M171 Fall 2018: Third Examination
Practice Problems over 4.1-4.5 and 4.7-4.8

You can check your answers in WebWork. Full solutions in WW available Friday evening.

Problem 1. Multiple choice: Suppose that $f''(x) < 0$ for x near a point a. Then the linearization of f at a is

1. an over approximation
2. an under approximation
3. unknown without more information.

Problem 2.

a. Find the linear approximation to $\sqrt{a+x}$ for x near zero. Here a is a positive constant.

b. Use your approximation to estimate $\sqrt{4+0.1}$

Problem 3. Consider the Monod Growth function

$$G(S) = \mu \frac{S}{K+S}.$$

a. Find the linear approximation to G at $S = 0$

b. Find the linear approximation to G at $S = K$

Problem 4. You want to calculate the depth of a fissure on Mars from the equation $s(t) = 4t^2$ by timing how long it takes a heavy stone you drop to hit the ice below. Here $s(t)$ is the distance in meters an object falls in t seconds under the influence of gravity on Mars. Suppose you measure the time to be 5 seconds. Use differentials to estimate the error in depth if there is a 0.1 second inaccuracy in measuring the time.

Problem 5. Suppose you are driving along a highway in a car with a broken speedometer at a nearly constant speed and you can record the number of seconds it takes to travel between two consecutive mile markers. If it takes 60 seconds, then your average speed is 1 mi/60 s or 60 mi/hr. The function

$$s(x) = \frac{3600}{60+x}$$

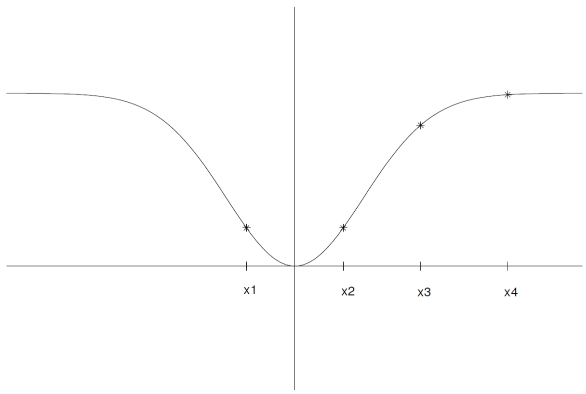
gives your average speed in mi/hr if you travel one mile in $60+x$ seconds. For example, if you drive one mile in 62 seconds, $x = 2$ and your average speed is $s(2) \approx 58.06$ mi/hr. If you travel one mile in 57 seconds, then $x = -3$ and your average speed is $s(-3) \approx 63.16$ mi/hr.

Because you don't want to use your phone or calculator while driving, you need an easy approximation to this function. Use linear approximation to derive such a formula.

Problem 6. Recall that Newton's method is used to find approximate solutions to $f(x) = 0$ by using the iteration: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Show how to setup Newton's method to find the cube root of b, i.e. show how would to setup Newton's method to compute a solution of $x^3 = b$.

Problem 7. Let $(a, f(a))$ be the point where the tangent line to the graph of $f(x) = x\cos(x)$ is horizontal. We use Newton's method to find $x_1, x_2, x_3, \ldots,$ the successive approximations to a. Give the formula that we use to compute x_{n+1} from x_n.

Problem 8. We will use each of the x_n below as the starting point for Newton's method. For which of them do you expect Newton's method to work and lead to the root of the function?



1. x_1 and x_2 only.
2. x_2 only.
3. x_1, x_2 and x_3 only.
4. All four.

Problem 9. Explain why Newton's method eventually fails when finding zeros of $f(x) = x^3 - 3x + 7$ with a starting value $x_1 = 2$.

Problem 10. Write the formula for Newton's method applied to $f(x) = x^{1/5}$, and use the given initial approximation $x_0 = 1$ to compute x_1 and x_2. What happens here? Does the method converge?

Problem 11. Find all critical points of the following functions (if they exist)

a. $f(x) = \dfrac{x^2}{2} - 3\sqrt[3]{x}$

b. $f(x) = (x^2 - 4)^{\frac{1}{3}}$

c. $f(x) = xe^{-x}$

d. $f(x) = \dfrac{x^2}{1+x}$

e. $f(x) = x^{1/3} - x$

Problem 12. Find the absolute maximum and absolute minimum values of the following functions on the given intervals. Specify the x-values where these extrema occur.

a. $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$

b. $h(x) = x + \sqrt{1 - x^2}$ on $[-1, 1]$

c. $f(x) = \dfrac{x}{\ln x}$ on $[\sqrt{e}, e^4]$

d. $f(x) = x \ln x$ on $(0, 1]$ (Hint: use L'H rule to figure out what happens as $x \to 0^+$.)

Problem 13. The power transmitted by a pulley belt system is given by the following equation

$$P(v) = (Fv - \rho A v^2)(1 - e^{-\mu\theta})$$

Where P is the power, v is the velocity of the belt, ρ is the density of the belt, A is the cross sectional area of the belt, F is the maximum force in the belt, μ is the coefficient of friction, and θ is the contact angle. Assuming the P is a function of v and that all the other parameters are constant and positive, determine the velocity that makes the power a maximum, and find a formula for this maximum power.

