Name ____________________ Score ____________________

Spring 2008

Math 451 Homework 3

Due Friday, March 7, 2008

1. On page 184 of Logan textbook: Number 1a.
2. On page 184 of Logan textbook: Number 1b.
3. On page 184 of Logan textbook: Number 5a.
4. On page 184 of Logan textbook: Number 5e.
5. On page 203 of Logan textbook: Number 1.
6. On page 203 of Logan textbook: Number 2.

30 pts.

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Homework #3, Problem #1

Section 3.4

(12) $J(y_1, y_2) = \int_0^{\pi/4} 4y_1^2 + y_2^2 + y_1' y_2' \, dx$

$y_1(0)=1,\ y_1(\pi/4)=0\ ;\ y_2(0)=0,\ y_2(\pi/4)=1$

The ELDEs are

$L_{y_1} - \frac{d}{dx}(L_{y_1'}) = 0$ (ELDE1)

$L_{y_2} - \frac{d}{dx}(L_{y_2'}) = 0$ (ELDE2)

$L_{y_1} = 8y_1 \quad , L_{y_1'} = y_2' \ ; \ L_{y_2} = 2y_2 \ , \ L_{y_2'} = y_1'$

$8y_1 - y_2'' = 0$ (ELDE1)

$2y_2 - y_1'' = 0$ (ELDE2)

Using $2y_2'' - y_1'''' = 0 \Rightarrow y_2'' = \frac{1}{2}y_1''''$

And subst. into (ELDE1)

$8y_1 - \frac{1}{2}y_1'''' = 0$

$y_1'''' - 16y_1 = 0$

Using Char. Eqn. $r^4 - 16 = 0$,

$(r^2-4)(r^2+4) = 0$

$r = \pm 2, r = \pm 2i$

$\therefore$

$y_1(x) = Ae^{-2x} + Be^{2x} + C\cos(2x) + D\sin(2x)$ (1)

From (ELDE2), $y_2 = \frac{1}{2}y_1''$

$\therefore \quad y_2(x) = \frac{1}{2}\left[4Ae^{-2x} + 4Be^{2x} - 4C\cos(2x) - 4D\sin(2x)\right]$

$$y_2(x) = 2Ae^{-2x} + 2Be^{2x} - 2C\cos(2x) - 2D\sin(2x) \qquad (2)$$

(1) $y_1(0) = 1 \Rightarrow A + B + C = 1$

(2) $y_1(\pi/4) = 0 \Rightarrow Ae^{-\pi/2} + Be^{\pi/2} + D = 0$

(3) $y_2(0) = 0 \Rightarrow 2A + 2B - 2C = 0$

(4) $y_2(\pi/4) = 1 \Rightarrow 2Ae^{-\pi/2} + 2Be^{\pi/2} - 2D = 1$

Solving the Linear System, we have

$-\frac{1}{2}(3) + (1) \Rightarrow 2C = 1 \Rightarrow \boxed{C = 1/2}$

$2(2) - (4) \Rightarrow 4D = -1 \Rightarrow \boxed{D = -1/4}$

$(1) \Rightarrow A = \frac{1}{2} - B$

Plug into (2): $\left(\frac{1}{2} - B\right)e^{-\pi/2} + Be^{\pi/2} = \frac{1}{4}$

$$B\left(e^{\pi/2} - e^{-\pi/2}\right) = \frac{1}{4} - \frac{1}{2}e^{-\pi/2}$$

$$\boxed{B = \frac{(1 - 2e^{-\pi/2})}{4(e^{\pi/2} - e^{-\pi/2})}}$$

$$\Rightarrow \boxed{A = \frac{1}{2} - \frac{(1 - 2e^{-\pi/2})}{4(e^{\pi/2} - e^{-\pi/2})}}$$

$$A = \frac{2(e^{\pi/2} - e^{-\pi/2})}{4(e^{\pi/2} - e^{-\pi/2})} - \frac{(1 - 2e^{-\pi/2})}{4(e^{\pi/2} - e^{-\pi/2})}$$

