1) Given \( f(x, y, z) = \sqrt{x^2 - y^2} \)
   - a) Find a unit vector that points in the direction in which \( f \) increases most rapidly at \( P(3, 2, 4) \).
     \[
     \nabla f(3, 2, 4) = \left< \frac{1}{2}(x^2 - y^2)^{-1/2}(2x), \frac{1}{2}(x^2 - y^2)^{-1/2}(-2y), \frac{1}{2}(x^2 - y^2)^{-1/2}(-y) \right>
     \]
     \[
     \nabla f(3, 2, 4) = \left< 3, -2, -1 \right>
     \]
     \[
     \frac{\left< 3, -2, -1 \right>}{\|\left< 3, -2, -1 \right>\|} = \frac{\left< 3, -2, -1 \right>}{\sqrt{14}}
     \]
     \[
     b) \text{what is the rate of change of } f \text{ at } P(3, 2, 4) \text{ in the direction found in a)?}
     \]
     \[
     \sqrt{14} = \|\nabla f(3, 2, 4)\|
     \]
   - c) Find an equation of the tangent plane to \( \sqrt{x^2 - y^2} = 1 \) at \( P(3, 2, 4) \).
     \[
     3(\bar{x} - 3) - 2(\bar{y} - 2) - 1(\bar{z} - 4) = 0
     \]
     \[
     \text{or} \quad 3\bar{x} - 2\bar{y} - \bar{z} = 1
     \]
   - d) Given \( \sqrt{x^2 - y^2} = 1 \), find \( \frac{dz}{dy} \) at \( P(3, 2, 4) \).
     \[
     \frac{dz}{dy} = -\frac{\frac{1}{2}(x^2 - y^2)^{-1/2}(-y)}{\frac{1}{2}(x^2 - y^2)^{-1/2}} = -(-2) = 1
     \]
   - e) without using a calculator, give me a good (linear, 1st order ...) approximation of \( \sqrt{(3.1)^2 - (1.9)(4.2)} \).
     \[
     \sqrt{(3.1)^2 - (1.9)(4.2)} \approx 1 + \left(3\right)\left(3.1 - 3\right) + \left(-2\right)\left(1.9 - 2\right) + \left(-1\right)\left(4.2 - 4\right)
     \]
     \[
     \approx 1 + 3(-1) - 2(-0.1) - 1(0.2)
     \]
     \[
     \approx 1 + 3 + 0.2 - 0.2
     \]
     \[
     \approx 1.3
     \]
2) Suppose that resistance $R$ is given by $R = \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1}$ and that $R_1$ and $R_2$ are changing at the rates of 2 and -3 ohms per second, respectively. At what rate is $R$ changing when $R_1 = 100$ ohms and $R_2 = 200$ ohms?

\[ \frac{dR}{dt} = \frac{2R}{dR_1/dt} + \frac{2R}{dR_2/dt} \]

\[ = - \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-2} \left( - \frac{1}{R_1^2} \right) \left( \frac{dR_1}{dt} \right) + \left( - \frac{1}{R_1} + \frac{1}{R_2} \right)^{-2} \left( - \frac{1}{R_2^2} \right) \left( \frac{dR_2}{dt} \right) \]

when $R_1 = 100$ and $R_2 = 200$:

\[ \frac{dR}{dt} = \frac{2}{(100)^2} \left( \frac{1}{100} + \frac{1}{200} \right)^{-2} - \frac{3}{(200)^2} \left( \frac{1}{100} + \frac{1}{200} \right)^{-2} \]

\[ = 2 \left( 1 + \frac{1}{2} \right)^{-2} - 3 \left( 2 + 1 \right)^{-2} = 2 \left( \frac{4}{9} \right) - 3 \left( \frac{1}{4} \right) = \frac{8}{9} \text{ ohms/second} \]

3) The picture below is a contour (level curve) plot of a function $z = f(x, y)$ of two variables. Assume that the distance between adjacent drawn curves is 1 unit.

a) Sketch in $\nabla f(2, 3)$, with appropriate direction and length.

b) Using a), estimate the rate of change of $f$ at P(2, 3) in the direction of $\langle 3, 4 \rangle$.

