2.7 Inferences for a Treatment Mean

- Recall: If we have a single sample of size n coming from a normally distributed population $N(\mu, \sigma^2)$, then $\dfrac{\overline{y} - \mu}{s/\sqrt{n}}$ follows a t-distribution with $n-1$ degrees of freedom. An analogous result is true when considering a treatment mean from a oneway ANOVA.
- That is, if the normality and homogeneity of variance assumptions for the oneway ANOVA are valid, then $\dfrac{\overline{y}_{i\cdot} - \mu_i}{\sqrt{MS_E/n_i}}$ follows a t-distribution with $N-a$ degrees of freedom (= d.f. for MS_E).
- Thus, to test $\mathrm{H}_0 : \mu_i = c$ against $\mathrm{H}_1 : \mu_i \neq c$ where c is a hypothesized value for μ_i,
  1. Calculate the test statistic $t = \dfrac{\overline{y}_{i\cdot} - c}{\sqrt{MS_E/n_i}}$.
  2. For a specified α, determine $t^* = t(1-\alpha/2, N-a)$, the critical t-value from the t-distribution with $N-a$ degrees of freedom (or find the p-value.)
  3. Reject H_0 if $t \geq t^*$ or if $t \leq -t^*$ (or compare α to the p-value.)
- To calculate a confidence interval for μ_i, calculate

2.8 Inferences for Differences Between Two Treatment Means

Estimation

- Consider the difference $D_{ij} = \mu_i - \mu_j$ between two of the treatment means μ_i and μ_j. This is also known as a **pairwise comparison** of two means.
- To estimate an unknown D_{ij}, we substitute the sample treatment means for μ_i and μ_j and get

$$\widehat{D}_{ij} =$$

- The expected value of $\widehat{D}_{ij}$: $E(\widehat{D}_{ij}) =$
- The variance of $\widehat{D}_{ij}$: $\sigma^2(\widehat{D}_{ij}) = \mathrm{Var}(\overline{y}_{i\cdot}) + \mathrm{Var}(\overline{y}_{j\cdot}) = \dfrac{\sigma^2}{n_i} + \dfrac{\sigma^2}{n_j} = \sigma^2\left(\dfrac{1}{n_i} + \dfrac{1}{n_j}\right)$
- Thus, the standard deviation of $\widehat{D}_{ij}$: $\sigma(\widehat{D}_{ij}) = \sigma\sqrt{\dfrac{1}{n_i} + \dfrac{1}{n_j}}$
- To estimate $\sigma(\widehat{D}_{ij})$, we replace σ with $\sqrt{MS_E}$. Thus, the standard error of $\widehat{D}_{ij}$ is

$$se(\widehat{D}_{ij}) = \sqrt{MS_E\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

Confidence Intervals and Hypothesis Tests

- Recall: If we have two independent samples of size n_1 and n_2 coming from a normally distributed populations having the same variance σ^2, then $\dfrac{(\overline{y_1} - \overline{y_2}) - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ follows a t-distribution with $n_1 + n_2 - 2$ degrees of freedom.
- The two sample variances are pooled together to get the pooled variance s_p^2, and the degrees of freedom from each sample are pooled together to get $(n_1 - 1) + (n_2 - 1) = n_1 + n_2 - 2$.

