Name ____________________ Score ______________

Spring 2008

Math 451 Homework 1

Due Friday, February 1, 2008

1. On page 158 of Logan textbook: Number 1. HINT: First show that $J(y) \leq \frac{2}{\pi}$ for all admissible functions y. Then show that $J(-t) = \frac{2}{\pi}$.
2. On page 158 of Logan textbook: Number 2.
3. On page 166 of Logan textbook: Number 1. For each of (a)-(c), if the set constitutes a linear space, verify each of the seven properties, and if the set does not constitute a linear space, show that one of the seven properties is violated. (note: there may be more than one of the seven properties violated; however, you only need to show that one of those properties does not hold in order to show that the set is not a linear space).
4. On page 166 of Logan textbook: Number 2.
5. On page 166 of Logan textbook: Number 3.
6. On page 166 of Logan textbook: Number 5(a).
7. On page 166 of Logan textbook: Number 5(c).
8. On page 166 of Logan textbook: Number 5(e).

Chapter 3 Section 1

(1.1)

$$J(y(\cdot)) = \int_0^1 \dot{y}\sin(\pi y) - (t+y)^2 \, dt$$

$$\leq \int_0^1 \dot{y}\sin(\pi y)\, dt \qquad \text{since } \int_0^1 (t+y)^2 dt \geq 0 \ \forall\, t, y(t)$$

$$= \int_0^1 \frac{d}{dt}\left[\frac{-1}{\pi}\cos(\pi y)\right] dt$$

$$= \frac{-1}{\pi}\cos(\pi y(t)) \Big|_{t=0}^{t=1}$$

$$= \frac{1}{\pi}\left(\cos(\pi y(0)) - \cos(\pi y(1))\right)$$

Since $y(0)$ and $y(1)$ are arbitrary, we bound the difference by using the bounds on $\cos(\pi y(0))$ and $\cos(\pi y(1))$

$$\leq \frac{1}{\pi}(1-(-1)) \qquad = 2/\pi$$

Hence $J(y(\cdot)) \leq 2/\pi \ \ \forall\, y \in V$.
And note that for $y(t) = -t$, we have

$$J(y(t)) = J(-t) = \int_0^1 -\sin(\pi(-t)) - (t+-t)^2 dt$$

$$= \int_0^1 \sin(\pi t)\, dt$$

$$= -\frac{1}{\pi}\cos(\pi t)\Big|_{t=0}^{t=1}$$

$$= -\frac{1}{\pi}(-1-1) \quad = 2/\pi$$

(Scratch $y(x) = x + c_1 x - c_1 x^2 + c_2 x^2 - c_2 x^3$)

Chapter 3, Section 1

(1.2) Let $y(x) = x + c_1 (x)(1-x) + c_2 x^2 (1-x)$

$$J(y) = \int_0^1 (1+x)(y')^2 dx$$

$$= \int_0^1 (1+x)[c_1 - 2c_1 x + 2c_2 x - 3c_2 x^2]^2 dx$$

$$= \int_0^1 (1+x)(c_1(1-2x) + c_2 x(2-3x))^2 dx$$

$$= \int_0^1 (1+x)[c_1^2 (1-2x)^2 + 2c_1 c_2 (1-2x)(2-3x) + c_2^2 x^2 (2-3x)^2] dx$$

$$= c_1^2 \int_0^1 (1+x)(1-2x)^2 + 2c_1 c_2 \int_0^1 (1+x)(1-2x)(2-3x)\, dx$$

$$+ c_2^2 \int_0^1 (1+x) x^2 (2-3x)^2 dx$$

Using Maple:

$$J(c_1, c_2) = \frac{7}{30} c_2^2 + \frac{17}{30} c_1 c_2 - \frac{1}{6} c_2 - \frac{1}{3} c_1 + \frac{1}{2} c_1^2 + \frac{3}{2}$$

