Name ____________________ Score ____________________

Fall 2007

Math 450 Midterm Exam

Due Wednesday, October 31, 2007

Please provide enough details of your work so that I can follow your reasoning. Without details, I cannot assign partial credit.

1. (25 points) At time $t = 0$ a small amount of chlorine gas, concentrated at a point in space, is allowed to diffuse outward. Suppose that there is a law $f(t, r, m, C, \kappa) = 0$ relating time t, radial distance r from the source, total mass m of the gas, concentration C of the gas, and diffusivity κ of the gas. C has units of mass per unit volume, and κ has units of area per unit time. Each of the quantities t, r, m, C, κ can be expressed in terms of fundamental units of time T, length L, and mass M. You are to

   a. Find the dimension matrix A.

   b. Use the Buckingham Pi Theorem (i.e., linear algebra) to show that there are exactly 2 independent dimensionless variables that can be formed from t, r, m, C, κ.

   c. Show that the 2 independent dimensionless variables can be selected to be

   $$\pi_1 = t^{3/2} m^{-1} C \kappa^{3/2}, \quad \pi_2 = t^{-1/2} r \kappa^{-1/2}.$$

2. (15 points) On page 67 of Logan textbook, Problem 1a.

   a. In addition to the characterization of the equilibrium point $(0, 0)$, sketch the phase plane. Clearly label the solution trajectory satisfying the initial condition $(x(0), y(0)) = (1, 0)$. Assess the long-term behavior of each component of the solution. That is, $x(t) \to ?$ and $y(t) \to ?$ as $t \to \infty$.

   b. Give the solution of the linear system for the initial conditions $(x(0), y(0)) = (1, 0)$. Verify your assertions of the long-term behavior from part a. Generate plots (**using plotting software of your choice**) of $x(t)$ and $y(t)$ for $t > 0$. Plot these on the same axes.

3. (10 points) On page 67 of Logan textbook, Problem 2. Assume that the parameter b is a real number.

4. (15 points) On page 67 of Logan textbook, Problem 4. Note, your sketch in the $pq-$ plane should have the $p-$axis as the horizontal axis and the $q-$ axis as the vertical axis.

5. (15 points) On page 79 of Logan textbook, Problem 1e.

6. (20 points) (The directions for this problem are taken from Problem 5, on page 31 of the Logan textbook. However, I am restating the problem and *adjusting* the notation and instructions a little bit.)

The dynamics of a nonlinear mass-spring system is described by

$$mx'' = -ax' - kx^3,$$

$$x(0) = 0, \quad mx'(0) = I,$$

where x is the displacement, $-ax'$ is a linear damping term, and $-kx^3$ is a nonlinear restoring force. Initially, the displacement is zero and the mass m is given an impulse I that starts the motion.

(a) Determine the dimensions of the constants I, a, k.

(b) Perform a dimensional analysis on the problem.

(c) Recast the problem into dimensionless form by selecting dimensionless variables

$$\tau = \frac{t}{Z} \quad \text{and} \quad u = \frac{x}{I/a},$$

where the characteristic time Z is yet to be determined. From your dimensional analysis in part (c), identify one possible characteristic time that one might use for Z.

(d) In the special case where the mass m is very small, choose an appropriate time scaling. In particular, choose a time scale so that once the problem is nondimensionalized, the ODE can be manipulated so that the parameters (including the small parameter m) appear in front of the nonlinear term in the model but the coefficients of $\frac{d^2u}{d\tau^2}$ and $\frac{du}{d\tau}$ do not involve any parameters from the model.