MATH 181 October 16, 2007 **EXAM 2**

Name KEY Section________

Prob	1	2	3	4	5	6	7	8	9	10	11	Total
Value	8	8	10	10	6	6	15	10	7	10	10	100
Points												

1. Does the equation $xe^x = 1 - 2x$ have a root in the interval $(0, 1)$? Justify your answer.

Let $f(x) = xe^x - 1 + 2x$. Then f is continuous on the interval $[0, 1]$. Also, $f(0) = -1 < 0$ and $f(1) = e + 1 > 0$. By the mean value theorem, there must be an x, $0 \le x \le 1$, so that $f(x) = 0$. Since $f(0) \ne 0 \ne f(1)$, this x must actually be in the open interval $(0, 1)$. So, yes, the equation $xe^x - 1 + 2x = 0$ has a root in the interval $(0, 1)$.

2. Find the horizontal asymptotes of $y = \frac{x-1}{\sqrt{4x^2+1}}$.

$$\lim_{x\to\infty} \frac{x-1}{\sqrt{4x^2+1}} = \lim_{x\to\infty} \frac{(x-1)(\frac{1}{x})}{\sqrt{4x^2+1}\,(\frac{1}{x})} = \lim_{x\to\infty} \frac{1-\frac{1}{x}}{\sqrt{4+\frac{1}{x^2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$\lim_{x\to-\infty} \frac{x-1}{\sqrt{4x^2+1}} = \lim_{x\to-\infty} \frac{(x-1)(\frac{1}{|x|})}{\sqrt{4x^2+1}\,(\frac{1}{|x|})} = \lim_{x\to-\infty} \frac{\frac{x}{|x|}-\frac{1}{|x|}}{\sqrt{4+\frac{1}{x^2}}} \quad \left(\text{for } x<0,\ \frac{x}{|x|} = -1\right)$$

$$= \lim_{x\to-\infty} \frac{-1-\frac{1}{|x|}}{\sqrt{4+\frac{1}{x^2}}} = \frac{-1}{\sqrt{4}} = -\frac{1}{2}.$$

So the horizontal asymptotes are $\boxed{y = \frac{1}{2} \text{ and } y = -\frac{1}{2}}$

3. Use the definition of the derivative (as a limit) to calculate $f'(1)$ given $f(x) = \frac{2}{x+1}$.

$$f'(1) = \lim_{h\to0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to0} \frac{\frac{2}{(1+h)+1} - \frac{2}{2}}{h} = \lim_{h\to0} \frac{\frac{2}{2+h} - 1}{h}$$

$$= \lim_{h\to0} \frac{\frac{2-(2+h)}{2+h}}{h} = \lim_{h\to0} \frac{-h}{h(2+h)} = \lim_{h\to0} \frac{-1}{2+h} = \boxed{-\frac{1}{2}}$$

4. Find an equation of the tangent line to the curve $y = 6x - 3 - (x^2 - x + 1)^3$ at the point $(1, 2)$.

$y' = 6 - 3(x^2 - x + 1)^2 (+2x - 1)$

$y'(1) = 6 - 3(1)^2(1) = 6 - 3 = 3$ = slope

tangent line : $\boxed{y - 2 = 3(x - 1)}$

5. Let $f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

(a) Is f continuous at $x = 0$? Explain.

$\boxed{\text{No}}$ $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1 \neq 0 = f(0)$

(b) Is f differentiable at $x = 0$? Explain.

$\boxed{\text{No}}$ were f differentiable at 0 then f would also be continuous at 0, but it's not, by (a).

6. Let $g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$

(a) Is g continuous at $x = 0$? Explain. $\boxed{\text{Yes}}$

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} x^2 = 0$

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 0 = 0$

So, $\boxed{\lim_{x \to 0} g(x) = 0 = g(0)}$

(b) Is g differentiable at $x = 0$? Explain.

$\boxed{\text{Yes}}$ $\lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{g(h)}{h}$.

Now, $\lim_{h \to 0^+} \frac{g(h)}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = \lim_{h \to 0^+} h = 0$

and $\lim_{h \to 0^-} \frac{g(h)}{h} = \lim_{h \to 0^-} \frac{0}{h} = 0$. So $\lim_{h \to 0} \frac{g(h)}{h} = 0$

That is, $g'(0)$ exists and is 0.

7. Find the derivatives.

(a) $y = \sqrt{\frac{x}{x+1}} = \left(\frac{x}{x+1} \right)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2} \left(\frac{x}{x+1} \right)^{-1/2} \left(\frac{1(x+1) - x(1)}{(x+1)^2} \right)$

(b) $s = te^{t^2}$

$\frac{ds}{dt} = e^{t^2} + te^{t^2}(2t)$

(c) $y = \frac{e^{2x}cos(3x)}{sin(3x)} = e^{2x} \cot 3x$

$\frac{dy}{dx} = 2e^{2x} \cot 3x + e^{2x}(-\csc^2 3x)(3)$

8. Find the derivatives.

(a) $f(x) = tan(x^2) + sin^2(3x) = \tan(x^2) + (\sin(3x))^2$

$f'(x) = (\sec^2(x^2))(2x) + 2(\sin(3x)\cos(3x))(3)$

(b) $g(x) = tan^{-1}(\sqrt{x})$

$g'(x) = \frac{1}{1+(\sqrt{x})^2} \left(\frac{1}{2} x^{-1/2} \right)$

9. Given $y = x\ln(\frac{1}{\sqrt{x}})$, find y' and y''.

$y = x \ln 1/\sqrt{x} = x \ln x^{-1/2} = -\frac{1}{2} x \ln x$

(a) $y' = -\frac{1}{2}\ln x - \frac{1}{2}x\left(\frac{1}{x}\right) = -\frac{1}{2} - \frac{1}{2}\ln x$

(b) $y'' = -\frac{1}{2x}$

10. Find the slope of the curve $x^2 + xy + 2y^2 = 3y + 1$ at the point $(1, 1)$.

$\frac{d}{dx}(x^2 + xy + 2y^2) = \frac{d}{dx}(3y+1)$

$2x + y + x\frac{dy}{dx} + 4y\frac{dy}{dx} = 3\frac{dy}{dx}$

$(x + 4y - 3)\frac{dy}{dx} = -2x - y$

$\frac{dy}{dx} = \frac{-2x - y}{x + 4y - 3}$. Slope $= \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-2-1}{1+4-3} = \boxed{\frac{-3}{2}}$

11. An object moves along a straight line in such a way that the distance (s, in feet) it has traveled after t seconds is given by $s = 1 + 5t - e^{-2t}$.

(a) Find the velocity (v) and acceleration (a) of the object at the time $t = 0$.

i. $v(0) = 5 + 2 = 7$ ft/sec $\qquad v = \frac{ds}{dt} = 5 + 2e^{-2t}$

ii. $a(0) = -4 \qquad a = \frac{d^2s}{dt^2} = \frac{dv}{dt} = -4e^{-2t}$

(b) Is there a limiting velocity as $t \to \infty$?

$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} 5 + 2e^{-2t} = \lim_{t\to\infty} 5 + \cancel{\frac{2}{e^{2t}}}^{\,0} = 5$

$\boxed{\text{Yes}}$, 5 ft/sec.