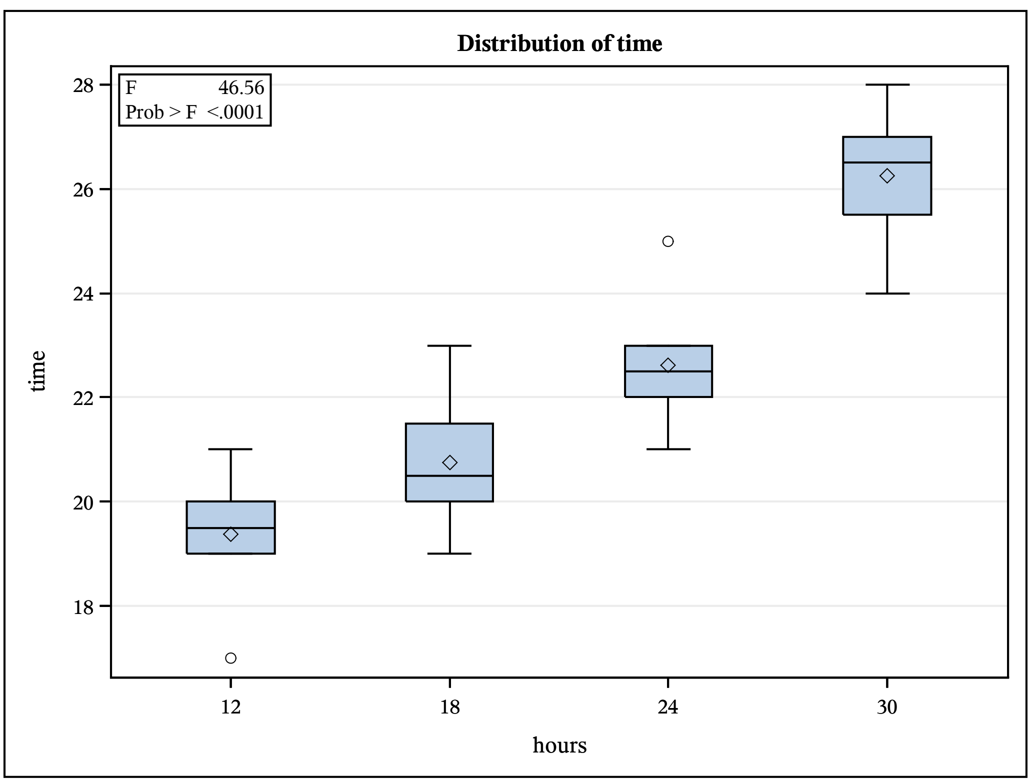
- An analogous result applies when comparing two means, μ_i and μ_j from a oneway ANOVA. We just replace s_p^2 with the MSE.
- That is, if the ANOVA assumptions are valid and if $\mu_i = \mu_j$ (i.e., $D_{ij} = \mu_i - \mu_j = 0$), then

$$t \ = \ \frac{\widehat{D}_{ij}}{se(\widehat{D}_{ij})} \ =$$

follows a t-distribution with $N - a$ degrees of freedom (= MS_E degrees of freedom).

- Thus, if you want to test $\text{H}_0 : \mu_i = \mu_j$ against $\text{H}_1 : \mu_i \neq \mu_j$
  1. Calculate the test statistic $t = \dfrac{\widehat{D}_{ij}}{se(\widehat{D}_{ij})}$
  2. For a specified α, determine $t^* = t(1 - \alpha/2, N - a)$, the critical t-value from the t-distribution with $N - a$ degrees of freedom (or find the p-value.)
  3. Reject H_0 if $t \geq t^*$ or if $t \leq -t^*$ (or compare α to the p-value.)
- To calculate a confidence interval for $\mu_i - \mu_j$, calculate

Revisiting the Sleep Deprivation Example: A study was conducted to determine the effects of sleep deprivation on hand-steadiness. The four levels of sleep deprivation of interest are 12, 18, 24, and 30 hours. 32 subjects were randomly selected and assigned to the four levels of sleep deprivation such that 8 subjects were assigned to each level. The response is the reaction time to the onset of a light cue. The response time is in hundredths of a second.



