Name ____________________ Score ____________________

Spring 2008

Math 451 Homework 6

Due Friday, April 25, 2008

1. Use Fourier Series techniques to solve the boundary value problem for the ordinary differential equation given by

$$-u'' + a^2 u = f(x), \quad 0 < x < \pi$$

$$u(0) = 0, \quad u(\pi) = 0,$$

where $f(x)$ is a continuous function on $[0, \pi]$. HINT: First find the eigenfunctions of the associated SLP problem $-u'' = \lambda u$ with zero boundary conditions. Next, expand the solution of your equation in terms of those eigenfunctions. You will also want to use a series expansion of $f(x)$ using those eigenfunctions. (*Do not find the eigenvalues from scratch! You may quote results in the notes.*)

2. Consider the 1D heat equation

$$\begin{aligned} u_t = u_{xx}, \qquad & 0 < x < 1,\ t > 0 \\ u(0,t) = 0,\ u(1,t) = 0, \qquad & t > 0 \\ u(x,0) = g(x), \qquad & 0 < x < 1, \end{aligned}$$

where

$$g(x) = \begin{cases} 1, & \text{if } 0 \le x < \frac{1}{2} \\ 0, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Use the Fourier Method to find the solution. (*Do not do this problem from scratch! You may quote results in the notes.*) Then use the first 3 terms of the series expansion to obtain an approximation $u_3(x,t)$ to the solution. Generate a time series plot of snapshots at three times $\bar{t} = 0, 0.1, 1$.

3. Use the Laplace Transform to solve the following IVP.

$$\frac{d^2u}{dt^2} + \frac{du}{dt} = f(t), \quad t > 0$$

$$u(0) = 0, \ \frac{du}{dt}(0) = 0$$

You should also assume that f is of exponential order, and you may need to do a partial fraction decomposition. (*thinly veiled hint*)

4. Verify the identity given below for the Fourier Transform (as it is defined in your text), and you may assume that $u = u(x,t) \in \mathcal{S}$.

$$[\mathcal{F}u_x](\xi, t) = (-i\xi)\hat{u}(\xi, t)$$

5. Use the fact that

$$e^{-|x|} = \begin{cases} e^{x}, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases}$$

to show that

$$\mathcal{F}(e^{-|x|}) = \frac{2}{1+\xi^2}.$$

6. Use the Fourier Transform to solve the following ODE.

$$-\frac{d^2u}{dx^2} + u = f(x), \quad -\infty < x < \infty$$

You should also assume that $f \in \mathcal{S}$, and you may quote results from previous problems on this assignment if you should find the need to do so. (*another thinly veiled hint*)

7. Recall that the Fourier Convolution is given by

$$f \star g = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

Use the change of variables $z = x - y$ to prove that

$$f \star g = g \star f.$$

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Homework #6, Problem #1

$$-u'' + a^2 u = f(x), \quad 0<x<\pi \qquad (*)$$
$$u(0)=0, \quad u(\pi)=0$$

f is continuous on $[0,\pi]$ and $a \in \mathbb{R}$

(A) First, find e-functions of
$$-u'' = \lambda u \quad , \quad 0<x<\pi$$
$$u(0)=0, u(\pi)=0$$

From the course notes on SLP's, the e-pairs are
$$\lambda_n = n^2 \quad , \quad \Phi_n(x) = \sin(n\pi), \; n \in \mathbb{Z}^+$$

(B) Express our original soln $u(x)$ in terms of a Fourier Series using the ϕ_n's from above. So, we want to find $C_n, n=1,2,\ldots$ so that

$$u(x) = \sum_{n=1}^{\infty} C_n \Phi_n(x)$$

solves our original eqn. (*)

- Note that we also use the Fourier Series for $f(x)$ given by

$$f(x) = \sum_{n=1}^{\infty} f_n \Phi_n(x), \text{ where } f_n = \frac{\int_0^{\pi} f(x)\Phi_n(x)\,dx}{\int_0^{\pi} [\Phi_n(x)]^2\,dx}, \; n=1,2,3,\ldots$$