$\nabla f(2, 3)$ has length = 2 and is in the direction of $\langle 1, 1 \rangle$. Thus

Thus at $\nabla f(2, 3) = \langle \sqrt{2}, \sqrt{2} \rangle$.

Then $D_3, 4 \cdot f(2, 3) = \langle \sqrt{2}, \sqrt{2} \rangle \cdot \langle 3, 4 \rangle = \frac{7\sqrt{2}}{5}$.

c) Suppose an object moves across P(2, 3) with velocity $\langle 3, 4 \rangle$. Using b), estimate the time rate of change of $f$.

Time rate of change of $f$

\[ = \text{(rate of change of } f \text{ in direction of motion)} \times \text{ (speed)} \]

\[ = \left( \frac{7\sqrt{2}}{5} \right) \left( 5 \right) = 7\sqrt{2}. \]

(OR, \[ \frac{df}{dt} = \nabla f \cdot \vec{v} = \langle \sqrt{2}, \sqrt{2} \rangle \cdot \langle 3, 4 \rangle = 7\sqrt{2} \)
4) Find all critical points of \( f(x,y) = x^2 + 4xy + y^2 - 2x + 8y + 3 \) and classify each as being a point at which \( f \) has a local (relative) max, min, or saddle.

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2x + 4y - 2 = 0 \\
\frac{\partial f}{\partial y} &= 4x + 2y + 8 = 0
\end{align*}
\]

\[-6x - 18 = 0 \quad x = -3 \]

\[-6 + 4y - 2 = 0 \quad y = \frac{2}{3} \]

so \( f \) has a saddle at \((-3, 2)\).

5a) Find the max and min of \( f(x,y) = 2x^2 + y^2 - 2x \) subject to \( x^2 + y^2 = 4 \).

Lagrange:

\[
\begin{align*}
4x - 2 &= 2\lambda x \\
2y &= 2\lambda y \\
x^2 + y^2 &= 4
\end{align*}
\]

\[
\begin{align*}
y - \lambda y &= 0 \\
g(1-\lambda) &= 0 \\
\lambda &= \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
\lambda &= 1 \\
x &= 2 \\
y &= 2
\end{align*}
\]

\[
\begin{align*}
\lambda &= \frac{1}{2} \\
x &= 1 \\
y &= \pm \sqrt{3}
\end{align*}
\]

Critical points: \((2, 0), (2, -\sqrt{3}), (2, \sqrt{3})\)

\[
\begin{align*}
&\max f(2, 0) = 8 - 4 = 4 \\
&\max f(2, -\sqrt{3}) = 2 + 3 - 2 = 3 \\
&\max f(2, \sqrt{3}) = 2 + 3 - 2 = 3
\end{align*}
\]

\[
\begin{align*}
&\min f(1, 0) = 2 + 0 - 2 = 0 \\
&\min f(1, \sqrt{3}) = 2 + 3 - 2 = 3 \\
&\min f(1, -\sqrt{3}) = 2 + 3 - 2 = 3
\end{align*}
\]

b) What are the absolute max and absolute min of \( f(x,y) = 2x^2 + y^2 - 2x \) on the region \( x^2 + y^2 \leq 4 \)?

\[
\begin{align*}
&\max \text{ on edge } = 12, \min \text{ on edge } = 3. \\
&\text{C.P.s inside region: } (\frac{1}{2}, 0)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 4x - 2 = 0 \\
\frac{\partial f}{\partial y} &= 2y = 0 \\
(\frac{1}{2}, 0) &= \frac{1}{2} + 0 - 1 = -\frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
x &= \frac{1}{2}, \quad y = 0 \quad \text{abs. max: } 12 \\
\text{abs. min: } -\frac{1}{2}
\end{align*}
\]