95% Confidence Intervals and t-tests for μ_i

$$\begin{aligned}\overline{y}_{i\cdot} \pm t^* \sqrt{MS_E/n_i} &= \overline{y}_{i\cdot} \pm t_{.975,28} \sqrt{1.526786/8} \\ &= \overline{y}_{i\cdot} \ \pm \ (2.048)(.43686) \quad = \quad \overline{y}_{i\cdot} \pm .895\end{aligned}$$

- 95% CI for μ_1 (Hours=12): $19.375 \pm .895 \quad \longrightarrow$
- 95% CI for μ_2 (Hours=18): $20.75 \ \pm .895 \quad \longrightarrow$
- 95% CI for μ_3 (Hours=24): $22.625 \pm .895 \quad \longrightarrow$
- 95% CI for μ_4 (Hours=30): $26.25 \ \pm .895 \quad \longrightarrow$

Let $\alpha = .05$. Test $H_0 : \mu_i = 23$ vs $H_1 : \mu_i \neq 23$ for $i = 1, 2, 3, 4$. The test statistic is:

$$t \;=\; \frac{\overline{y}_{i\cdot} - c}{\sqrt{MS_E/n_i}} \;=\; \frac{\overline{y}_{i\cdot} - 23}{.43686} \quad \text{for } i = 1, 2, 3, 4$$

Decision Rule: Reject $H_0 : \mu_i = 23$ if $|t| \geq 2.048$.

- For $H_0 : \mu_1 = 23$ (Hours=12): $t = -8.30 \quad \longrightarrow$
- For $H_0 : \mu_2 = 23$ (Hours=12): $t = -5.15 \quad \longrightarrow$
- For $H_0 : \mu_3 = 23$ (Hours=12): $t = -0.86 \quad \longrightarrow$
- For $H_0 : \mu_4 = 23$ (Hours=12): $t = +7.44 \quad \longrightarrow$

95% Confidence Intervals and t-tests for $\mu_i - \mu_j$

For a difference $D_{ij} = \mu_i - \mu_j$, the standard error is

$$\begin{aligned} se(\widehat{D}_{ij}) &= \sqrt{MS_E \left(\frac{1}{n_i} + \frac{1}{n_j} \right)} \\ &= \sqrt{1.526786 \left(\frac{1}{8} + \frac{1}{8} \right)} \; = \end{aligned}$$

and for a 95% confidence interval for $\mu_i - \mu_j$, calculate

$$\begin{aligned} \widehat{D}_{ij} \ \pm \ t^* se(\widehat{D}_{ij}) &= \widehat{D}_{ij} \ \pm \ (2.048)(.61782) \\ &= (\overline{y}_i - \overline{y}_j) \ \pm \ 1.265 \\ &= \end{aligned}$$

- 95% CI for $\mu_4 - \mu_1$ (Hours=30,12): $(26.25 - 19.375) \pm 1.265 \quad \longrightarrow$
- 95% CI for $\mu_3 - \mu_2$ (Hours=24,18): $(22.625 - 20.75) \pm 1.265 \quad \longrightarrow$

Contrast	DF	Contrast SS	Mean Square	F Value	Pr > F
Linear Trend	1	202.5000000	202.5000000	132.63	<.0001
Quadratic Trend	1	10.1250000	10.1250000	6.63	0.0156
Cubic Trend	1	0.6250000	0.6250000	0.41	0.5275

Parameter	Estimate	Standard Error	t Value	Pr > |t|	95% Confidence Limits	
12 hour effect	-2.8750000	0.37833340	-7.60	<.0001	-3.6499808	-2.1000192
18 hour effect	-1.5000000	0.37833340	-3.96	0.0005	-2.2749808	-0.7250192
24 hour effect	0.3750000	0.37833340	0.99	0.3301	-0.3999808	1.1499808
30 hour effect	4.0000000	0.37833340	10.57	<.0001	3.2250192	4.7749808
12 hour mean	19.3750000	0.43686178	44.35	<.0001	18.4801292	20.2698708
18 hour mean	20.7500000	0.43686178	47.50	<.0001	19.8551292	21.6448708
24 hour mean	22.6250000	0.43686178	51.79	<.0001	21.7301292	23.5198708
30 hour mean	26.2500000	0.43686178	60.09	<.0001	25.3551292	27.1448708
12 vs 18 hrs	1.3750000	0.61781585	2.23	0.0343	0.1094616	2.6405384
12 vs 30 hrs	6.8750000	0.61781585	11.13	<.0001	5.6094616	8.1405384
18 vs 24 hrs	1.8750000	0.61781585	3.03	0.0052	0.6094616	3.1405384
Linear Trend	22.5000000	1.95370527	11.52	<.0001	18.4980162	26.5019838
Quadratic Trend	2.2500000	0.87372356	2.58	0.0156	0.4602584	4.0397416
Cubic Trend	1.2500000	1.95370527	0.64	0.5275	-2.7519838	5.2519838