Plugging our Fourier Series into (*), we get

$$-\left(\sum_{n=1}^{\infty} c_n\phi_n(x)\right)'' + a^2\left(\sum_{n=1}^{\infty} c_n\phi_n(x)\right) = \left(\sum_{n=1}^{\infty} f_n\phi_n(x)\right) \quad \forall x\in(0,\pi)$$

$$\sum_{n=1}^{\infty} c_n\underbrace{(-\phi_n''(x))} + \sum_{n=1}^{\infty} a^2 c_n\phi_n(x) = \sum_{n=1}^{\infty} f_n\phi_n(x) \quad " \quad "$$

Since λ_n, ϕ_n are e-pairs of SLP

$$\sum_{n=1}^{\infty} c_n\lambda_n\phi_n(x) + \sum_{n=1}^{\infty}(a^2c_n - f_n)\phi_n(x) = 0 \quad " \quad "$$

$$\sum_{n=1}^{\infty}\left[c_n\lambda_n + a^2c_n - f_n\right]\phi_n(x) = 0 \quad \forall x\in(0,\pi)$$

Since, the ϕ_n are linearly independent, we must have

$$c_n\lambda_n + a^2c_n - f_n = 0 \quad \forall\, n=1,2,3,\dots$$

$$c_n = \frac{f_n}{\lambda_n + a^2} = \frac{f_n}{n^2+a^2}, \quad \forall\, n=1,2,3,\dots$$

Hence, we have fully determined the soln.

$$u(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \text{ with } c_n = \frac{f_n}{n^2+a^2}, \quad n=1,2,3,\dots$$

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Homework #6, Problem #2

- Using notes in class from Example 6.24 with $l=1$, and I.C. denoted by $g(x)$, we get a soln of the form:

$$u(x,t) = 2\sum_{n=1}^{\infty}\left(\int_0^1 g(\xi)\sin(n\pi\xi)\,d\xi\right)e^{-n^2\pi^2 t}\sin(n\pi x)$$

As an approximation, we choose to use

$$\tilde{u}(x,t) = 2\sum_{n=1}^{3}\left(\int_0^1 g(\xi)\sin(n\pi\xi)\,d\xi\right)e^{-n^2\pi^2 t}\sin(n\pi x)$$

$$= 2\Big[e^{-\pi^2 t}\sin(\pi x)\int_0^{1/2}\sin(\pi\xi)\,d\xi$$

$$+ e^{-4\pi^2 t}\sin(2\pi x)\int_0^{1/2}\sin(2\pi\xi)\,d\xi$$

$$+ e^{-9\pi^2 t}\sin(3\pi x)\int_0^{1/2}\sin(3\pi\xi)\,d\xi\Big]$$

Note: $\displaystyle\int_0^{1/2}\sin(n\pi\xi)\,d\xi = \frac{1}{n\pi}\int_0^{n\pi/2}\sin(u)\,du$ $\quad u = n\pi\xi$, $du = n\pi\,d\xi$

