Ph.D. Comprehensive Exam in Dynamical Systems
Differential Equations
May 30, 2007

Instructions: Show all work, but be concise. If you find an obvious typo, please correct it and proceed. If you find a serious error or genuine ambiguity, please bring it to the monitor's attention. If you need a definition or help with terminology, such assistance will be provided (in most cases). You have up to four hours and are expected to work the majority of the problems.

1. Let $\dot{x} = f(x)$. Suppose f is a *gradient vector field* on R^n. Assume solutions exist globally on R^n. Prove that if there is a periodic orbit then it is a rest point.

2. The simple pendulum is given by the second order equation

$$\ddot{\theta} - k\ \sin(\theta) = 0$$

for parameter $k > 0$. Supply a hamiltonian function $H(\theta, \dot{\theta})$ for this system. Use this function to describe and determine the solutions near the origin $(0, 0)$ in phase space $(\theta, \dot{\theta})$. Now suppose the equation becomes

$$\ddot{\theta} - \delta\,\dot{\theta} - k\ \sin(\theta) = 0,$$

where $\delta > 0$.
Show that any solution sufficiently close to the origin tends to the origin as $t \to \infty$.

3. Compute the time required for the solution to

$$\dot{x} = x(1 - y)\dot{y} = y(x - 1)$$

with initial condition $(x, y) = (1, 0)$ to arrive at $(x, y) = (2, 0)$. Show this system has a section or "Poincaré map" $h(y)$ from a neighborhood of $(1, 0)$ on the line $x = 1$ into a neighborhood of $(2, 0)$ on the line $x = 2$. Compute $h'(0)$.

4. Draw the phase portrait for the system

$$\ddot{x} - x^2 + x^3 = 0.$$

Is the solution through $x(0) = 1/2$ and $\dot{x}(0) = 0$ periodic? (Justify your answer, of course.)

5. The flow box theorem roughly asserts that away from a rest point solutions can be "straightened out". State this famous theorem precisely and provide a proof.

DO NOT WRITE ON BACK

SKETCHES OF SOLUTIONS.

1. Since the system is gradient there exists a real function which strictly increases along solution curves. But such a function would be multiple valued along a nontrivial periodic solution.

2. A hamiltonian is given by

$$H(\theta, \dot{\theta}) = \dot{\theta}^2/2 - k\cos(\theta)$$

whose level curves are

$$\dot{\theta}^2/2 - k\cos(\theta) = H_0.$$

Near the origin, which is an isolated rest point, $H_0 \approx -k$. If $x = \theta$ and $y = \dot{\theta}$, $dy/dx = -k \sin(x)/y$, which is zero on the y-axis and infinity on the x-axis. Also this derivative is negative in the interior first quadrant. By the implicit function theorem (note $H_y = y$) and symmetry these arcs reflect across the axes. Hence, near the origin, the level curves are closed curves that contain no rest points.

For the second partk, note the divergence is $-\delta < 0$.

3. First we note that $y = 0$ is an invariant manifold, since y is a factor of $\dot{y}$. So the solution through $(1, 0)$ satisfies $\dot{x} = x$, and $x = \exp(t)$ with $y = 0$. So it takes one second to pass from $x = 1$ to $x = 2$. Let $t(\xi)$ denote the time it takes for a solution to pass from $x = 1, y = \xi$ to $x = 2$. For the Poincaré map, we have

$$h'(\xi) = (d/d\xi)y(t(\xi)) = \dot{y}(t(\xi))\, dt/d\xi = y(t(\xi))(x(t(\xi)) - 1)\, dt/d\xi.$$

Setting $\xi = 0$,

$$h'(0) = h(0)(2 - 1)dt/d\xi|_{\xi=0} = 0.$$

4.

5. See any text.