The SAS output above was generated by the **SOLUTION** option in the **MODEL** statement and by the following SAS code which appears after the MODEL statement:

```
ESTIMATE '12 hour mean' INTERCEPT 1 hours 1 0 0 0;
ESTIMATE '18 hour mean' INTERCEPT 1 hours 0 1 0 0;
ESTIMATE '24 hour mean' INTERCEPT 1 hours 0 0 1 0;
ESTIMATE '30 hour mean' INTERCEPT 1 hours 0 0 0 1;

ESTIMATE '12 vs 18 hrs' hours -1  1  0  0;
ESTIMATE '12 vs 30 hrs' hours -1  0  0  1;
ESTIMATE '18 vs 24 hrs' hours  0 -1  1  0;

ESTIMATE 'Linear Trend'    hours  -3  -1  1 3;
ESTIMATE 'Quadratic Trend' hours   1  -1 -1 1;
ESTIMATE 'Cubic Trend'     hours  -1   3 -3 1;

CONTRAST 'Linear Trend'    hours  -3  -1  1 3;
CONTRAST 'Quadratic Trend' hours   1  -1 -1 1;
CONTRAST 'Cubic Trend'     hours  -1   3 -3 1;
```

2.9 Inferences for Linear Combinations and Contrasts of Treatment Means

- Let $\sum_{i=1}^{a} c_i\mu_i$ be a **linear combination** of the treatment means (i.e., at least one $c_i \neq 0$).
- A **contrast** Γ is a comparison involving a linear combination of two or more treatment means subject to the restriction that the sum of the coefficients = 0. That is,

$$\Gamma = \sum_{i=1}^{a} c_i\mu_i \qquad \text{where} \qquad \sum_{i=1}^{a} c_i = 0$$

- Any pairwise comparison of means $\Gamma = \mu_i - \mu_j$ is an example of a contrast.

<u>**Estimation**</u>

- $\widehat{\Gamma} = \sum_{i=1}^{a} c_i\overline{y}_{i\cdot}$ is an unbiased estimate of the linear combination $\Gamma = \sum_{i=1}^{a} c_i\mu_i$.
- The expected value of $\widehat{\Gamma}$: $E(\widehat{\Gamma}) = \sum_{i=1}^{a} c_i\mu_i = \Gamma$. Thus, $\widehat{\Gamma}$ is an unbiased estimator of Γ.
- Recall that the variance of a sum of independent statistics equals the sum of the variances of the statistics. Therefore, the variance of $\widehat{\Gamma}$ is

$$\sigma^2(\widehat{\Gamma}) = \sum_{i=1}^{a} \text{Var}(c_i\overline{y}_{i\cdot}) = \sum_{i=1}^{a} c_i^2\text{Var}(\overline{y}_{i\cdot}) =$$

- The standard deviation of $\widehat{\Gamma}$: $\sigma(\widehat{\Gamma}) = \sigma\sqrt{\sum_{i=1}^{a} \frac{c_i^2}{n_i}}$.
- Using MS_E to estimate σ^2, the standard error of $\widehat{\Gamma}$ is $se(\widehat{\Gamma}) =$.
- These estimates will be useful when we are interested in generating confidence intervals and testing hypotheses about $\Gamma = \sum_{i=1}^{a} c_i\mu_i$.

<u>**Confidence Intervals and Hypothesis Tests**</u>

- If the assumptions for the single factor ANOVA are valid, then

$$\frac{\widehat{\Gamma} - \Gamma}{se(\widehat{\Gamma})} = \frac{\widehat{\Gamma} - \Gamma}{\sqrt{MS_E\sum(c_i^2/n_i)}}$$

follows a t-distribution with $N - a$ degrees of freedom (= MS_E degrees of freedom).