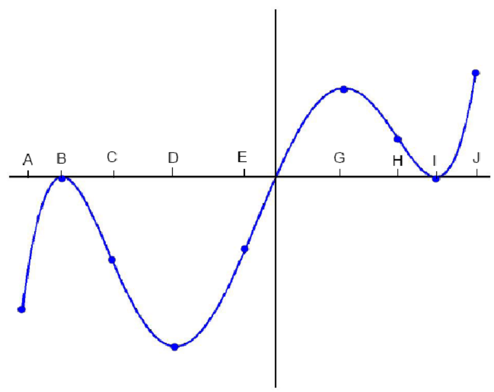
Problem 14. The Reaction $R(x)$ of a patient to a drug dose of size x depends on the type of drug. For a certain drug, it was determined that a good description of the relationship is:

$$R(x) = Ax^2(B - x), \qquad 0 \le x \le B$$

where A and B are positive constants. The Sensitivity of the patients body to the drug is defined to be $R'(x)$.

a. For what value of x is the reaction a maximum, and what is that maximum reaction value?
b. For what value of x is the sensitivity a maximum? What is that maximum sensitivity?

Problem 15. The following graph is the graph of $f'(x)$. Where do the points of inflection of $f(x)$ occur and on which intervals is f concave up? Does f have a local max or min at $x = 0$?



Problem 16. Sketch the graph of a function for which:

$$f(0) = 0, \qquad \begin{array}{l} f'(x) > 0 \text{ for } x < 0, \\ f'(x) > 0 \text{ for } x > 0, \end{array} \qquad \begin{array}{l} \lim_{x\to-\infty} f(x) = 1, \\ \lim_{x\to\infty} f(x) = 0, \end{array} \qquad \begin{array}{l} f''(x) > 0 \text{ for } x < 0, \\ f''(x) < 0 \text{ for } x > 0. \end{array}$$

Problem 17. Given that $f(x) = \frac{x^2}{x^2+1}$, $f'(x) = \frac{2x}{(x^2+1)^2}$, and $f''(x) = \frac{2-6x^2}{(x^2+1)^3}$. Make a sketch of the graph the function f. Be sure to include local extrema and points of inflection.

Problem 18. Multiple Choice.

The second derivative, $f''(x)$, of a function $f(x)$ is negative everywhere. We know that $f(0) = 0$ and $f'(0) = 0$. What must be true about $f(1)$?

a. $f(1)$ is negative
b. $f(1)$ is positive
c. $f(1)$ is zero
d. Not enough information to conclude anything about $f(1)$.

Problem 19. Suppose that $g(x)$ is differentiable for all x and that $-5 \leq g'(x) \leq 3$ for all x. Assume also that $g(0) = 4$. Based on this information, use the Mean Value Theorem to determine the largest and smallest possible values for $g(2)$.

Problem 20. The fastest drag racers can reach a speed of 300 mi/hr over a quarter-mile strip in 4.45 seconds (from a standing start), Complete the following sentence about such a drag racer: At some point during the race, the maximum acceleration of the drag racer is at least ______ mi/hr/sec.

Problem 21. Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$.

Problem 22. Evaluate the following limits.

a. $\displaystyle\lim_{x\to\pi} \frac{x-\pi}{\sin(x-\pi)}$

b. $\displaystyle\lim_{x\to\infty} x^2 \sin\left(\frac{1}{2x^2}\right)$

c. $\displaystyle\lim_{x\to 0^+} \frac{x}{\ln(x)}$

d. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x+1}}{\sqrt{2x+1}}$

e. $\displaystyle\lim_{x\to\pi/2} \frac{1-\sin(x)}{\cos(x)}$

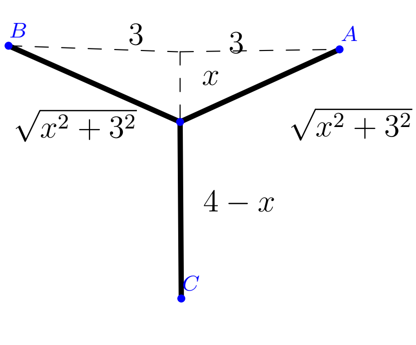
f. $\displaystyle\lim_{x\to\infty} \frac{x\ln(x)}{x^a}$; $a > 1$.

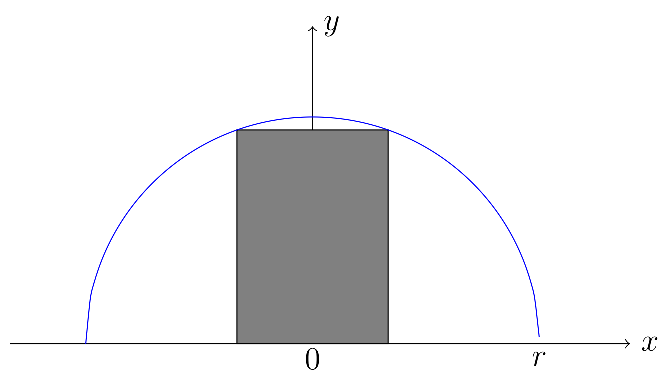
g. $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h}$

h. $\displaystyle\lim_{x\to 1} \frac{\ln x}{1-x}$

i. $\displaystyle\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x$

Problem 23. The bottom of the legs of a three-legged table are the vertices of an isosceles triangle with sides 5,5,6. The legs are to be braced at the bottom by three wires in the shape of a **Y**. What is the minimum length of wire needed? Show it is a minimum.





Problem 24.
An architect plans to build a rectangular window under a arch that is a semi-circle. Find the maximum area of a rectangle inscribed in semi-circle of radius r.