Name ____________________ Score ____________________

Spring 2008

Math 451 Homework 2

Due Friday, February 15, 2008

1. On page 167 of Logan textbook: Number 10.
2. On page 167 of Logan textbook: Number 11.
3. On page 168 of Logan textbook: Number 12.
4. On page 175 of Logan textbook: Number 1a,b,c.
5. On page 175 of Logan textbook: Number 7.
6. Find the extremal for the functional

$$J(y) = \frac{1}{2}\int_0^1 [y'(x)^2 + y(x)^2]\,dx - 2y(1)$$

subject to the constraints $y \in C^2[0,1]$ and $y(0) = 0$.
Hint: Show that the extremal for the functional satisfies the boundary value problem

$$\begin{aligned} y'' - y = 0, &\quad 0 < x < 1, \\ y(0) = 0, &\quad y'(1) = 2. \end{aligned}$$

Hint^2: Find the first variation of J and set it $= 0$. BE SURE THAT YOU CLEARLY DEFINE THE SET OF ADMISSIBLE VARIATIONS for this problem. Then derive the ELDE first by considering $\delta J = 0$ on a subset of the set of admissible variations which makes the funny point evaluation at $x = 1$ go away. After you derive the ELDE, then go back and consider the original expression on the full set of admissible variations and derive that Neumann boundary condition $y'(1) = 2$.

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HOMEWORK #2, Problem #1
p. 167, #10

1. $J(y)=\int_0^1 x^2-y^2+(y')^2\,dx$, $y\in C^2[0,1]$
Calculate ΔJ and $\delta J(y,h)$ with $y(x)=x$, $h(x)=x^2$

$$\Delta J = J(y+\varepsilon h)-J(y)$$

$$=\int_0^1 x^2-(y+\varepsilon h)^2+(y'+\varepsilon h')^2\,dx - \int_0^1 x^2-y^2+(y')^2\,dx$$

$$=\int_0^1 x^2-(x+\varepsilon x^2)^2+(1+2\varepsilon x)^2\,dx - \int_0^1 x^2-x^2+(1)^2\,dx$$

$$=\int_0^1 x^2-[x^2+2\varepsilon x^3+\varepsilon^2x^4]+[1+4\varepsilon x+4\varepsilon^2x^2]\,dx - \int_0^1 1\,dx$$

$$=\int_0^1 x^2-x^2-2\varepsilon x^3-\varepsilon^2x^4+1+4\varepsilon x+4\varepsilon^2x^2\,dx - 1$$

$$=\int_0^1 -2\varepsilon x^3-\varepsilon^2x^4+4\varepsilon x+4\varepsilon^2x^2\,dx + \int_0^1 1\,dx - 1$$

$$=-\frac{2\varepsilon}{4}x^4-\frac{\varepsilon^2}{5}x^5+2\varepsilon x^2+\frac{4\varepsilon^2}{3}x^3\Big|_{x=0}^{x=1}$$

$$=-\tfrac{1}{2}\varepsilon-\tfrac{1}{5}\varepsilon^2+2\varepsilon+\tfrac{4}{3}\varepsilon^2$$

$$=\frac{3}{2}\varepsilon+\frac{17}{15}\varepsilon^2$$

$$\Delta J = \frac{3}{2}\varepsilon+\frac{17}{15}\varepsilon^2$$

Let $H = C^2[0,1]$ be the set of admissible variations

$$\delta J(y,h) = \frac{d}{d\varepsilon} J(y+\varepsilon h)\Big|_{\varepsilon=0} \qquad \forall\, h \in H$$

$$= \frac{d}{d\varepsilon} \int_0^1 x^2 - (y+\varepsilon h)^2 + (y'+\varepsilon h')^2\, dx \Big|_{\varepsilon=0} \qquad \forall\, h \in H$$

$$= \int_0^1 -2(y+\varepsilon h)h + 2(y'+\varepsilon h')h'\, dx \Big|_{\varepsilon=0} \qquad \forall\, h \in H$$

