1) Given $f(x, y, z) = x^2 z (2y + 3z)^2$, calculate the gradient of $f$ at $(2, -1, 1)$ and use it to calculate the following:

a) $\nabla f(2, -1, 1) = \langle 4, -16, -4 \rangle$

b) $\frac{\partial f}{\partial x} = 2x z (2y + 3z)^2$

c) $\frac{\partial f}{\partial y} = 4x^2 (2y + 3z)^2$

d) $\frac{\partial f}{\partial z} = x^2 (2y + 3z)^2 + x^2 (2y + 3z)^2$

3) Maximum rate of change of $f$ at $(2, -1, 1)$ is $\nabla f(2, -1, 1) \cdot \langle 1, 0, 1 \rangle = 4\sqrt{18} = \frac{12\sqrt{2}}{3}$

5) Rate of change of $f$ at $(2, -1, 1)$ in the direction $\langle 2, 1, 2 \rangle = \frac{8 - 16 - 8}{3} = -\frac{16}{3}$

5) Unit vector $\vec{u}$ so that $\nabla f(2, -1, 1) \cdot \vec{u} = 0$.

6) Find an equation for the tangent plane to the surface $x^2 z (2y + 3z)^2 = 4$ at the point $(2, -1, 1)$.

7) If $x$, $y$, and $z$ are related by $x^2 z (2y + 3z)^2 = 4$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $x = 2$, $y = -1$ and $z = 1$.

8) If $x$, $y$, and $z$ are related by $x^2 z (2y + 3z)^2 = 4$, use linear approximation to find an approximate value for $z$ when $x = 2.1$ and $y = -1.2$.

9) Tangent plane approx: $L(x, y) = z_0 + \left( \frac{\partial z}{\partial x} \right) (x - x_0) + \left( \frac{\partial z}{\partial y} \right) (y - y_0)$

So, $L(2.1, -1.2) = 1 - 1(x - 2) - 4(y + 1) = 1.7$
1) Below is a level curve picture for the function \( f(x,y) \).

\[
\begin{array}{c}
\text{(2,1)} \\
0 \\
1 \\
2 \\
3 \\
4
\end{array}
\]

Circle the best answer:

a) \( f_x(2,1) \) equals i) -2, ii) \( \frac{1}{2} \), iii) 2

b) \( f_y(2,1) \) equals i) -1, ii) 0, iii) 1

c) \( f_{yx}(2,1) \) is i) < 0, ii) \( \geq 0 \), iii) > 0

d) \( f_{xy}(2,1) \) is i) < 0, ii) \( \geq 0 \), iii) > 0

e) \( D_{-1,2} f(2,1) \) is i) < 0, ii) \( \geq 0 \), iii) > 0

3) Suppose a conical cow's volume increases at the rate of \( \frac{\pi}{10} \) ft\(^3\)/day

If the cow's height is increasing at the rate of \( \frac{1}{50} \) ft/day when the cow's radius is 3 ft and height is 4 ft, at what rate is the radius changing?

\[
\begin{align*}
\frac{dV}{dt} &= \frac{2V}{3H} \frac{dH}{dt} + \frac{dV}{dR} \frac{dR}{dt} \\
\frac{dV}{dt} &= \left( \frac{1}{3} \pi r^2 \right) \frac{dH}{dt} + \left( \frac{2}{3} \pi r H \right) \frac{dR}{dt}
\end{align*}
\]

Put in: \( \frac{\pi}{10} \) ft\(^3\)/day, 1/50 ft/day, 3 ft for \( r \), 4 ft for \( H \) to get:

\[
\frac{\pi}{10} = \frac{\pi}{3} \left( 3^2 \left( \frac{1}{50} \right) + \frac{2\pi}{3} \left( 3 \right) \left( 4 \right) \frac{dR}{dt} \right)
\]

Thus \( \frac{dR}{dt} = \frac{\pi}{10} - \frac{3\pi}{50} = \frac{2}{8.50} = \frac{1}{200} \) ft/day.
4) Find all critical points of \( f(x,y) = x^2 + 3xy + y^2 - 4x - y \) and classify (rel. max, rel. min, or saddle).

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2x + 3y - 4 = 0 \\
\frac{\partial f}{\partial y} &= 3x + 2y - 1 = 0
\end{align*}
\]

\[-3 \begin{vmatrix} 2x + 3y &= 4 \\ 3x + 2y &= 1 \end{vmatrix} + 2 \begin{vmatrix} 2x + 3y &= 4 \\ 3x + 2y &= 1 \end{vmatrix} = -5y = -10
\]

\[
y = 2
\]

\[\text{Now} 2x = -2 \quad x = -1
\]

So \((-1, 2)\) is the only critical point. Since \(D(-1, 2) < 0\), \(f\) has a saddle at \((-1, 2)\).

5) Find the max and min of \( f(x,y) = xy \) on the ellipse \( x^2 + 2y^2 = 8 \).

Lagrange:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\
\frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\
g(x,y) &= 8
\end{align*}
\]

\[
\begin{vmatrix} y &= \lambda (2x) \\
x &= \lambda (4y) \\
x^2 + 2y^2 &= 8
\end{vmatrix}
\]

\[
\text{Note that } x \neq 0, y \neq 0, \lambda \neq 0 \text{ (3rd eqn to be satisfied).}
\]

So we may divide the 1st eqn by the second:

\[
y = \frac{1}{2} y \quad \text{Max. } 2y^2 = x^2. \text{ Together with } x^2 + 2y^2 = 8
\]

we have \( x^2 + 2y^2 = 8, x = 4, y = \pm 2 \). \( \text{Max. } x^2 + 2y^2 = 8 \)

\[
(x, y) = (2, \pm 2), (-2, \pm 2)
\]

Extra credit: Estimate the max and min of \( f(x,y) = xy \) subject to \( x^2 + 2y^2 = 8 \) without explicitly solving this new problem.

\[
\frac{\partial f}{\partial x} = \lambda \quad \text{at} (2, \pm 2) \quad \lambda = \frac{\sqrt{2}}{4}
\]

\[
\text{So new max = old max} + (\lambda) (\Delta C) = 2\sqrt{2} + \frac{\sqrt{2}}{4} (8.1 - 8)
\]

\[
\text{New max} \approx 2\sqrt{2} + \frac{\sqrt{2}}{4} = (2 + \frac{1}{4})\sqrt{2}
\]

Similarly, \( \lambda = -\frac{\sqrt{2}}{4} \) at \((-2, \pm 2)\). New min \( -2\sqrt{2} + \frac{\sqrt{2}}{4} \).