Find c_1, c_2 so that

$$\nabla J(c_1, c_2) = 0$$

$$\frac{17}{30} c_2 + c_1 - \frac{1}{3} = 0 \qquad \Rightarrow c_1 = \frac{1}{3} - \frac{17}{30} c_2$$

$$\frac{14}{30} c_2 + \frac{17}{30} c_1 - \frac{1}{6} = 0$$

Algebra:

$$\frac{14}{30} c_2 + \frac{17}{30}\left(\frac{1}{3} - \frac{17}{30} c_2\right) = \frac{1}{6}$$

$$\left(\frac{14}{30} - \frac{17^2}{30^2}\right) c_2 = \frac{1}{6} - \frac{17}{90}$$

$$\left(\frac{420}{900} - \frac{289}{900}\right) c_2 = \frac{15}{90} - \frac{17}{90}$$

$$c_1 = \frac{1}{3} - \frac{17}{30}\left(\frac{-20}{131}\right)$$

$$= \frac{1}{3} - \frac{17 \cdot (-2)}{3 \cdot 131}$$

$$c_1 = \frac{131 + 34}{3(131)} = \frac{165}{3(131)} = \frac{165}{393}$$

$$c_2 = \frac{-2}{90} \cdot \frac{900}{131} = \frac{-20}{131} \approx -.152672, \quad c_1 \approx .419847$$

(1.2 cont')

Determine Hessian Matrix (matrix of 2nd partial derivatives) (2)

$$H = \begin{bmatrix} 1 & \frac{17}{30} \\ \frac{17}{30} & \frac{14}{30} \end{bmatrix}$$, and H is SPD

In particular, $\det(H) = \frac{14}{30} - \left(\frac{17}{30}\right)^2$

$$= \frac{14}{30} - \frac{289}{900}$$

$$= \frac{420-289}{900} = \frac{131}{900} > 0$$

Hence, J is a quadratic functional, & concave up everywhere and we are guaranteed that

$(c_1, c_2) = (165/393, -20/131) \approx (.419847, -.152672)$

is the unique local min

Math 451 - Spring '08
Homework #1
p. 166 #1

③ (a) Let $V = \{p(x) : p(x)$ is a polynomial of degree $\leq 2\}$ is a Linear Space.

<u>Properties</u>: Let $p(x), q(x), r(x) \in V$. These elements of the set must have the form

$$p(x) = p_0 + p_1 x + p_2 x^2 \quad \text{where } p_0, p_1, p_2 \in \mathbb{R}$$
$$q(x) = q_0 + q_1 x + q_2 x^2 \quad \text{where } q_0, q_1, q_2 \in \mathbb{R}$$
$$r(x) = r_0 + r_1 x + r_2 x^2 \quad \text{where } r_0, r_1, r_2 \in \mathbb{R}.$$

1. The binary operation + is defined as follows
$$p(x) + q(x) = (p_0 + q_0) + (p_1 + q_1)x + (p_2 + q_2)x^2$$
And note that the closure of the real #s $\mathbb{R}$ guarantees that $p_0 + q_0 \in \mathbb{R}$, $p_1 + q_1 \in \mathbb{R}$ and $p_2 + q_2 \in \mathbb{R}$. And hence, $p(x) + q(x) \in V$ whenever $p(x), q(x) \in V$. So, V is closed under addition.

2. <u>Commutativity</u>:
$$\begin{aligned} p(x) + q(x) &= (p_0 + q_0) + (p_1 + q_1)x + (p_2 + q_2)x^2 \\ &= (q_0 + p_0) + (q_1 + p_1)x + (q_2 + p_2)x^2 \\ &= q(x) + p(x) \end{aligned}$$
Since p_i's, q_i's are in $\mathbb{R}$ & $\mathbb{R}$ is commutative