$$= \int_0^1 -2yh + 2y'h'\, dx \qquad \forall\, h \in H$$

And

$$\delta J(x,x^2) = \int_0^1 -2x(x^2) + 2(1)(2x)\, dx$$

$$= \int_0^1 -2x^3 + 4x\, dx$$

$$= -\tfrac{1}{2}x^4 + 2x^2 \Big|_{x=0}^{x=1}$$

$$= -\tfrac{1}{2} + 2 - 0$$

$$\delta J(x,x^2) = \frac{3}{2}$$

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HOMEWORK #2, Problem #2

p.167, #11

(2) $J(y) = \int_0^1 (1+x)(y')^2\,dx$ where $y \in C^2[0,1]$ and $y(0) = y(1) = 0$.
Let $y_0 = \dfrac{\ln(1+x)}{\ln(2)}$, and show that $\delta J(y_0, h) = 0$

Let $H = \{h \in C^2[0,1] : h(0) = h(1) = 0\}$, then $\forall h \in H$

$$\delta J(y_0, h) = \frac{d}{d\varepsilon} J(y_0 + \varepsilon h)\Big|_{\varepsilon=0} \qquad \forall h \in H$$

$$= \frac{d}{d\varepsilon} \int_0^1 (1+x)(y_0' + \varepsilon h')^2\,dx \Big|_{\varepsilon=0} \qquad ''$$

$$= \int_0^1 (1+x)\left[2(y_0' + \varepsilon h')h'\right]dx \Big|_{\varepsilon=0} \qquad ''$$

$$= \int_0^1 2(1+x)\,y_0'(x)\,h'(x)\,dx \qquad \forall h \in H$$

$$= \int_0^1 2(1+x)\left[\frac{1}{\ln(2)}\left(\frac{1}{1+x}\right)\right] h'(x)\,dx \qquad ''$$

$$= \frac{2}{\ln(2)} \int_0^1 h'(x)\,dx \qquad ''$$

$$= \frac{2}{\ln(2)}\left[h(1) - h(0)\right] \qquad ''$$

$$= 0 \qquad \forall h \in H$$

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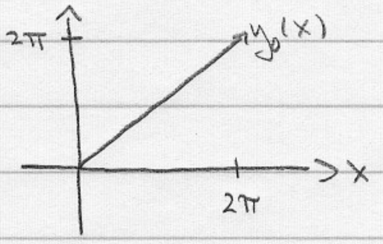
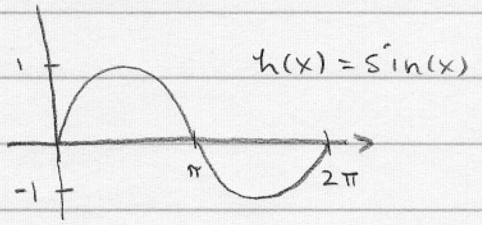
Homework #2, Problem #3

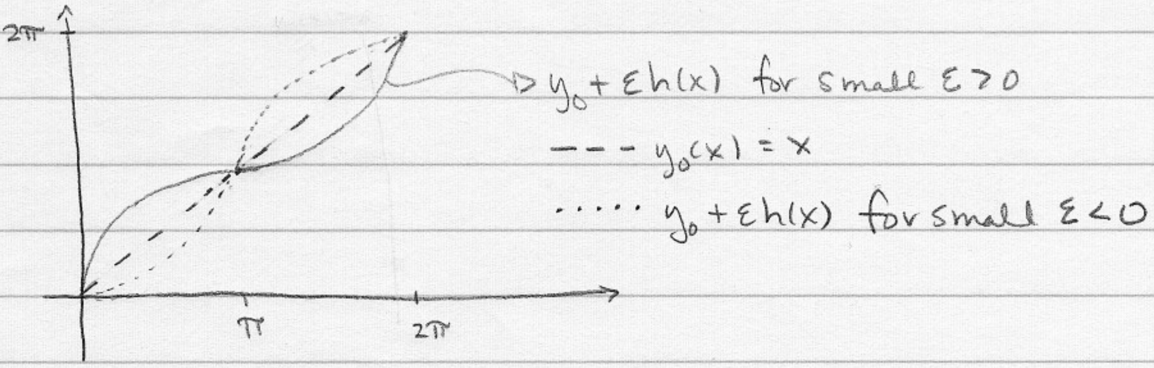
p. 168, #12

3 $\quad J[y] = \int_0^{2\pi} [y'(x)]^2 \, dx$

- Sketch $y_0(x) = x$ and the family $y_0(x) + \varepsilon h(x)$ where $h(x) = \sin(x)$.