- Thus, if you want to test $\text{H}_0 : \Gamma = 0$ against $\text{H}_1 : \Gamma \neq 0$
  1. Calculate the test statistic $t = \dfrac{\widehat{\Gamma}}{se(\widehat{\Gamma})}$.
  2. For a specified α, determine $t^* = t(1 - \alpha/2, N - a)$, the critical t-value from the t-distribution with $N - a$ degrees of freedom (or find the p-value.)
  3. Reject H_0 if $t \geq t^*$ or if $t \leq -t^*$ (or compare α to the p-value.)

- A confidence interval for Γ is given by $\quad \widehat{\Gamma} \pm t^* se(\widehat{\Gamma})$
- On computer output, you may get the results for a $(1, N-a)$ degrees of freedom F-test. However, the sum of squares for Γ is $SS_\Gamma = MS_\Gamma$. In other words,

$$SS_\Gamma \;=\; MS_\Gamma \;=\; \frac{\widehat{\Gamma}^2}{\sum_{i=1}^a c_i^2/n_i}.$$

Thus, $F \;=\; \dfrac{MS_{trt}}{MS_E} \;=\; \dfrac{\widehat{\Gamma}^2}{MS_E \sum_{i=1}^a c_i^2/n_i} \;=\; \dfrac{\widehat{\Gamma}^2}{se(\widehat{\Gamma})} \;=\; t^2.$

SOME RESULTS CONCERNING THE MEAN SQUARE FOR A CONTRAST

- $E(\hat{\Gamma}-\Gamma)^2 = E\left[\left(\sum_{i=1}^a c_i\bar{y}_{i\cdot} - \sum_{i=1}^a c_i\mu_i\right)^2\right] = E\left\{\left[\sum c_i(\bar{y}_{i\cdot}-\mu_i)\right]^2\right\}$

$$= E\left\{\left[c_1(\bar{y}_{1\cdot}-\mu_1) + c_2(\bar{y}_{2\cdot}-\mu_2) + \cdots + c_a(\bar{y}_{a\cdot}-\mu_a)\right]^2\right\}$$

$$= E\left[\sum_{i=1}^a c_i^2(\bar{y}_{i\cdot}-\mu_i)^2 + 2\sum_{i<j} c_i c_j(\bar{y}_{i\cdot}-\mu_i)(\bar{y}_{j\cdot}-\mu_j)\right]$$

$$= \sum_{i=1}^a c_i^2\, E\left[(\bar{y}_{i\cdot}-\mu_i)^2\right] + 2\sum_{i<j} c_i c_j\, E\left[(\bar{y}_{i\cdot}-\mu_i)(\bar{y}_{j\cdot}-\mu_j)\right]$$

$$= \sum_{i=1}^a c_i^2\, \mathrm{VAR}(\bar{y}_{i\cdot}) + 2\sum_{i<j} c_i c_j\, \mathrm{Cov}(\bar{y}_{i\cdot}, \bar{y}_{j\cdot})$$

BUT $\bar{y}_{i\cdot}$ AND $\bar{y}_{j\cdot}$ ARE INDEPENDENT $\forall i,j$

$\Rightarrow \mathrm{COV}(\bar{y}_{i\cdot}, \bar{y}_{j\cdot}) = 0 \quad \forall i,j$

$$= \sum_{i=1}^a c_i^2 \frac{\sigma^2}{n_i} + 0$$

(USING THE HOV ASSUMPTION)

$$= \sigma^2 \sum \frac{c_i^2}{n_i}$$

THEN

$$E\left[\frac{(\hat{\Gamma}-\Gamma)^2}{\sum \frac{c_i^2}{n_i}}\right] = \sigma^2$$