$$= -\frac{1}{n\pi}\left[\cos(n\pi/2) - \cos(0)\right]$$

$$= \frac{1}{n\pi}\left[1 - \cos(n\pi/2)\right]$$

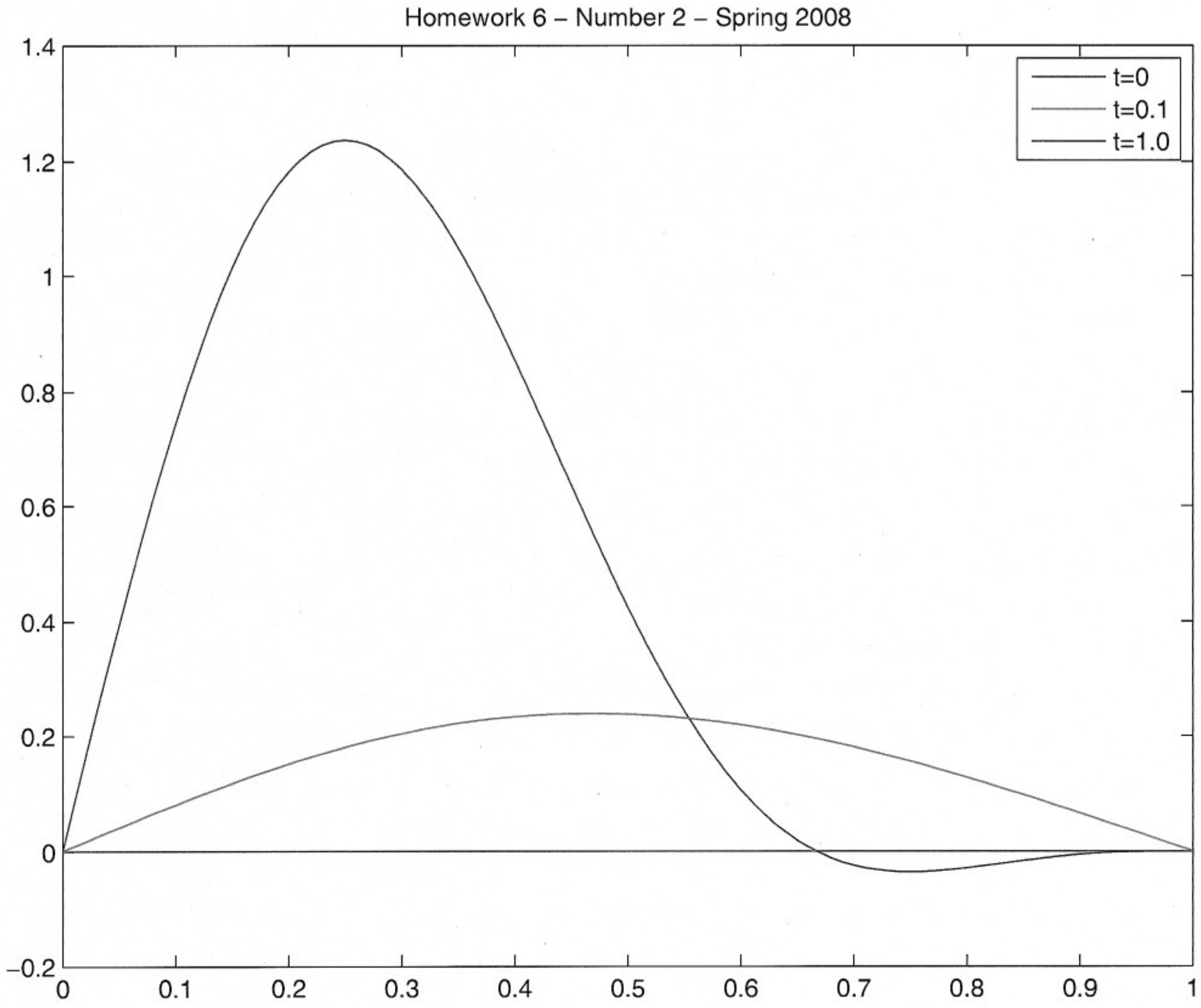
$$\int_0^{1/2}\sin(n\pi\xi)\,d\xi = \begin{cases} \frac{1}{\pi}, & n=1 \\ \frac{1}{\pi}, & n=2 \\ \frac{1}{3\pi}, & n=3 \end{cases}$$

Hence, we can write our approximation in a nice form:

(2)

$$\tilde{u}(x,t) = \frac{2}{\pi} e^{-\pi^2 t} \sin(\pi x) + \frac{2}{\pi} e^{-4\pi^2 t} \sin(2\pi x) + \frac{2}{3\pi} e^{-9\pi^2 t} \sin(3\pi x)$$

Now we generate a time series plot: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t=0
t=0.1
t=1.0
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

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HW #5, Problem # 3

$\ddot{u} + \dot{u} = f(t), \quad t > 0$ (I'm using $\cdot = \frac{d}{dt}$ notation)

$u(0) = 0, \quad \dot{u}(0) = 0$

Applying Laplace Transform:

$$\mathcal{L}[\ddot{u}] + \mathcal{L}[\dot{u}] = \mathcal{L}[f]$$

$$s^2 U(s) + 0 - 0 + sU(s) - 0 = F(s)$$

$$U(s) = \frac{1}{s(s+1)} F(s)$$

We want to use the Convolution Theorem:

Note: $\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t}$

Then

$$u(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s+1)} F(s)\right]$$

and

$$u(t) = \int_0^t [1 - e^{-(t-\tau)}] f(\tau)\, d\tau$$

using the Convolution theorem.