<u>Associativity</u>:
$$\begin{aligned} p(x) + (q(x) + r(x)) &= p(x) + (q_0 + r_0) + (q_1 + r_1)x + (q_2 + r_2)x^2 \\ &= [p_0 + (q_0 + r_0)] + [p_1 + (q_1 + r_1)]x + [p_2 + (q_2 + r_2)]x^2 \end{aligned}$$

$\Rightarrow$

And by the associativity property of $\mathbb{R}$, we can rewrite

$$[p_0+(q_0+r_0)] = [(p_0+q_0)+r_0]$$
$$[p_1+(q_1+r_1)] = [(p_1+q_1)+r_1]$$
$$[p_2+(q_2+r_2)] = [(p_2+q_2)+r_2]$$

So that the last eqn on the previous page becomes

$$\begin{aligned} p(x)+(q(x)+r(x)) &= [(p_0+q_0)+r_0] + [(p_1+q_1)+r_1]x + [(p_2+q_2)+r_2]x^2 \\ &= [(p_0+q_0)] + [p_1+q_1]x + [p_2+q_2]x^2 + r(x) \\ &= (p(x)+q(x)) + r(x) \end{aligned}$$

So, Associativity holds.

3. The zero element in V is given by

$$Z(x) = 0 + 0x + 0x^2,$$

and it has the property that

$$\begin{aligned} p(x) + Z(x) &= (p_0+0) + (p_1+0)x + (p_2+0)x^2 \\ &= p_0 + p_1 x + p_2 x^2 \\ &= p(x) \qquad \forall\, p(x) \in V \end{aligned}$$

Hence, $Z(x) =$ "0" is the zero element of this space. Note that this element is unique! The argument follows by the uniqueness of the additive identity in $\mathbb{R}$. That is, 0 is the unique additive identity in $\mathbb{R}$.

4. Since $p(x) = p_0 + p_1 x + p_2 x^2 \in V$, then $p_0, p_1, p_2 \in \mathbb{R}$. And $-p_0, -p_1, -p_2 \in \mathbb{R}$ also. Then the polynomial

$$v(x) = -p_0 - p_1 x - p_2 x^2 \in V \text{ also.}$$

AND

$$\begin{aligned} p(x) + v(x) &= (p_0 - p_0) + (p_1 - p_1)x + (p_2 - p_2)x^2 \\ &= 0 + 0x + 0x^2 = Z(x) = \text{zero element in } V. \end{aligned}$$

Then $v(x)$ is the inverse of $p(x)$ in the set V.
Hence, every element in V has an additive inverse in the set V. And we denote $v(x) = -p(x)$.

5. For each $p(x) \in V$ and $\alpha \in \mathbb{R}$,
$$\alpha p(x) = \alpha p_0 + \alpha p_1 x + \alpha p_2 x^2$$
is well-defined, and $\alpha p_0, \alpha p_1, \alpha p_2 \in \mathbb{R}$
Hence, $\alpha p(x) \in V$ also.

6. For $\alpha, \beta \in \mathbb{R}$,
$$\begin{aligned} \alpha(\beta p(x)) &= \alpha(\beta p_0 + \beta p_1 x + \beta p_2 x^2) \\ &= \alpha\beta p_0 + \alpha\beta p_1 x + \alpha\beta p_2 x^2 \\ &= (\alpha\beta) p_0 + (\alpha\beta) p_1 x + (\alpha\beta) p_2 x^2 \\ &= (\alpha\beta) p(x). \end{aligned}$$
since $\alpha, \beta, p_0, p_1, p_2 \in \mathbb{R}$

$$\begin{aligned} (\alpha+\beta) p(x) &= (\alpha+\beta) p_0 + (\alpha+\beta) p_1 x + (\alpha+\beta) p_2 x^2 \\ &= \alpha p_0 + \beta p_0 + \alpha p_1 x + \beta p_1 x + \alpha p_2 x^2 + \beta p_2 x^2 \\ &\vdots \\ &= \alpha p(x) + \beta p(x) \end{aligned}$$
↳ by distributive property of $\mathbb{R}$

$$\begin{aligned} \alpha(p(x) + q(x)) &= \alpha\left((p_0+q_0) + (p_1+q_1)x + (p_2+q_2)x^2\right) \\ &= (\alpha p_0 + \alpha q_0) + (\alpha p_1 + \alpha q_1)x + (\alpha p_2 + \alpha q_2)x^2 \\ &\vdots \\ &= \alpha p(x) + \alpha q(x) \end{aligned}$$
↳ by distributive property of $\mathbb{R}$

