n_{a-1} \\ \hline n_1 & n_1 & & & 0 \\ n_2 & & n_2 & & \\ \vdots & & & \ddots & \\ n_{a-1} & 0 & & & n_{a-1} \end{array}\right]$$

$$(X'X)^{-1} = \left[\begin{array}{c|cccc} \frac{1}{n_a} & \frac{-1}{n_a} & \frac{-1}{n_a} & \cdots & \frac{-1}{n_a} \\ \hline \frac{-1}{n_a} & \frac{n_1+n_a}{n_1 n_a} & \frac{1}{n_a} & \cdots & \frac{1}{n_a} \\ \frac{-1}{n_a} & \frac{1}{n_a} & \frac{n_2+n_a}{n_2 n_a} & \cdots & \frac{1}{n_a} \\ \vdots & & & & \\ \frac{-1}{n_a} & \frac{1}{n_a} & \frac{1}{n_a} & \cdots & \frac{n_{a-1}+n_a}{n_{a-1} n_a} \end{array}\right]$$

AND $(X'X)^{-1}X'y = \begin{bmatrix} \bar{y}_{a\cdot} \\ \bar{y}_{1\cdot} - \bar{y}_{a\cdot} \\ \bar{y}_{2\cdot} - \bar{y}_{a\cdot} \\ \vdots \\ \bar{y}_{a-1\cdot} - \bar{y}_{a\cdot} \end{bmatrix} = \begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \\ \vdots \\ \hat{\tau}_{a-1} \end{bmatrix}$ WITH $\hat{\tau}_a = 0$.