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HOMEWORK #6, Problem #4

$$[\mathcal{F} u_x](\xi,t) = \int_{-\infty}^{\infty} u_x(x,t)e^{i\xi x}\,dx$$

$$= e^{i\xi x}u(x,t)\Big|_{x=-\infty}^{x=+\infty} - \int_{-\infty}^{\infty} u(x,t)\,i\xi e^{i\xi x}\,dx$$

$\boxed{u = e^{i\xi x} \quad v = u(x,t) \quad du = i\xi e^{i\xi x}\,dx \quad dv = u_x(x,t)\,dx}$

$$= \underbrace{0 - 0}_{} - i\xi \int_{-\infty}^{\infty} u(x,t)e^{i\xi x}\,dx$$

$$= -i\xi\,\mathcal{F}u$$

$$-i\xi\,\hat{u}(\xi,t)$$

These term $\to 0$ as $x \to \pm\infty$ because

(a) $e^{i\xi x}$ is bounded as $x \to \pm\infty$ and

(b) $u \in \mathcal{S} \Rightarrow u(x,t) \to 0$ as $x \to \pm\infty$

Note that this assumes that the function $u(x,t) \in \mathcal{S}$. That is, it's in the Schwarz Class as mentioned in your text.

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HOMEWORK #6, Problem #5

Note: $e^{-|x|} = \begin{cases} e^{x}, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases}$

$$\mathcal{F} e^{-|x|} = \int_{-\infty}^{\infty} e^{-|x|} e^{i\xi x}\, dx$$

$$= \int_{-\infty}^{0} e^{x} e^{i\xi x}\, dx + \int_{0}^{\infty} e^{-x} e^{i\xi x}\, dx$$

$$= \int_{-\infty}^{0} e^{x(1+i\xi)}\, dx + \int_{0}^{\infty} e^{x(i\xi-1)}\, dx$$

$$= \frac{1}{1+i\xi} e^{x(1+i\xi)} \Big|_{x=-\infty}^{x=0} + \frac{1}{i\xi-1} e^{x(\xi i-1)} \Big|_{x=0}^{x=\infty}$$

$$= \frac{1}{1+i\xi} - 0 + 0 - \frac{1}{i\xi-1}$$

↳ Since $e^{x(1+i\xi)} = e^{x} e^{i\xi x} \to 0$ as $x \to -\infty$

Since $e^{x(\xi i-1)} = e^{x\xi i} e^{-x} \to 0$ as $x \to +\infty$

$$= \frac{1}{1+\xi i} - \frac{1}{i\xi-1}$$

$$= \frac{i\xi - 1 - (i\xi+1)}{(i\xi+1)(i\xi-1)}$$

$$= \frac{-2}{i^2\xi^2-1}$$

$$= \frac{2}{1+\xi^2}$$

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HW #6, Problem #6

Use Fourier Transform to solve

$$-u''(x) + u(x) = f(x), \quad -\infty < x < \infty$$

Require that $u(x)$ and $u'(x) \longrightarrow 0$ as $x \rightarrow \pm\infty$

$$-\mathcal{F}u'' + \mathcal{F}u = \mathcal{F}f$$

$$-(-i\xi)^2 \hat{u}(\xi) + \hat{u}(\xi) = \hat{f}(\xi)$$

$$\xi^2 \hat{u} + \hat{u} = \hat{f}$$

$$\hat{u}(\xi)[\xi^2 + 1] = \hat{f}$$

$$\hat{u}(\xi) = \frac{1}{(\xi^2+1)} \hat{f}(\xi)$$

$$\hat{u}(\xi) = \frac{1}{2}\left[\frac{2}{\xi^2+1}\right] \hat{f}(\xi)$$

$$\mathcal{F}^{-1}(\hat{u}(\xi)) = \frac{1}{2} \mathcal{F}^{-1}\left(\frac{2}{\xi^2+1} \cdot \hat{f}\right)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-y|} f(y)\, dy$$

$$u(x) = \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x-y|} f(y)\, dy$$

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HOMEWORK #6, Problem #7

$$f*g = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$$

Let $z = x-y$, $dz = -dy$. Then for any fixed, arbitrary x, as $y \to \infty$, $z \to -\infty$. And as $y \to -\infty$, $z \to +\infty$. Hence, we have

$$
\begin{aligned}
f*g &= \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \\
&= \int_{\infty}^{-\infty} f(z)g(x-z)(-dz) \\
&= -\int_{\infty}^{-\infty} g(x-z)f(z)\,dz \\
&= \int_{-\infty}^{\infty} g(x-z)f(z)\,dz \\
&= g*f
\end{aligned}
$$