7. For the scalar $1 \in \mathbb{R}$, $1p(x) = 1 \cdot p_0 + 1 \cdot p_1 x + 1 p_2 x^2$
$$= p_0 + p_1 x + p_2 x^2 = p(x) \quad \forall p \in V.$$

3 (b) Let $V = \{f \in C[0,1] : f(0) = 0\}$ is a Linear Space.
Let $f, g, h \in V$ and $\alpha, \beta \in \mathbb{R}$

1. $f(x) + g(x)$ is cont. on $[0,1]$ and
$(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$
Hence, $f + g \in V$.

2. $f(x) + g(x) = g(x) + f(x)$ for all $f, g \in C[0,1]$ and
$0 = f(0) + g(0) = g(0) + f(0) = 0$
• Hence, addition is commutative

$f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x) \quad \forall f, g, h \in C[0,1]$ and
$0 = f(0) + (g(0) + h(0)) = (f(0) + g(0)) + h(0) = 0 \quad \forall f, g, h \in V$.
• Hence, addition is associative

3. The zero polynomial $z(x) \equiv 0 \ \forall x \in [0,1]$ is cont. on $[0,1]$
and $z(0) = 0$, then $z \in V$. Also $p(x) + 0 = p(x) \ \forall p \in V$.

4. For each $p(x) \in C[0,1]$, $-p(x) \in C[0,1]$, and $-p(x)$
is the inverse of $p(x)$ in $C[0,1]$. Furthermore, $-p(0) = 0$
Hence, $-p(x)$ is the inverse of $p(x)$ in V.

5. For any $\alpha \in \mathbb{R}$, $\alpha p(x)$ is Cont. on $[0,1]$, and
$\alpha p(0) = \alpha(0) = 0$. Hence $\alpha p \in V$.

6. Foreach $\alpha, \beta \in \mathbb{R}$, $\alpha(\beta p(x)) = \alpha\beta p(x) \in C[0,1]$
and $\alpha(\beta p(0)) = \alpha\beta p(0) = 0$. Hence, $\alpha\beta p(x) \in V$.

6. (cont') $(\alpha+\beta)p(x) = \alpha p(x) + \beta p(x) \in C[0,1]$

and $0 = (\alpha+\beta)p(0) = \alpha p(0) + \beta p(0) = 0 + 0 = 0.$

Hence, $(\alpha+\beta)p(x) \in V.$

$\alpha(p(x)+q(x)) = \alpha p(x) + \alpha q(x) \in C[0,1]$

and $0 = \alpha(p(0)+q(0)) = \alpha(0+0) = 0$

Hence, $\alpha(p+q) \in V.$

7. $1p(x) = p(x) \ \forall\ p \in C[0,1]$

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3 (c) Let $V = \{f \in C[0,1] : f(1) = 1\}$ is NOT A LINEAR SPACE.

Let $f, g \in V$ then $f(x)+g(x) \in C[0,1]$.

However, $f(1) + g(1) = 1 + 1 = 2$

Hence, $f(x)+g(x) \notin V$, and V is NOT A LINEAR SPACE.

Math 451 - Spring '08
Homework #1
p. 166 #2

(4) (a) Prove that $\|y\|_M = \max_{a \le x \le b} |y(x)| \quad \forall y \in C[a,b]$. (3.7)
is a norm.

Note that the norm is well-defined since any continuous function attains a maximum and a minimum on any closed interval $[a,b]$.