THUS, IF $H_0: \Gamma = 0$ IS TRUE,

$$E\left[\frac{\hat{\Gamma}^2}{\sum \frac{c_i^2}{n_i}}\right] = E\left[\frac{(\sum c_i \bar{y}_{i\cdot})^2}{\sum \frac{c_i^2}{n_i}}\right] = E[MS_\Gamma] = \sigma^2$$

- RECALL

(1) $E[\hat{\Gamma}] = \Gamma$, $Var(\hat{\Gamma}) = \sigma^2 \sum \frac{c_i^2}{n_i}$

(ASSUMING HOV)

(2) EACH $\overline{y}_i \sim N(\mu_i, \sigma^2/n_i)$

(ASSUMING NORMALITY)

THUS,

(3) $\hat{\Gamma} \sim N\left(\Gamma, \sigma^2 \sum \frac{c_i^2}{n_i}\right)$

(4) STANDARDIZING $\hat{\Gamma}$ YIELDS

$$Z = \frac{\hat{\Gamma} - \Gamma}{\sqrt{\sigma^2 \sum \frac{c_i^2}{n_i}}} \sim N(0,1)$$

(5) IF $H_0: \Gamma = 0$ IS TRUE, THEN

$$Z = \frac{\hat{\Gamma}}{\sqrt{\sigma^2 \sum \frac{c_i^2}{n_i}}} = \frac{\sum c_i \overline{y}_{i\cdot}}{\sqrt{\sigma^2 \sum \frac{c_i^2}{n_i}}} \sim N(0,1)$$

(6) THEN

$$Z^2 = \frac{(\sum c_i \overline{y}_{i\cdot})^2}{\sigma^2 \sum \frac{c_i^2}{n_i}} \sim \chi^2(1)$$

BUT $Z^2 = MS_\Gamma / \sigma^2$

$$\Rightarrow \frac{MS_\Gamma / \sigma^2}{MS_E / \sigma^2} = \frac{MS_\Gamma}{MS_E} = F_\Gamma \sim F(1, N-a)$$

IF $H_0: \Gamma = 0$ IS TRUE

2.9.1 Orthogonal Contrasts

- Let $c_1, c_2, \ldots, c_a$ and $d_1, d_2, \ldots, d_a$ be the coefficients of two contrasts Γ_1 and Γ_2.
- If $\sum_{i=1}^{a} c_i d_i = 0$ then Γ_1 and Γ_2 are **orthogonal contrasts**.
- In a oneway ANOVA with a factor levels, any set of $a-1$ mutually orthogonal contrasts partition the SS_{trt} into $a-1$ single degree of freedom components. Therefore, all tests of orthogonal contrasts are independent.
- If the a levels are equally spaced on a numerical scale, then there exists a set of $a-1$ orthogonal contrasts that test for polynomial effects (linear, quadratic, cubic, etc.) up to order $a-1$. The contrast coefficients can be found in Table IX of the text.

CONTRAST EXAMPLE: COMPARE THE AVERAGE OF THE 12 AND 18 HOUR MEANS TO THE AVERAGE OF THE 24 AND 30 HOUR MEANS, THAT IS, DOES

$$\frac{\mu_1+\mu_2}{2} = \frac{\mu_3+\mu_4}{2} \;? \quad \text{OR, DOES} \quad \frac{\mu_1+\mu_2}{2} - \frac{\mu_3+\mu_4}{2} = 0 \;?$$

OR, DOES

$$\underbrace{\tfrac{1}{2}\mu_1 + \tfrac{1}{2}\mu_2 - \tfrac{1}{2}\mu_3 - \tfrac{1}{2}\mu_4}_{\text{CONTRAST } \Gamma} = 0$$