other simplifications

$$\boxed{A = \frac{2e^{\pi/2} - 1}{4(e^{\pi/2} - e^{-\pi/2})}}$$

$$y_1(x) = Ae^{-2x} + Be^{2x} + \frac{1}{2}\cos(2x) - \frac{1}{4}\sin(2x)$$

$$y_2(x) = 2Ae^{-2x} + 2Be^{2x} - \cos(2x) + \frac{1}{2}\sin(2x)$$

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Homework #3, Problem #2

Section 3.4

(1.b) $J(y) = \int_0^1 (1+(y'')^2)\,dx$

$y(0)=0,\ y'(0)=1,\ y(1)=1,\ y'(1)=1$

$L_y = 0,\ L_{y'} = 0,\ L_{y''} = 2y''$

ELDE is:

$$0 - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(2y'') = 0$$

$$2y'''' = 0$$

Integrating, we get

$$y(x) = ax^3 + bx^2 + cx + d$$

$y(0)=0 \Rightarrow d=0$

$y'(0)=1 \Rightarrow c=1 \qquad \Rightarrow y(x) = ax^3 + bx^2 + 1$

$y(1)=1 \Rightarrow a+b+1=1 \qquad *$

$y'(1)=1 \Rightarrow 3a+2b+1=1$

$$\begin{aligned} a+b&=0 \\ 3a+2b&=0 \end{aligned} \Rightarrow \boxed{a=0,\ b=0}$$

$\therefore$

$y(x) = x$

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Homework #3, Problem #3

Section 3.4

(5.2) $J(y) = \int_0^1 (y')^2 + y^2 \, dx, \quad y(0)=1, \; y(1) = \text{unspecified}$

$L_y = 2y, \; L_{y'} = 2y'$

(ELDE)

$$2y - \frac{d}{dx}(2y') = 0$$

$$y - y'' = 0$$

$$\left.\begin{array}{l} y'' - y = 0 \\ y(0)=1, \; y'(1)=0 \end{array}\right\} \text{ELDE}$$

$$r^2 - 1 = 0 \implies r = \pm 1$$

$$y(x) = c_1 e^{-x} + c_2 e^{x}$$

$$y(0)=1 \implies c_1 + c_2 = 1$$

$$y'(1)=0 \implies -c_1 e^{-1} + c_2 e = 0 \implies c_2 = c_1 e^{-1} \cdot e^{-1}$$

$$\implies c_2 = c_1 e^{-2}$$

$\therefore$

$$c_1 + c_1 e^{-2} = 1$$

$$c_1 = \frac{1}{1+e^{-2}}, \quad c_2 = \frac{e^{-2}}{1+e^{-2}}$$

$\therefore$

$$y(x) = \frac{1}{1+e^{-2}} e^{-x} + \frac{e^{-2}}{1+e^{-2}} e^{x}$$

(Your soln may look different depending on your algebra manipulations + simplifications.)

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Homework #3, Problem #4
Section 3.4

(5.e.) $J(y) = \int_0^1 (y')^2 dx + [y(1)]^2$, $y(0)=1$, $y(1)=$ unspecified

Does not fit into the Natural BC. Framework, so we must derive it. Let $H = \{h \in C^2[0,1]$ and $h(0)=0\}$

$$\delta J = \frac{d}{d\varepsilon}\int_0^1 [y'+\varepsilon h']^2 dx + [y(1)+\varepsilon h(1)]^2 \Big|_{\varepsilon=0}$$

$$= \int_0^1 2[y'+\varepsilon h']h'\, dx + 2[y(1)+\varepsilon h(1)]h(1) \Big|_{\varepsilon=0}$$

$$= \int_0^1 2y'h'\,dx + 2y(1)h(1)$$

If y is an extremal, then

$$\delta J(y,h) = \int_0^1 2y'h'\,dx + 2y(1)h(1) = 0 \quad \forall\, h \in H \quad (*)$$

Then it follows that

$$\int_0^1 2y'h'\,dx = 0 \quad \forall\, h \in H \text{ with } h(1)=0.$$

$$2y'h\Big|_{x=0}^{x=1} - \int_0^1 2y''h\,dx = 0 \quad \text{"} \quad \text{"}$$

$\therefore$

$$\int_0^1 y''h\,dx = 0 \ \forall \quad \text{"} \quad \text{"}$$

Then by F.L.C.V. $y''=0 \quad \forall x \in [0,1]$

Returning to (*)

$$\int_0^1 2y'h'\,dx + 2y(1)h(1) = 0 \quad \forall\, h \in H$$

$$2y'h\Big|_{x=0}^{x=1} - \underbrace{\int_0^1 2y''h\,dx}_{=0} + 2y(1)h(1) = 0 \quad \text{" "}$$