- $\mathcal{J}(\varepsilon) \equiv J[y_0 + \varepsilon h(x)]$

$$= \int_0^{2\pi} \left([x + \varepsilon \sin x]' \right)^2 dx$$

$$= \int_0^{2\pi} [1 + \varepsilon \cos(x)]^2 \, dx$$

$$= \int_0^{2\pi} 1 + 2\varepsilon \cos x + \varepsilon^2 \cos^2 x \, dx$$

$$= 2\pi + 2\varepsilon \underbrace{\int_0^{2\pi} \cos(x) \, dx}_{=0} + \varepsilon^2 \int_0^{2\pi} \cos^2 x \, dx$$

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Homework #2, Problem #3 (cont'd)

$$J(\varepsilon) = 2\pi + \varepsilon^2 \int_0^{2\pi} \cos^2(x)dx$$

$$= 2\pi + \varepsilon^2 \int_0^{2\pi} \tfrac{1}{2}(1+\cos(2x))dx$$

$$= 2\pi + \tfrac{1}{2}\varepsilon^2 \left[\int_0^{2\pi} 1 + \cos(2x)dx \right]$$

$$= 2\pi + \tfrac{1}{2}\varepsilon^2 \left[x + \tfrac{1}{2}\sin(2x) \right]_{x=0}^{x=2\pi}$$

$$= 2\pi + \tfrac{1}{2}\varepsilon^2 \left[2\pi + 0 - 0 - 0 \right]$$

$$J(\varepsilon) = 2\pi + \pi\varepsilon^2$$

- Show $J'(0) = 0$

- First, By Calculus,

$$J'(0) = J'(\varepsilon)\Big|_{\varepsilon=0} = 2\pi\varepsilon\Big|_{\varepsilon=0} = 0.$$

- Alternatively, note that in your text, (3,11)

$$J'(0) = \tfrac{d}{d\varepsilon} J(y_0 + \varepsilon h)\Big|_{\varepsilon=0}$$

$$= \tfrac{d}{d\varepsilon}\left[2\pi + \pi\varepsilon^2\right]\Big|_{\varepsilon=0} = 2\pi\varepsilon\Big|_{\varepsilon=0} = 0.$$

- Alternatively, this problem is equivalent to asking you to compute $\delta J(y_0, h) = \delta J(x, \sin(x))$ "the 1st variation of J at x in the direction of $\sin(x)$."

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Homework # 2, Problem #4
Pg. 175 #1 a, b, c.

Find extremals:

(4) (a) $J(y) = \int_0^1 y'\,dx$, $y \in C^2[0,1]$, $y(0)=0$, $y(1)=1$

ELDE: $L_y = 0$, $L_{y'} = 1$

$$L_y - \frac{d}{dx}(L_{y'}) = 0$$
$$-\frac{d}{dx}(1) = 0$$
$$0 = 0$$

← A true statement, but this tells me nothing about an extremal.

Examine $J(y)$ more closely:
For any $y \in C^2[0,1]$ with $y(0)=0$, $y(1)=1$, we have

$$J(y) = \int_0^1 y'(x)\,dx = y(x)\Big|_{x=0}^{x=1} = y(1)-y(0) = 1-0 = 1$$

Hence, $J(y) = 1$ for all y in the set. This means that J is the constant functional, $J(y) = 1$ $\forall\, y \in C^2[0,1]$, $y(0)=0$, $y(1)$. Then, using the defn. of $\delta J(y,h)$, we have that $\forall\, h \in C^2[0,1]$ with $h(0)=h(1)=0$,

$$\delta J(y,h) = \frac{d}{d\varepsilon}\int_0^1 (y'+\varepsilon h')\,dx\Big|_{\varepsilon=0} = \int_0^1 h'(x)\,dx = h(1)-h(0) = 0.$$

Hence, any $y \in C^2[0,1]$ with $y(0)=0$, $y(1)=1$ is an extremal of J.

