Name ______________________ Score ________________

Spring 2008

Math 451 Homework 4

Due Friday, April 4, 2008

1. On page 365 of Logan textbook: Number 2.
2. On page 366 of Logan textbook: Number 10.
3. On page 366 of Logan textbook: Number 11.
4. On page 372 of Logan textbook: Number 5.
5. On page 372 of Logan textbook: Number 6.

Math 451 - Spring '08 ①

HOMEWORK #4, Problem #1

Section 6.2, pg. 365, #2

Use the Energy Method to show that a soln. to I.B.V.P

$$u_t - k u_{xx} = 0, \quad 0<x<l, \quad 0<t<T$$

$$u(x,0) = f(x), \quad 0<x<l \qquad \text{(I.C.)}$$

$$u_x(0,t) = 0, \quad u(l,t) = h(t) \qquad \text{(Bdry Cond.)}$$

is unique. Here, we assume $k>0$.

pf: Assume existence of a soln is given. Let $u_1(x,t)$ and $u_2(x,t)$ be solns of the I.B.V.P. above.

Define

$$w(x,t) = u_1(x,t) - u_2(x,t)$$

Since we assume u_1, u_2 are continuously diff'ble w.r.t. t and twice continuously diff'ble w.r.t. x, then $w(x,t)$ retains these smoothness properties as well. And $w(x,t)$ satisfies the following I.B.V.P.

$$w_t - k w_{xx} = (u_1 - u_2)_t - k[u_1 - u_2]_{xx}$$

$$= (u_1)_t - (u_2)_t - k(u_1)_{xx} + k(u_2)_{xx}$$

$$= (u_1)_t - k(u_1)_{xx} - [(u_2)_t - k(u_2)_{xx}]$$

$$= 0 - 0$$

$$= 0. \qquad , \text{for } 0<x<l, \ 0<t<T$$

I.C.: $w(x,0) = u_1(x,0) - u_2(x,0) = f(x) - f(x) = 0.$

B.C.s: $w_x(0,t) = (u_1)_x(0,t) - (u_2)_x(0,t) = 0 - 0 = 0$

$w(l,t) = u_1(l,t) - u_2(l,t) = h(t) - h(t) = 0.$

So, $w(x,t)$ satisfies the follow I.B.V.P.

$$(*)\quad \left[\begin{array}{ll} w_t - k w_{xx} = 0, & 0<x<l,\ 0<t<T \\ w(x,0) = 0, & 0<x<l \\ w_x(0,t) = 0 \text{ and } w(l,t) = 0, & 0<t<T \end{array}\right]$$

Define the Energy Function

$$E(t) = \int_0^l [w(x,t)]^2\,dx$$

Note that $E(0) = \int_0^l 0\,dx = 0$. And $E(t) \geq 0$ for all $t \geq 0$.
Now, we examine the derivative

$$E'(t) = \frac{d}{dt}\int_0^l [w(x,t)]^2\,dx$$

By Chain Rule

$$= \int_0^l 2[w(x,t)]\,w_t(x,t)\,dx$$

By 1st Eqn in (*)

$$= \int_0^l 2[w(x,t)][k w_{xx}(x,t)]\,dx$$

$$= 2k\left[w(x,t)w_x(x,t)\Big|_{x=0}^{x=l} - \int_0^l [w_x(x,t)]^2\,dx \right]$$

$$= 2k\left[\underbrace{w(l,t)}_{=0}w_x(l,t) - w(0,t)\underbrace{w_x(0,t)}_{=0} - \int_0^l [w_x(x,t)]^2\,dx \right]$$

So,

(3)

$$E'(t) = -2k \int_0^{\ell} [w_x(x,t)]^2 dx \leq 0 \quad \text{for all } t > 0.$$

Hence $E(t)$ is a non-increasing function $\forall\, t > 0$.
And $E(0) = 0$ and $E(t) \geq 0\ \forall\, t > 0$ must mean that $E(t) = 0$ for all $0 < t < T$. That is

$$\int_0^{\ell} [w(x,t)]^2 dx = 0 \quad \forall\ t \in (0,T)$$

Since, $w(x,t)$ is a smooth function, then this implies that

$$\begin{aligned} w(x,t) &= 0 & \forall\ 0 < x < 1,\ 0 < t < T \\ u_1(x,t) - u_2(x,t) &= 0 & \text{"} \qquad \text{"} \\ u_1(x,t) &= u_2(x,t) & \text{"} \qquad \text{"} \end{aligned}$$

Hence, the soln to the original I.B.V.P. is unique.

Math 451 - Spring '08

HOMEWORK #4, Problem #2

Section 6.2, pg. 366, #10

Consider the I.B.V.P.

$$u_t - Du_{xx} = f(x), \quad 0<x<l, \quad t>0$$
$$u_x(0,t) = A, \quad u_x(l,t) = B, \quad t>0$$
$$u(x,0) = u_0(x), \quad 0<x<l$$

Show that if the soln is independent of time, ie. $u = u(x)$, then $DA - DB = \int_0^l f(x)dx$.
Give a physically meaningful interpretation of this condition.