<u>Properties:</u> Let $y, z \in C[a,b]$ and $\alpha \in \mathbb{R}$

1. If $y(x) \equiv 0$ on $[a,b]$, then
$$\|y\|_M = \max_{a \le x \le b} |y(x)| = \max_{a \le x \le b} |0| = 0$$

Hence, $\|y\|_M = 0$

Now, we show that if $\|y\|_M = 0$, then $y(x) \equiv 0 \ \forall x \in [a,b]$.
We use a contrapositive argument. So, we assume $y(x) \neq 0$ for some $x \in [a,b]$. Then, if $y(x) = d$ for some $x \in [a,b]$ and for some $d \neq 0$, it must be the case that
$$\|y\|_M = \max_{a \le x \le b} |y(x)| \geq |d| > 0$$
so that $\|y\|_M \neq 0$
Hence, $y(x) \not\equiv 0 \Rightarrow \|y\|_M \neq 0$
This implies that if $\|y\|_M = 0$, then $y(x) = 0$.

2. $\|\alpha y(x)\|_M = \max_{a \le x \le b} |\alpha y(x)| = |\alpha| \max_{a \le x \le b} |y(x)| = |\alpha| \|y\|_M$
for all $\alpha \in \mathbb{R}$ and $y \in C[a,b]$.

④ (cont')

3. $\|y+z\|_M = \max_{a\le x\le b}\{|y(x)+z(x)|\}$

$\le \max_{a\le x\le b}\{|y(x)|+|z(x)|\}$ by Δ ineq. for each x.

$\le \max_{a\le x\le b}|y(x)| + \max_{a\le x\le b}|z(x)|$

$= \|y\|_M + \|z\|_M$

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④ (6.) Prove that $\|y\|_1 = \int_a^b |y(x)|\,dx \quad \forall y \in C[a,b]$ (3,8)
is a norm

Note that the norm is well-defined since any continuous function (and its absolute value fctn) are integrable over the interval $[a,b]$.

Properties: Let $y, z \in C[a,b]$ and $\alpha \in \mathbb{R}$

① If $y(x) \equiv 0 \ \forall x \in [a,b]$, then

$$\|y\|_1 = \int_a^b |y(x)|\,dx = \int_a^b 0\,dx = 0$$

Now, we suppose that $y(x)$ is NOT the zero function. Then $\exists$ a pt. $d \in (a,b) \ni y(d) \neq 0$. Since $y(x)$ is continuous, then $\exists$ an open interval containing d and contained in $[a,b]$ so that $a<c<d<e<b$ with

$$|y(x)|>0 \text{ for all } x \in (c,e)$$

Then

$$\|y\|_1 = \int_a^b |y(x)|\,dx = \int_a^c |y(x)|\,dx + \int_c^e |y(x)|\,dx + \int_e^b |y(x)|\,dx > 0$$

Hence, $\|y\|_1 \neq 0$. this shows that $\|y\|_1 = 0 \Rightarrow y(x) \equiv 0 \ \forall x \in [a,b]$.
This shows that Property ① is satisfied.

④ (b) (cont')

② $\|\alpha y\|_1 = \int_a^b |\alpha y(x)|\,dx = |\alpha| \int_a^b |y(x)|\,dx = |\alpha|\,\|y\|_1$

$\forall \alpha \in \mathbb{R}$, $\forall y \in C[a,b]$.

③ $\|y+z\|_1 = \int_a^b |y(x)+z(x)|\,dx \leq \int_a^b |y(x)| + |z(x)|\,dx$

$$= \int_a^b |y(x)|\,dx + \int_a^b |z(x)|\,dx$$

$$= \|y\|_1 + \|z\|_1$$

Hence (3.8) defines a norm.