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Homework #2, Problem #4
(Cont'd)

(4) (b) $J(y)=\int_0^1 yy'\,dx$, $y\in C^2[0,1]$, $y(0)=0$, $y(1)=1$

- ELDE: $L_y = y'$, $L_{y'} = y$

$\Rightarrow$

$$y' - \frac{d}{dx}(y) = 0$$
$$y' - y' = 0$$
$$0 = 0$$

Again, a true statement, but it doesn't help us find extremals.

- Examine $J(y)$:

For any $y\in C^2[0,1]$, $y(0)=0$, $y(1)=1$,

$$J(y)=\int_0^1 yy'\,dx = \int_0^1 \frac{d}{dx}\left[\tfrac{1}{2}y^2\right]dx = \tfrac{1}{2}[y(x)]^2\Big|_{x=0}^{x=1} = \tfrac{1}{2}[1-0] = \tfrac{1}{2}$$

Again, $J(y)=\frac{1}{2}$ is the constant functional.

- Examine $\delta J(y,h)$

For any $h\in C^2[0,1]$, $h(0)=0$, $h(1)=0$, we have

$$\delta J(y,h) = \frac{d}{d\varepsilon}\int_0^1 (y+\varepsilon h)(y'+\varepsilon h')\,dx\Big|_{\varepsilon=0}$$
$$= \int_0^1 (y+\varepsilon h)h' + (y'+\varepsilon h')h\Big|_{\varepsilon=0}\,dx$$
$$= \int_0^1 yh' + y'h\,dx$$

$\rightarrow$

I.B.P., we get

$u = h(x) \quad v = y(x)$
$du = h'(x) \quad dv = y'(x)\,dx$

$$\delta J(y,h) = \int_0^1 y h'\,dx + h(x)y(x)\Big|_{x=0}^{x=1} - \int_0^1 y h'\,dx$$

$$= \int_0^1 y h' - y h'\,dx + \underbrace{h(1)y(1)}_{=0} - \underbrace{h(0)y(0)}_{=0}$$

$$= 0$$

Hence, $\delta J(y,h) = 0 \quad \forall\ y \in C^2[0,1]$ with $y(0)=0,\ y(1)=1$
Thus, any y in this set is an extremal.

(c.) $J(y) = \int_0^1 x y y'\,dx,\ y \in C^2[0,1],\ y(0)=0,\ y(1)=1$

ELDE : $L_y = xy' \quad , L_{y'} = xy$

$\Rightarrow$

$$xy' - \frac{d}{dx}(xy) = 0$$
$$xy' - [xy' + y] = 0$$
$$-y = 0$$
$$y(x) \equiv 0$$

$\to$ the zero function is our only candidate for an extremal.

But $y(x) \equiv 0 \ \forall x \in [0,1]$ fails to satisfy the Bdry Condition $y(1)=1$. So, J has no extremals in the set described above.

HW#2 | P. 175, #7 |

(5) (a) Find extremal for

$$J(y) = \int_0^1 (1+x)[y']^2 dx, \quad y \in C^2[0,1], \ y(0)=0, \ y(1)=1$$

$L(x,y,y') = (1+x)(y')^2$

$L_y = 0, \quad L_{y'} = 2(1+x)y'$

E.L.D.E:

$$\frac{d}{dx}\left(2(1+x)y'\right) = 0 \qquad \forall x \in (0,1]$$

$$2(1+x)y' = C \qquad \text{"} \quad \text{"}$$

$$y' = \frac{\tilde{C}}{1+x}$$

$$y(x) = \tilde{C}\int \frac{1}{1+x}\,dx + D$$

$$y(x) = \tilde{C}\ln(1+x) + D$$

$y(0)=0 \Rightarrow 0 = D$

$y(1)=1 \Rightarrow 1 = \tilde{C}\ln(2) \Rightarrow \tilde{C} = \frac{1}{\ln(2)}$

