Name ______________________ Score ____________________

Spring 2008

Math 451 Homework 5

Due Friday, April 18, 2008

1. Find all radially symmetric solutions to Laplace's equation in $\mathbb{R}^2$, that is in two spatial dimensions.

2. Let D denote an annular domain in $\mathbb{R}^2$ described by

$$D = \{(r, \theta) \colon 1 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$$

Find the steady-state temperature distribution, $u(x, y, t)$, in D if the temperature at the inner circle $r = 1$ is fixed at 0 degrees and the temperature at the outer circle $r = 2$ is fixed at 10 degrees. Note, you may assume that the temperature distribution in the region is *radially symmetric.*

   (a) Sketch the region D.

   (b) Formulate the ODE that must be solved taking into account the assumptions that I stated above. You may assume that the heat conductivity coefficient is $k = 1$. Don't forget to clearly state the boundary conditions.

   (c) Solve the BVP you gave above.

3. On page 372 of Logan textbook: Number 4.

4. Find the eigenvalues and **normalized** eigenfunctions for the SLP given by

$$-y''(x) = \lambda y(x), \quad 0 < x < \ell$$

$$y'(0) = 0, \;\; y'(\ell) = 0$$

5. On page 224 of Logan textbook: Number 2

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Homework #5, Problem #1

Find all radially symmetric solns. to Laplace's Eqn in $\mathbb{R}^2$.

Converting to polar coordinates to get

$$\Delta u = 0$$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \forall\ (r,\theta)$$

Using the assumption that u is radially symmetric, we enforce $u_\theta = 0$ and $u_{\theta\theta} = 0$ consequently.
This reduces to

$$u_{rr} + \frac{1}{r}u_r = 0$$

where we now consider $u = u(r)$.
Let $z = u_r$, then the eqn reduces to

$$z_r + \frac{1}{r}z = 0 \quad , \text{for } r \neq 0$$

$$\frac{dz}{dr} = -\frac{1}{r}z$$

$$\int \frac{1}{z}\frac{dz}{dr} = -\int \frac{1}{r}dr$$

$$\ln|z| = -\ln|r| + C$$

$$\ln|z| = \ln|\tfrac{1}{r}| + C$$

$$z(r) = C\frac{1}{r}$$

$\therefore$

$$\frac{du}{dr} = C\frac{1}{r}$$

$$u(r) = C\int \frac{1}{r}dr + D$$

$$u(r) = C\ln|r| + D$$

So, for $r > 0$, the radially symmetric solutions of Laplace's Eqn are

$$u(r) = C\ln(r) + D$$

for arbitrary constants C and D

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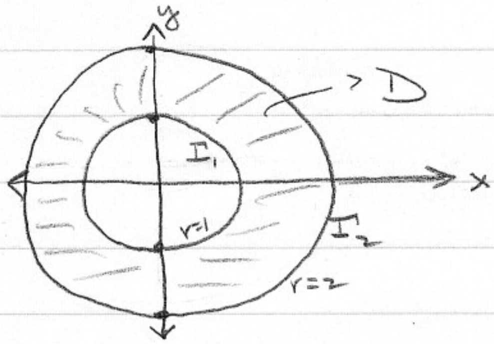
Homework #5, Number 2

(2) a.)



b.) Begin with $u = u(x,y,t)$, and $k=1$ in the heat eqn

$$u_t = u_{xx} + u_{yy}$$

And assuming steady-state temperature, we wish to solve

$$\Delta u = 0 \quad , u|_{\Gamma_1} = 0, u|_{\Gamma_2} = 10$$

c.) For this geometry, we pose Laplace's Eqn in 2D in polar coordinates. And we use the previous problem to note that this eqn reduces to solving for the radially symmetric soln whose general form is given by

$$u(r) = C\ln(r) + D$$

B.C.s

If $r=1$, $u(1) = 0 = C\ln(1) + D = D$

$\therefore D = 0$

If $r=2$, $u(2) = 10 = C\ln(2) \Rightarrow C = 10/\ln(2)$

$$\therefore u(r) = \frac{10}{\ln(2)} \ln(r)$$

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Homework #5, Problem #3
Page 372, #4 ①

(4) Let $q>0$ and cont. in $\bar{V} \in \mathbb{R}^3$, f be cont. in $\bar{V}$, and g be cont. on ∂V. Prove that the B.V.P.

$$(*) \qquad \Delta u - q(x)u = f(x), \quad x \in V$$
$$\frac{du}{dn} = g(x), \quad x \in \partial V$$

can have at most one soln in $C^1(\bar{V}) \cap C^2(V)$.