THEN $\hat{\Gamma} = \frac{1}{2}\bar{y}_{1\cdot} + \frac{1}{2}\bar{y}_{2\cdot} - \frac{1}{2}\bar{y}_{3\cdot} - \frac{1}{2}\bar{y}_{4\cdot}$

$$= \tfrac{1}{2}(19.375 + 20.75 - 22.625 - 26.25) = -4.375$$

$\Rightarrow$ A 95% CONFIDENCE INTERVAL FOR Γ IS

$$\hat{\Gamma} \pm t^* \, s.e.(\hat{\Gamma}) \Rightarrow -4.375 \pm 2.048\sqrt{MSE \cdot \Sigma c_i^2/n_i}$$

$$\Rightarrow -4.375 \pm 2.048\sqrt{1.526786\left(\frac{(1/2)^2+(1/2)^2+(-1/2)^2+(-1/2)^2}{8}\right)}$$

$$\Rightarrow -4.375 \pm 2.048\,(.43686)$$

$$\Rightarrow -4.375 \pm .895 \quad \text{OR} \quad (-5.27, -3.48)$$

ORTHOGONAL CONTRASTS EXAMPLE

LINEAR TREND $\Gamma_L = -3\mu_1 - \mu_2 + \mu_3 + 3\mu_4$

QUADRATIC TREND $\Gamma_Q = \mu_1 - \mu_2 - \mu_3 + \mu_4$

CUBIC TREND $\Gamma_C = -\mu_1 + 3\mu_2 - 3\mu_3 + \mu_4$

$\rightarrow \hat{\Gamma}_L = -3(19.375) - 20.75 + 22.625 + 3(26.25) = 22.525$

$\hat{\Gamma}_Q = 19.375 - 20.75 - 22.625 + 26.25 = 2.25$

$\hat{\Gamma}_C = -19.375 + 3(20.75) - 3(22.625) + 26.25 = 1.25$

FOR Γ_L AND Γ_Q, $\sum \frac{c_i^2}{n_i} = \frac{20}{8} = 2.5$

FOR Γ_Q, $\sum \frac{c_i^2}{n_i} = \frac{4}{8} = .5$

THEREFORE

$$s.e.(\hat{\Gamma}_L) = s.e.(\hat{\Gamma}_c) = \sqrt{MSE \sum \frac{c_i^2}{n_i}} = \sqrt{1.526786(2.5)} = 1.9537$$

AND

$$s.e.(\hat{\Gamma}_Q) = \sqrt{MSE \sum \frac{c_i^2}{n_i}} = \sqrt{1.526786(.5)} = .8737$$

HYPOTHESIS TESTING

$H_0: \Gamma_L = 0$

$H_A: \Gamma_L \neq 0$

$$MS(\Gamma_L) = \frac{(\sum c_i \bar{y}_{i.})^2}{\sum n_i c_i^2} = \frac{(8\hat{\Gamma}_L)^2}{8(9+1+1+9)} = \frac{(8(22.525))^2}{160}$$

1 d.f.

$$= \frac{180.2^2}{160} = 202.95025$$

$$\Rightarrow F_0 = \frac{MS(\Gamma_L)}{MSE} = \frac{202.95025}{1.5267875} = 132.962 \Rightarrow t_0 = \sqrt{F_0} = 11.529$$

$H_0: \Gamma_Q = 0$

$H_A: \Gamma_Q \neq 0$

$$MS(\Gamma_Q) = \frac{(8\hat{\Gamma}_Q)^2}{8(1+1+1+1)} = \frac{[8(2.25)]^2}{32} = \frac{18^2}{32} = 10.125$$

$$\Rightarrow F_0 = \frac{MS(\Gamma_Q)}{MSE} = \frac{10.125}{1.5267875} = 6.63157 \Rightarrow t_0 = \sqrt{F_0} = 2.575$$

$H_0: \Gamma_c = 0$

$H_A: \Gamma_c \neq 0$

$$MS(\Gamma_c) = \frac{(8\hat{\Gamma}_c)^2}{8(1+9+9+1)} = \frac{(8(1.25))^2}{160} = \frac{100}{160} = .625$$

$$\Rightarrow F_0 = \frac{MS(\Gamma_c)}{MSE} = \frac{.625}{1.5267875} = .4094 \Rightarrow t_0 = \sqrt{F_0} = .6398$$