Math 451 - Spring '08

Homework #1

p.166 #3

(5.) In $C'[0,1]$, compute $\|y_1 - y_2\|_w$ and $\|y_1 - y_2\|_m$

when $y_1(x) = 0$ and $y_2(x) = \frac{1}{100}\sin(1000x)$

- $\|y_1 - y_2\|_m = \max_{0 \le x \le 1} \left|\frac{1}{100}\sin(1000x)\right| = \frac{1}{100} = .01$

- $\|y_1 - y_2\|_w = \max_{0 \le x \le 1} \left|\frac{1}{100}\sin(1000x)\right| + \max_{0 \le x \le 1} \left|10\sin(1000x)\right|$

$= \frac{1}{100} + 10$

$= 10.01$

Math 451 - Spring '08

HOMEWORK #1

p. 166 (5a.)

(6) Let $y_1, y_2 \in C'[a,b]$ and let $\alpha \in \mathbb{R}$

$J(y) = \int_a^b y(x)y'(x)\,dx$ is NOT A LINEAR FUNCTIONAL.

$$J(y_1+y_2) = \int_a^b (y_1+y_2)(y_1'+y_2')\,dx$$

$$= \int_a^b y_1 y_1'\,dx + \int_a^b y_2 y_2'\,dx + \int_a^b y_2 y_1'\,dx + \int_a^b y_1 y_2'\,dx$$

$$= J(y_1) + J(y_2) + \int_a^b y_2 y_1' + y_1 y_2'\,dx$$

Note that we only need to choose $y_1, y_2 \in C'[a,b]$ so that the extra term is nonzero. Let

$y_1(x) = x$ and $y_2(x) = 1$

Then

$$\int_a^b 1(1)\,dx + \int_a^b x(0)\,dx = b-a$$

Hence, for $y_1(x) = x$ and $y_2(x) = 1 \in C'[a,b]$

$$J(y_1+y_2) = J(y_1) + J(y_2) + b-a \neq J(y_1) + J(y_2)$$

And J is NOT A LINEAR FUNCTIONAL.

Math 451 - Spring '08

HOMEWORK #1

p.166 (5c.)

(7.) Let $y_1, y_2 \in C'[a,b]$ and Let $\alpha \in \mathbb{R}$

$J(y) = e^{y(a)}$ is NOT A LINEAR FUNCTIONAL

$J(y_1+y_2) = e^{y_1(a)+y_2(a)} = e^{y_1(a)} \cdot e^{y_2(a)} = J(y_1)J(y_2) \underbrace{\neq J(y_1)+J(y_2)}$

Choose $y_1(x) = x-a$ and $y_2(x) = 1$ ← generally

$y_1(a) = 0$ $y_2(a) = 1$

$J(y_1+y_2) = e^{0+1} = e$

and

$J(y_1)+J(y_2) = e^0 + e^1 = 1+e$

Hence $J(y_1+y_2) \neq J(y_1) + J(y_2)$, and

J is NOT a LINEAR FUNCTIONAL.

Math 451 - Spring '08

HOMEWORK #1

p.166 (5e)

(8) Let $y_1, y_2 \in C'[a,b]$ and let $\alpha \in \mathbb{R}$

$J(y) = \int_a^b y(x)\sin(x)\,dx$ IS A LINEAR FUNCTIONAL.

$$J(y_1+y_2) = \int_a^b [y_1(x)+y_2(x)]\sin(x)\,dx$$

$$= \int_a^b y_1(x)\sin(x)\,dx + \int_a^b y_2(x)\sin(x)\,dx$$

$$= J(y_1) + J(y_2) \qquad \forall\, y_1, y_2 \in C'[a,b]$$

And

$$J(\alpha y_1) = \int_a^b \alpha y_1(x)\sin(x)\,dx$$

$$= \alpha \int_a^b y_1(x)\sin(x)\,dx$$

$$= \alpha J(y_1) \qquad \forall\, y_1 \in C'[a,b] \text{ and } \forall\, \alpha \in \mathbb{R}$$

So, J is a Linear Functional