Hopf Bifurcations - An introduction

Consider the planar system

$$\frac{dx}{dt} = f(x,y;\mu) \quad , \tag{1}$$

$$\frac{dy}{dt} = g(x,y;\mu) \quad , \tag{2}$$

where μ is a parameter. Alternately, we have the notations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

Further, let $\bar{\mathbf{x}}(\mu) = (\bar{x}(\mu), \bar{y}(\mu))$ be the equilibria. The Jacobian of the vector field $\mathbf{f}(\mathbf{x})$ at $\bar{\mathbf{x}}$ is

$$\mathbf{Df}(\bar{\mathbf{x}}) = \begin{bmatrix} f_x(\bar{x},\bar{y}) & f_y(\bar{x},\bar{y}) \\ g_x(\bar{x},\bar{y}) & g_y(\bar{x},\bar{y}) \end{bmatrix}$$

The eigenvalues of $\mathbf{Df}(\bar{\mathbf{x}})$ are functions of the parameter μ. In terms of the trace $Tr\mathbf{Df}$ and determinant $det\mathbf{Df}$, the eigenvalues of the Jacobian are:

$$\lambda_{\pm}(\mu) = \frac{Tr\mathbf{Df} \pm \sqrt{(Tr\mathbf{Df})^2 - 4det\mathbf{Df}}}{2}$$

In this summary we consider the special case where at some parameter value $\mu = \mu_0$

$$Tr\mathbf{Df}(\bar{\mathbf{x}}(\mu_0)) = 0 \tag{3}$$

$$det\mathbf{Df}(\bar{\mathbf{x}}(\mu_0)) > 0 \tag{4}$$

When these two conditions are satisfied, the eigenvalues of the Jacobian are purely imaginary. If, in addition to (3)-(4) being satisfied, the *transversality condition*

$$\frac{d}{d\mu}\left\{\mathbb{R}e\left(\lambda_+(\mu)\right)\right\}|_{\mu=\mu_0} \neq 0 \tag{5}$$

is satisfied, then a *Hopf* bifurcation occurs at the bifurcation point $(\bar{\mathbf{x}}(\mu_0)), \mu_0)$ (here, $\mathbb{R}e(z)$ is the real part of z). At such a Hopf bifurcation for some μ near μ_0, small amplitude oscillations (limit cycles) exist. The amplitude of these oscillations approaches zero as μ approaches μ_0. Though Hopf theory guarantees the existence of such periodic orbits for $\mu \simeq \mu_0$, it does not guarantee the existence of the oscillations for μ further away from μ_0. Often, however, the periodic orbits persist and grow in amplitude as $|\mu - \mu_0|$ increases.

At $\mu = \mu_0$ the linearized system (linearization of (1)-(2) about $\bar{\mathbf{x}}$)

$$\frac{dz}{dt} = \mathbf{Df}(\bar{\mathbf{x}})z. \quad , \quad z = (z_1, z_2) \in \mathbb{R}^2 \tag{6}$$

has a center at $z = 0$. Therefore, solutions $z(t)$ have the form

$$z(t) = c_1\vec{\zeta}_1 \cos\omega t + c_2\vec{\zeta}_2 \sin\omega t$$

for some real constants c_k and constant vectors $\zeta_k,\ k = 1, 2$. Given the assumed conditions (3)-(4), $\lambda_\pm = \pm\omega i$ where $i^2 = -1$ and

$$\omega = \sqrt{det\mathbf{Df}} \tag{7}$$

By Hopf theory, if (3)-(5), are satisfied then for every μ with $|\mu - \mu_0|$ sufficiently small, there exists a T-periodic orbit (limit cycle) $\mathbf{x}_p(t;\mu)$ which satisfy (1)-(2). The period $T = T(\mu)$ and Hopf theory also guarantees

$$\lim_{|\mu-\mu_0|\to 0} T(\mu) = \frac{2\pi}{\omega} \tag{8}$$

In other words, for μ very nearly equal μ_0, the period of the (emergent) periodic orbits of (1)-(2) nearly equals the period of the concentric periodic orbits of the linearized system (6).

If the Jacobian has the very special form:

$$\mathbf{Df}(\bar{\mathbf{x}}_\mathbf{0}) = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} \quad , \quad \bar{\mathbf{x}}_\mathbf{0} = \bar{\mathbf{x}}(\mu_0)$$

then a third-order Taylor Series expansion of (1)-(2) about $\bar{\mathbf{x}}$ yields a system of the form:

$$\frac{dz_1}{dt} = (d\mu + a(z_1^2 + z_2^2))z_1 - (\omega + c\mu + b(z_1^2 + z_2^2))z_2 \tag{9}$$

$$\frac{dz_1}{dt} = (\omega + c\mu + b(z_1^2 + z_2^2))z_1 + (d\mu + a(z_1^2 + z_2^2))z_2 \tag{10}$$

which when expressed in polar coordinates is

$$\frac{dr}{dt} = (d\mu + ar^2)r \tag{11}$$

$$\frac{d\theta}{dt} = (\omega + c\mu + br^2) \tag{12}$$

for constants $a, b, c, d, \omega,\ z_1 = r\cos\theta, z_2 = r\sin\theta$. Note the equation for $r(t)$ is not coupled to the equation for θ. Furthermore, depending on the signs of the constants a and d, this third-order system possesses periodic orbits along the locus

$$\mu = -ar^2/d \quad d \neq 0$$

It can be shown that

$$d = \frac{d}{d\mu}\left\{\mathbb{R}e\left(\lambda_+(\mu)\right)\right\}|_{\mu=\mu_0}$$

so that the existence of periodic orbits local to the bifurcation point depends on $d \neq 0$. This is just the transversality condition (5).