$$2y'(1)h(1) + 2y(1)h(1) = 0 \quad \text{" "}$$

$$[y'(1) + y(1)]h(1) = 0$$

Since $h(1)$ is unrestricted, we must have

$$y'(1) + y(1) = 0$$

Hence, our extremal must satisfy

$$y'' = 0$$

$$y(0) = 1 \;,\quad y'(1) + y(1) = 0$$

Solving

$$y(x) = Ax + B$$

$y(0) = 1 \;\Rightarrow\; B = 1 \;\Rightarrow\; y(x) = Ax + 1$

$y'(1) + y(1) = 0 \;\Rightarrow\; A + A + 1 = 0$

$$2A = -1$$

$$A = -\tfrac{1}{2}$$

$\therefore$

$$y(x) = 1 - \tfrac{1}{2}x$$

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Homework #3, Problem #5
Section 3.6

① Find extremals of

$$J(y)=\int_0^\pi [y']^2 dx, \quad y(0)=y(\pi)=0$$

subject to

$$W(y)=\int_0^\pi (y)^2 dx = 1$$

Let $H=\{(h_1,h_2): h_1,h_2\in C^2[0,\pi], h_1(0)=h_1(\pi)=h_2(0)=h_2(\pi)=0\}$

Define $\mathcal{J}(\varepsilon_1,\varepsilon_2) = J(y+\varepsilon_1 h_1+\varepsilon_2 h_2)$

and

$$\mathcal{W}(\varepsilon_1,\varepsilon_2) = W(y+\varepsilon_1 h_1+\varepsilon_2 h_2)$$

If y is an extremal, then by LMR, $\exists\ \lambda$ so that

$$\frac{\partial \mathcal{J}}{\partial \varepsilon_1}(0,0) + \lambda \frac{\partial \mathcal{W}}{\partial \varepsilon_1}(0,0) = 0 \qquad (1)$$

$$\frac{\partial \mathcal{J}}{\partial \varepsilon_2}(0,0) + \lambda \frac{\partial \mathcal{W}}{\partial \varepsilon_2}(0,0) = 0 \qquad (2)$$

$$\mathcal{W}(0,0)=1$$

<u>Observe</u>: $\frac{\partial}{\partial \varepsilon_1}\mathcal{J}(0,0) = \int_0^\pi 2y'h_1'\,dx, \quad \frac{\partial \mathcal{J}}{\partial \varepsilon_2}(0,0) = \int_0^\pi 2y'h_2'\,dx$

$$\frac{\partial \mathcal{W}}{\partial \varepsilon_1}(0,0) = \int_0^\pi 2yh_1\,dx, \quad \frac{\partial \mathcal{W}}{\partial \varepsilon_2}(0,0) = \int_0^\pi 2yh_2\,dx$$

Eqn (1) Leads to $\int_0^\pi 2y'h_1'dx + 2\lambda\int_0^\pi yh_1\,dx = 0 \ \forall\ h_1\in C^2[0,\pi]$, $h_1(0)=h_1(\pi)=0$

$\Rightarrow \int_0^\pi [2\lambda y - 2y'']h_1(x)\,dx = 0 \ \forall\ h_1\in C^2[0,\pi],\ h_1(0)=h_1(\pi)=0.$

By F.L.C.V. $\quad y''-\lambda y=0 \quad \forall x\in[0,\pi].$

Hence, the ELDE is

$$y'' - \lambda y = 0 \quad , \forall x \in [0, \pi]$$
$$y(0) = y(\pi) = 0$$

Char. Eqn. $r^2 - \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}$

If $\lambda > 0$, then B.cs $\Rightarrow$ $0 = A + B$

$$0 = Ae^{-\sqrt{\lambda}\pi} + Be^{+\sqrt{\lambda}\pi}$$

So $B = -A \Rightarrow 0 = A(e^{-\sqrt{\lambda}\pi} - e^{\sqrt{\lambda}\pi}) \Rightarrow A = 0 \Rightarrow B = 0$

So $y(x) \equiv 0$ doesn't satisfy constraint in (3).

If $\lambda = 0$, $y(x) = A + Bx$, and B.C.'s $\Rightarrow$

$$0 = A \text{ and } B\pi = 0 \Rightarrow B = 0$$

Again, $y(x) \equiv 0$ doesn't satisfy (3).