$$\therefore \boxed{y(x) = \frac{1}{\ln(2)}\ln(1+x)}$$

(b) If the B.C. $y'(1)=0$ is used, we would get

$$y(x) = \tilde{C}\ln(1+x) \quad \& \quad y'(x) = \tilde{C}\cdot\frac{1}{1+x}$$

$y'(1)=0 \Rightarrow 0 = \tilde{C}\cdot\frac{1}{2} \Rightarrow \tilde{C}=0$

so, $\boxed{y(x) \equiv 0}$

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HOMEWORK #2, Problem #6

(6) $J(y) = \int_0^1 \frac{1}{2}\left[(y'(x))^2 + (y(x))^2\right] dx - 2y(1)$
for all
$y \in C^2[0,1]$ and $y(0)=0$

$J(y+\varepsilon h) = \int_0^1 \frac{1}{2}\left[(y'+\varepsilon h')^2 + (y+\varepsilon h)^2\right] dx - 2(y(1)+\varepsilon h(1))$
$\forall\, h \in D = \{y \in C^2[0,1] : y(0)=0\}$
Then
$\frac{d}{d\varepsilon} J(y+\varepsilon h) = \int_0^1 (y'+\varepsilon h')h' + (y+\varepsilon h)h \, dx - 2h(1)$

$\delta J(y,h) = \frac{d}{d\varepsilon} J(y+\varepsilon h)\Big|_{\varepsilon=0}$
$= \int_0^1 y'h' + yh \, dx - 2h(1) \quad \forall\, h \in D.$

Find $y \in C^2[0,1], y(0)=0$ satisfying $\delta J(y,h)=0 \;\forall\, h \in D$
(1) $0 = \int_0^1 y'(x)h'(x) + y(x)h(x)\, dx - 2h(1)$
In particular, it's true $\forall\, h \in D$ with $h(1)=0$
Hence, we have
$\int_0^1 y'(x)h'(x) + y(x)h(x)\, dx = 0$
$\underbrace{y'(x)h(x)\Big|_{x=0}^{x=1}}_{=0} - \int_0^1 y''(x)h(x) + y(x)h(x) = 0$

$\int_0^1 (-y'' + y)h(x)\, dx = 0 \quad \forall\, h \in D$ with $h(1)=0$
$\therefore$ By F.L.C.V.
(*) $-y'' + y = 0 \quad \forall\, x \in (0,1)$
Hence, y must satisfy the ELDE given in (*)
Returning to (1), we see that if we consider this eqn on all of D and I.B.P., we have

$0 = y'(1)h(1) + \int_0^1 (-y'' + y)h(x)\, dx - 2h(1) \quad \forall\, h \in D$

(6 cont'd)

But using (*), we have that ②

$$0 = (y'(1) - 2)\,h(1) \quad \forall\, h \in D$$

Hence, we must have

$$y'(1) - 2 = 0 \quad \text{or} \quad y'(1) = 2.$$

Therefore, the extremal of J satisfy

$$-y'' + y = 0$$

$$y(0) = 0, \; y'(1) = 2$$

Soln: $-\lambda^2 + 1 = 0$

$\lambda^2 - 1 = 0 \qquad \lambda = \pm 1$

$y(t) = c_1 e^{-t} + c_2 e^{t}$

$y(0) = 0 = c_1 + c_2 \qquad \Rightarrow \quad c_2 = -c_1$

$y'(1) = 2 = -c_1 e^{-1} + c_2 e$

$2 = c_2 e^{-1} + c_2 e$

$\Rightarrow$

$$c_2 = \frac{2}{e + e^{-1}}, \quad c_1 = \frac{-2}{e + e^{-1}} \qquad (**)$$

And the extreme pt. of J is given by

$$y(t) = c_1 e^{-t} + c_2 e^{t},$$

Where c_1, c_2 are given in (**)