→ If u is independent of t, then $u_t = 0$, and u satisfies

$$-Du_{xx} = f(x), \quad 0<x<l$$

$$\int_0^l -Du_{xx}\,dx = \int_0^l f(x)\,dx$$

$$-D\left[u_x\Big|_{x=0}^{x=l} \right] = \int_0^l f(x)dx$$

$$-D\left[u_x(l) - u_x(0)\right] = \int_0^l f(x)dx$$

$$-D\left[A - B\right] = \int_0^l f(x)dx$$

$$BD - AD = \int_0^l f(x)dx$$

→ next page

Physical Interpretation:

Since we began with a diffusion eqn, we can think our soln $u=u(x)$ as the steady-state or equilibrium soln of our original Problem. Furthermore, recall that the "Flux" for the diffusion eqn is defined to be $J(x,t) = -Du_x(x,t)$. So,

$-Du_x(0,t) = -DA$ represents Flux at left end
$-Du_x(l,t) = -DB$ " " right end

Then $BD - AD$ = Flux at left end − Flux at right end represents net flux through the domain in the steady state of the soln. Hence, the total amount of heat created in the domain by the source term $f(x)$ should equal the net flux through the boundary. This is simply Thm 6.19 for 1 spatial dimension.

Math 451 - Spring '08
HOMEWORK #4, Problem #3
Section 6.2, pg. 366, #11

Solve the I.B.V.P.

$$u_t - ku_{xx} = 0, \quad x>0, \quad t>0$$
$$u(0,t)=1, \quad u(\infty,t)=0, \quad t>0$$
$$u(x,0)=0, \quad x>0$$

by assuming a soln. of the form $u(x,t)=U(z)$
Where $z(t) = x/\sqrt{kt} = x(kt)^{-1/2}$

$$\rightarrow \quad u_t = U'(z)\cdot\frac{dz}{dt} = U'(z)\left[-\tfrac{1}{2}x(kt)^{-3/2}\cdot k\right]$$

$$u_x = U'(z)\cdot\frac{dz}{dx} = U'(z)\left[(kt)^{-1/2}\right]$$

$$u_{xx} = \left[U'(z)\cdot\frac{dz}{dx}\right]_x = \left[U'(z)\cdot\underbrace{\frac{d^2z}{dx^2}}_{=0} + \left(U''(z)\cdot\underbrace{\frac{dz}{dx}}_{(kt)^{-1/2}}\right)\cdot\underbrace{\frac{dz}{dx}}_{(kt)^{-1/2}}\right]$$

$$u_{xx} = U''(z)\,(kt)^{-1}$$

Then $\quad u_t - ku_{xx} = 0$

$$\Rightarrow$$

$$U'(z)\left[-\tfrac{1}{2}kx(kt)^{-3/2}\right] - k\left[U''(z)(kt)^{-1}\right] = 0.$$

$$U'(z)\left[kx(kt)^{-3/2}\right] + 2k(kt)^{-1}U''(z) = 0.$$

$$U'(z)\left[x k^{-1/2} t^{-3/2}\right] + 2t^{-1}U''(z) = 0.$$

$$U'(z)\left[\underbrace{x(kt)^{-1/2}}_{z}t^{-1}\right] + 2t^{-1}U''(z) = 0$$

Since $t>0$, we can multiply by t and re-arrange to get:

$$2v''(z) + zv'(z) = 0$$

$$\frac{d^2v}{dz^2} + \frac{1}{2}z \cdot \frac{dv}{dz} = 0$$

We can solve this via "Reduction of Order"
Let $y(z) = \frac{dv}{dz}$, then our eqn reduces to

$$y' + \frac{1}{2}z\,y = 0$$

$$y' = -\frac{1}{2}zy$$

$$\int \frac{1}{y}\frac{dy}{dz}\,dz = \int -\frac{1}{2}z\,dz$$

$$\int \frac{1}{y}\,dy = -\frac{1}{4}z^2 + C$$

$$\ln|y| = -\frac{1}{4}z^2 + c$$

$$y(z) = Ce^{-\frac{1}{4}z^2}$$

Substituting

$$\frac{dv}{dz} = Ce^{-\frac{1}{4}z^2}$$

$$v(z) = C_1\int_0^z e^{-\frac{1}{4}s^2}\,ds + C_2$$

And substituting for z, we get

$$u(x,t) = C_1 \int_0^{x/\sqrt{kt}} e^{-\frac{1}{4}s^2}\, ds + C_2$$

$$u(0,t) = 1 \Rightarrow C_1(0) + C_2 = 1 \Rightarrow C_2 = 1$$

$$u(\infty,t) = 0 \Rightarrow \lim_{x\to+\infty} u(x,t) = 0.$$

$$\lim_{x\to+\infty} C_1 \int_0^{x/\sqrt{kt}} e^{-\frac{1}{4}s^2}\, ds + 1 = 0$$

$$C_1 \lim_{x\to+\infty} \int_0^{x/\sqrt{kt}} e^{-\frac{1}{4}s^2}\, ds = -1$$

$$C_1\sqrt{\pi} = -1 \Rightarrow C_1 = \frac{-1}{\sqrt{\pi}}$$

$$\Leftarrow$$

<u>Note:</u> $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$

So, $\int_0^{\infty} e^{-\frac{1}{4}s^2}\, ds = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\frac{1}{4}s^2}\, ds$ by symmetry of the function.