If $\lambda < 0$, $y(x) = A\cos(\sqrt{|\lambda|}\,x) + B\sin(\sqrt{|\lambda|}\,x)$

B.cs $\Rightarrow$ $A = 0$ and $B\sin(\sqrt{|\lambda|}\,\pi) = 0$.

Since $B = 0$ gives us the trivial soln, then we choose λ so that

$$\sqrt{|\lambda|}\,\pi = n\pi$$
$$|\lambda| = n^2$$
$$\lambda = -n^2, \quad n = 1, 2, \ldots$$

Hence, we have an entire set of extremals

$$y_n(x) = B_n \sin(nx), \quad n = 1, 2, \ldots$$

Then, applying eqn (3), we have

$$\int_0^\pi B_n^2 \sin^2(nx)\,dx = 1 \qquad \text{Let } u = nx,\ du = n\,dx$$
$$B_n^2 \left(\tfrac{1}{n}\right) \int_0^{n\pi} \sin^2(u)\,du = 1$$
$$B_n^2 \left[\tfrac{1}{2}u - \tfrac{1}{4}\sin(2u)\right]_{u=0}^{u=n\pi} = n \qquad \Longrightarrow$$

③

$$B_n^2 \left[\tfrac{1}{2} n\pi\right] = n$$

$$B_n^2 = \frac{2}{\pi}$$

$$B_n = \pm\sqrt{2/\pi}$$

Therefore, the extremals are

$$y_n(x) = \pm\sqrt{2/\pi}\,\sin(nx)\ , \quad n = 1, 2, \ldots$$

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Homework #3, Problem #6
Section 3.6

② Find extremals for the isoperimetric problem

$$J(y) = \int_0^1 (y')^2 + x^2 \, dx \quad , y(0)=0 \, , y(1)=0$$

Subject to

$$W(y) = \int_0^1 y^2(x) \, dx = 2$$

Let $H = \{(h_1,h_2) : h_1, h_2 \in C^2[0,1], \ h_1(0)=h_1(1)=h_2(0)=h_2(1)=0\}$

Define $\mathcal{J}(\varepsilon_1,\varepsilon_2) = J(y+\varepsilon h_1 + \varepsilon h_2)$

and

$$\mathcal{W}(\varepsilon_1,\varepsilon_2) = W(y+\varepsilon_1 h_1 + \varepsilon_2 h_2)$$

If y is a local min for the problem, then by the LMR $\exists \ \lambda \in \mathbb{R}$ so that

$$\frac{\partial \mathcal{J}(0,0)}{\partial \varepsilon_1} + \lambda \frac{\partial \mathcal{W}(0,0)}{\partial \varepsilon_1} = 0 \qquad (1)$$

$$\frac{\partial \mathcal{J}(0,0)}{\partial \varepsilon_2} + \lambda \frac{\partial \mathcal{W}(0,0)}{\partial \varepsilon_2} = 0 \qquad (2)$$

$$\mathcal{W}(0,0) = \tfrac{1}{2} \qquad (3)$$

Observe: $\frac{\partial \mathcal{J}}{\partial \varepsilon_1}(0,0) = \int_0^1 2y'h_1' \, dx \, , \ \frac{\partial \mathcal{J}}{\partial \varepsilon_2}(0,0) = \int_0^1 2y'h_2' \, dx \quad \forall (h_1,h_2) \in H$

$$\frac{\partial \mathcal{W}}{\partial \varepsilon_1}(0,0) = \int_0^1 2y h_1 \, dx \, , \ \frac{\partial \mathcal{W}}{\partial \varepsilon_2}(0,0) = \int_0^1 2y h_2 \, dx \quad \forall (h_1,h_2) \in H$$

Then we get the following eqns for (1), (2)

(1) $\int_0^1 2y'h_1'\,dx + 2\lambda \int_0^1 y h_1\,dx = 0$

(2) $\int_0^1 2y'h_2'\,dx + 2\lambda \int_0^1 y h_2\,dx = 0 \qquad \forall\, (h_1,h_2) \in H$

Simplifying (1) using I.B.P., we get

$$\int_0^1 [2\lambda y - 2y'']\, h_1\,dx = 0 \qquad \forall\ h_1 \in C^2[0,1],\ h_1(0) = h_1(1) = 0$$