proof: Assume that u_1 and u_2 are solns of $(*)$, then

$$\Delta u_1 - q(x)u_1 = f(x) \quad \text{and} \quad \Delta u_2 - q(x)u_2 = f(x), \; x \in V$$
$$\frac{du_1}{dn} = g(x) \qquad\qquad \frac{du_2}{dn} = g(x), \; x \in \partial V$$

Define $w(x) = u_1(x) - u_2(x)$, then $w(x)$ satisfies the PDE.

$$\Delta w - q(x)w = 0 \quad , x \in V$$
$$\frac{dw}{dn} = 0 \quad , x \in \partial V$$

$$\int_V w\Delta w + \nabla w \cdot \nabla w \, d\vec{x} = \int_{\partial V} w \frac{dw}{dn} \, dS = 0.$$

$$\int_V q(x)[w(x)]^2 + \nabla w \cdot \nabla w \, d\vec{x} = 0$$

But $q(x) > 0 \; \forall x \in \bar{V}$, $[w(x)]^2 \geq 0 \; \forall x \in V$, and $\nabla w \cdot \nabla w = \|\nabla w\|^2 \geq 0 \; \forall x \in V$.
Hence, we must have

$$q(x)[w(x)]^2 + \nabla w \cdot \nabla w = 0 \; \forall x \in V$$

In particular,

$$q(x)\,[w(x)]^2 = 0 \quad \forall x \in V$$

$$\Rightarrow \quad w(x) \equiv 0 \quad \forall x \in V$$

$$\Rightarrow \quad u_1(x) = u_2(x) \;\forall x \in V.$$

Also note that $\nabla w \equiv 0 \;\forall x \in V \Rightarrow$
$\Rightarrow$ w is constant $\forall \in V$, and since we
assume $w \in C^1(\bar{V})$, then $w(x) \equiv 0 \;\forall x \in \bar{V}$. //

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Homework #5 - Number 4

1

(4.) $-y'' = \lambda y \quad , \quad 0 < x < l,$
$y'(0) = 0 \quad , \quad y'(l) = 0$

Char. Eqn: $-r^2 - \lambda = 0$
$r^2 + \lambda = 0$
$r = \pm\sqrt{-\lambda}$

Cases: (1.) $\lambda < 0$
(2) $\lambda = 0$
(3) $\lambda > 0$

Case (1) If $\lambda < 0$, then let $\gamma = \sqrt{-\lambda} \in \mathbb{R}$.

Gen Soln:
$y(x) = ae^{-\gamma x} + be^{\gamma x}$
$y'(x) = -a\gamma e^{-\gamma x} + b\gamma e^{\gamma x}$

B.C.s

$y'(0) = 0 = -a\gamma + b\gamma \Rightarrow b - a = 0 \Rightarrow b = a$
$y'(l) = 0 = -a\gamma e^{-\gamma l} + b\gamma e^{\gamma l}$
Using $b = a$, we get
$-b\gamma e^{-\gamma l} + b\gamma e^{\gamma l} = 0$
$b\gamma(e^{\gamma l} - e^{-\gamma l}) = 0$
$\gamma \neq 0$, and $e^{\gamma l} - e^{-\gamma l} > 0$, so
$\boxed{b = 0 \text{ and } a = 0}$

$\therefore \lambda < 0 \Rightarrow$ only $y(x) \equiv 0$

Case ② If $\lambda = 0$, then $r = 0$

Gen Soln: $y(x) = a + bx$

$y'(x) = b$

$y'(0) = b$ and $y'(\ell) = b \Rightarrow b = 0$

This implies that any y of the form

$y(x) = a$, for some constant a

is an e-function for $\lambda = 0$.

Case ③ If $\lambda > 0$, then $r = \pm i\sqrt{|\lambda|}$. Let $\gamma = \sqrt{|\lambda|} = \sqrt{\lambda}$

Gen Soln: $y(x) = a\cos(\gamma x) + b\sin(\gamma x)$

$y'(x) = -a\gamma\sin(\gamma x) + b\gamma\cos(\gamma x)$

B.C.s

$y'(0) = 0 = b\gamma \Rightarrow b = 0$

$y'(\ell) = 0 = -a\gamma\sin(\gamma\ell)$

Since $a = 0$ yields the trivial soln, we choose γ so that

$\gamma\ell = n\pi$, $n = 1, 2, \ldots$ ($n=0$ puts us in case ②)

$\gamma = \frac{n\pi}{\ell}$

$\sqrt{\lambda} = \frac{n\pi}{\ell}$

$\lambda = \frac{n^2\pi^2}{\ell^2}$, $n = 1, 2, \ldots$

Define $\lambda_0 = 0$, $\varphi_0 = a$

$\lambda_n = \frac{n^2\pi^2}{\ell^2}$, $\varphi_n = a_n\cos(\frac{n\pi}{\ell}x)$, $n \in \mathbb{Z}^+$