$$= \frac{1}{2}\lim_{a\to\infty}\int_{-a}^{a} e^{-\frac{1}{4}s^2}\, ds$$

$u = \frac{1}{2}s$

$du = \frac{1}{2}ds$

$$= \frac{1}{2}\lim_{a\to\infty}\int_{-\frac{1}{2}a}^{\frac{1}{2}a} e^{-u^2}\, 2\,du$$

$$= \lim_{a\to+\infty}\int_{-\frac{1}{2}a}^{\frac{1}{2}a} e^{-u^2}\, du$$

$$= \int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$$

So, $u(x,t) = 1 - \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{kt}} e^{-\frac{1}{4}s^2}\, ds$ //

Math 451 - Spring '08
HOMEWORK #4, Problem #1
Section 6.3, pg. 372, #5

Find all radially symmetric solutions to Laplace's EQN in $\mathbb{R}^3$. Radially symmetric implies that $u = u(r)$. So, u is indepedent of the angular variables θ, ϕ. If we look at Laplace's EQN in spherical coordinates, we look for solns of

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r^2}\cdot\frac{1}{\sin\phi}\cdot\frac{\partial}{\partial\phi}(\sin(\phi)u_\phi) + \frac{1}{r^2}\cdot\frac{1}{\sin^2\phi}u_{\theta\theta} = 0$$

of the form $u = u(r)$. This means that $u_\theta = u_\phi = 0 \ \forall\, r, \theta, \phi$
Hence, the p.d.e. reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0 \qquad \text{(an ODE for } u(r)\text{)}$$

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$$

$$r^2\frac{du}{dr} = C$$

$$\frac{du}{dr} = Cr^{-2}$$

$$u(r) = C\int r^{-2}\,dr + D$$

$$u(r) = -Cr^{-1} + D$$

Hence, the radially symmetric solns. of Laplace's EQN in $\mathbb{R}^3$ have the form $u(r) = C_1 r^{-1} + C_2$

Math 451 - Spring '08 ①

HOMEWORK #4, Problem #5

Section 6.3, pg. 372 #6

Prove that if the Dirichlet Problem

$$-\Delta u = \lambda u \quad , \quad x \in \Omega$$
$$u = 0 \quad , \quad x \in \partial\Omega$$

has a nontrivial solution, then the constant λ must be positive.

→ Let ϕ be a nontrivial soln of the Dirichlet problem. Applying Green's Identity (see (iv) on page 352) using $u = w = \phi$, we have

$$\int_\Omega \phi \Delta \phi + \nabla\phi \cdot \nabla\phi \, d\vec{x} = \int_{\partial\Omega} \phi \cdot \frac{d\phi}{dn} dS$$

$$\int_\Omega \phi(-\lambda\phi) + \nabla\phi \cdot \nabla\phi \, d\vec{x} = 0 \qquad \text{since } \phi(\vec{x}) = \vec{0} \ \forall x \in \partial\Omega$$

$$-\lambda \int_\Omega [\phi(\vec{x})]^2 dx + \int_\Omega \|\nabla\phi\|^2 d\vec{x} = 0$$

$$-\lambda \int_\Omega [\phi(\vec{x})]^2 d\vec{x} = -\int_\Omega \|\nabla\phi\|^2 d\vec{x}$$

$$\lambda = \frac{\int_\Omega \|\nabla\phi\|^2 d\vec{x}}{\int_\Omega [\phi(\vec{x})]^2 d\vec{x}}$$

Remark that $\phi(\vec{x})$ is nontrivial, so it's not the zero function.

Hence, $\int_\Omega [\phi(x)]^2 dx > 0$.

The numerator $\int_\Omega \|\nabla\phi\|^2 d\vec{x} > 0$ as well.
Justification is as follows:
Clearly, $\|\nabla\phi\|^2 \geq 0$, and if $\|\nabla\phi\| = 0$, then $\nabla\phi(\vec{x}) = \vec{0}$ must be true $\forall x \in \Omega$.
But this implies that $\phi(\vec{x})$ is a constant function. But $\phi(\vec{x})$ must be continuous, and $\phi(\vec{x})$ satisfies $\phi(\vec{x}) = 0 \;\forall x \in \partial\Omega$.
Hence, $\phi(\vec{x})$ must be $= 0 \;\forall x \in \Omega$ (Since ϕ is constant) But this contradicts that $\phi(x)$ is nontrivial. Hence, we must have $\nabla\phi(\vec{x}) \neq \vec{0}$ and $\|\nabla\phi\|^2 > 0$. Hence

$$\lambda = \frac{\int_\Omega \|\nabla\phi\|^2 d\vec{x}}{\int_\Omega [\phi(\vec{x})]^2 d\vec{x}} > 0$$