And by F.L.C.V. (Lemma 3.1), we have

$$2\lambda y - 2y'' = 0 \qquad \forall\ x \in [0,1]$$
$$y'' - \lambda y = 0 \qquad \text{"} \qquad \text{"}$$

Note Eqn (2) gives the same result.
Hence, our ELDE is

$$y'' - \lambda y = 0 \qquad \forall x \in [0,1]$$
$$y(0) = 0,\ y(1) = 0$$

$$r^2 - \lambda = 0 \implies r = \pm\sqrt{\lambda}$$

<u>Cases:</u>

① If $\lambda > 0$, then $y(x) = Ae^{-\sqrt{\lambda}x} + Be^{\sqrt{\lambda}x}$
And B.C.s $\implies$ $0 = A + B$
$0 = Ae^{-\sqrt{\lambda}} + Be^{\sqrt{\lambda}}$, and solving these
$\implies B(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}) = 0 \implies B = 0 \implies A = 0$
So $y(x) \equiv 0$ is our only candidate
This will not satisfy our constraint that $\int_0^1 [y(x)]^2\,dx = 2$.
Hence, $\lambda > 0$ leads to no extremals.

Case 2: If $\lambda = 0$, then $y(x) = A + Bx$ ③

And B.C.s $\Rightarrow$ $0 = A$

$0 = A + B \quad \Rightarrow B = 0$

So $y(x) \equiv 0$ is our only candidate, and, again, this will not satisfy our constraint in eqn (3).

Case 3: If $\lambda < 0$, then $y(x) = A\cos(\sqrt{|\lambda|}\,x) + B\sin(\sqrt{|\lambda|}\,x)$

And B.C.s $\Rightarrow$ $0 = A$

$0 = B\sin(\sqrt{|\lambda|})$

Since $B = 0$ would $\Rightarrow$ $y(x) \equiv 0$, and we already know that this doesn't yield extremals, we choose to leave B arbitrary and note that we can choose λ so that

$$\sqrt{|\lambda|} = n\pi \quad \text{for any } n \in \mathbb{Z} \setminus \{0\}$$

$$|\lambda| = n^2\pi^2$$

$$-\lambda = n^2\pi^2 \quad \text{so } \lambda = -n^2\pi^2 \ \forall\, n \in \mathbb{Z}^+$$

leads to extremals of the form

$$y_n(x) = B_n \sin(n\pi x), \quad n \in \mathbb{Z}^+$$

Then we know that y_n must satisfy eqn (3).

This gives

$$\int_0^1 B_n^2 \sin^2(n\pi x)\,dx = 2$$

Let $u = n\pi x$, $du = n\pi\,dx$

$$B_n^2 \frac{1}{n\pi} \cdot \int_0^{n\pi} \sin^2(u)\,du = 2$$

$$B_n^2 \frac{1}{n\pi}\left[\tfrac{1}{2}u - \tfrac{1}{4}\sin(2u)\right]_{u=0}^{u=n\pi} = 2$$

$$B_n^2 \frac{1}{n\pi}\left[\tfrac{1}{2}n\pi\right] = 2$$

$$B_n^2 = 4 \qquad B_n = \pm 2$$

Hence, we have an entire family of extremals of the form ④.

$$\boxed{y_n(x) = \pm 2\sin(n\pi x), \quad n = 1, 2, \ldots}$$

Plugging these into the cost function, we get

$$J(y_n(x)) = \int_0^1 [\pm 2n\pi \cos(n\pi x)]^2 + x^2\,dx$$

$$= 4n^2\pi^2 \int_0^1 \cos^2(n\pi x)\,dx + \int_0^1 x^2\,dx$$

$$= 4n^2\pi^2 \left(\frac{1}{n\pi}\right) \int_0^{n\pi} \cos^2(u)\,du + \frac{1}{3}x^3\Big|_{x=0}^{x=1}$$

$$= 4n\pi \left[\frac{1}{2}u + \frac{1}{4}\sin(2u)\right]_{u=0}^{u=n\pi} + \frac{1}{3}$$

$$= 4n\pi \left[\frac{1}{2}n\pi + 0\right] + \frac{1}{3}$$

$$= 4n^2\pi^2 + \frac{1}{3}$$

Hence, $J(y_n)$ is minimized when $n=1$.

Hence, our minimizers are $y(x) = \pm 2\sin(\pi x)$.