And now we normalize:

Choose a so that $\|\psi_0\|_2 = 1$ (3.)

$$\|\psi_0\|_2^2 = \int_0^l [\psi_0]^2\,dx = 1 \quad \Rightarrow \quad \int_0^l a^2\,dx = 1$$

$$a^2 l = 1$$

$$a = \sqrt{1/l}$$

$\therefore$

$$\boxed{\psi_0(x) = 1/\sqrt{l} \quad \text{with } \lambda_0 = 0.}$$

For $n \geq 1$, we choose a_n so that $\|\psi_n\|_2 = 1$

$$\|\psi_n\|_2^2 = \int_0^l a_n^2 \cos^2\left(\frac{n\pi}{l}x\right) dx \Rightarrow a_n^2 \int_0^l \cos^2\left(\frac{n\pi}{l}x\right) dx = 1$$

$$\Rightarrow a_n = 1/\sqrt{l/2}$$

$$\Rightarrow a_n = \sqrt{2/l}$$

And $\boxed{\psi_n(x) = \sqrt{2/l}\,\cos\left(\frac{n\pi}{l}\right) \text{ with } \lambda_n = \frac{n^2\pi^2}{l^2}}$, $n \in \mathbb{Z}^+$.

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Homework #5, Problem #5
Page 224, Problem # 2

(5) Show that the SLP

$$-y'' = \lambda y \ , \quad 0 < x < l$$
$$y'(0) = 0 \ , \quad y(l) = 0$$

with mixed Dirichlet and Neumann BC's has e-values

$$\lambda_n = \left(\frac{(1+2n)\pi}{2l} \right)^2$$

with corresponding e-functions $y_n(x) = \cos\left(\frac{(1+2n)\pi x}{2l} \right)$

for $n = 0, 1, 2, \ldots$

Note that from the previous problem
Char Eqn: $-r^2 - \lambda = 0 \implies r = \pm\sqrt{-\lambda}$

Cases: (1) $\lambda < 0$
(2) $\lambda = 0$
(3) $\lambda > 0$

Case (1): If $\lambda < 0$, Let $\gamma = \sqrt{-\lambda}$ and
Gen. Soln: $y(x) = a e^{-\gamma x} + b e^{\gamma x}$
$$y'(x) = -a\gamma e^{-\gamma x} + b\gamma e^{\gamma x}$$

B.C.'s $y'(0) = 0 \implies -a\gamma + b\gamma = 0 \implies (-a+b)\gamma = 0 \implies a = b$
$y(l) = 0 \implies a e^{-\gamma l} + b e^{\gamma l} = 0$

$\rightarrow$

Using $a=b$, the second eqn becomes

$$a(e^{-\gamma l}+e^{\gamma l})=0$$

$$ae^{-\gamma l}(1+e^{2\gamma l})=0 \quad \Rightarrow a=0$$

$$\Rightarrow b=0$$

So, $\lambda<0$ gives only trivial solns.

Case 2: $\lambda=0 \Rightarrow y(x)=a+bx$, $y'(x)=b$

B.C.s

$$y'(0)=0 \quad \Rightarrow b=0 \quad \Rightarrow y(x)=a$$

$$y(l)=0 \quad \Rightarrow y(l)=a=0 \quad \Rightarrow a=0$$

Hence, if $\lambda=0$, we only get the trivial soln. $y(x)\equiv 0$.

Case 3: $\lambda>0 \Rightarrow r=\pm\sqrt{-\lambda}=\pm i\sqrt{\lambda}$

Gen Soln:

$$y(x)=a\cos(\sqrt{\lambda}x)+b\sin(\sqrt{\lambda}x)$$

$$y'(x)=-a\sqrt{\lambda}\sin(\sqrt{\lambda}x)+b\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

B.C.s $y'(0)=0 \Rightarrow b\sqrt{\lambda}=0 \Rightarrow b=0.$

$$y(l)=0 \Rightarrow a\cos(\sqrt{\lambda}l)=0$$

Choose λ so that

$$\sqrt{\lambda}l=\pi/2+n\pi, \quad n=0,1,2,\ldots$$

$$\sqrt{\lambda}l=\frac{(1+2n)\pi}{2}$$

$$\Rightarrow \quad \lambda_n=\left[\frac{(1+2n)\pi}{2l}\right]^2, \quad n=0,1,2,\ldots$$

And the corresponding eigenfunctions are

$$y_n(x) = \cos\left(\frac{(1+2n)\pi x}{2l}\right), \quad n=0,1,2,\ldots$$