Math 451 Midterm Exam

Due Friday, March 28, 2008

Please provide enough details of your work so that I can follow your reasoning. Without details, I cannot assign partial credit.

1. *(20 points)* Page 204 of Logan text, Problem 6.

2. *(10 points)* Assume that $k(x,t), d(x,t)$ are given functions that are sufficiently smooth. Show that the operator

$$\mathcal{L}u = u_t - (k(x,t)u_x)_x + d(x,t)u$$

   is a linear operator.

3. *(10 points)* Determine whether the following PDE is linear or nonlinear.

$$u_{tx} + u^2 = \sin x,$$

4. (a) *(10 points)* Determine regions of the xt-plane where the following equation is **hyperbolic, parabolic** or **elliptic**.

$$tu_{tt} + u_{xx} = 0,$$

   (b) *(10 points)* Determine regions of the xt-plane where the following equation is **hyperbolic, parabolic** or **elliptic**.

$$u_{tt} + (1 + x^2)u_x - u_t = e^{tx},$$

5. *(10 points)* Page 175 of Logan text, Problem 2b.

6. *(10 points)* Page 184 of Logan text, Problem 1d.

7. *(20 points)* Page 346 of Logan text, Problem 5. Hint: This problem is stating that any function u that is only a function of the radius-variable explicitly must satisfy that equation. That is, any function of only r (so that the θ variable doesn't appear explicitly in the function expression for u) is a solution of the PDE.

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Midterm Exam - #1
Page 204, Problem #6

1. Minimize

$$J(y) = \int_a^b p(x)[y'(x)]^2 + q(x)[y(x)]^2 dx$$

s.t.

$$W(y) = \int_a^b r(x)[y(x)]^2 dx = 1$$

where p, q, r are given functions and $y(a) = y(b) = 0$.

Soln: Define $H = \{(h_1, h_2) : h_i \in C[a,b] \text{ and } h_i(a) = h_i(b) = 0, i = 1, 2\}$
Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$, and define

$$\mathcal{J}(\varepsilon_1, \varepsilon_2) = J(y + \varepsilon_1 h_1 + \varepsilon_2 h_2)$$
$$\mathcal{W}(\varepsilon_1, \varepsilon_2) = W(y + \varepsilon_1 h_1 + \varepsilon_2 h_2)$$

Let y be a local minimum for J st. $W(y) = 1$
Then $(0,0)$ is a local min for $\mathcal{J} : \mathbb{R}^2 \to \mathbb{R}^1$.
Therefore, by LMR, we must have

$$\frac{\partial \mathcal{J}(0,0)}{\partial \varepsilon_1} + \lambda \frac{\partial \mathcal{W}(0,0)}{\partial \varepsilon_1} = 0 \qquad (1)$$

$$\frac{\partial \mathcal{J}(0,0)}{\partial \varepsilon_1} + \lambda \frac{\partial \mathcal{W}(0,0)}{\partial \varepsilon_2} = 0 \qquad (2)$$

$$\mathcal{W}(0,0) = 1 \qquad (3)$$

$$\frac{\partial \mathcal{J}}{\partial \varepsilon_1}(0,0) = \frac{\partial}{\partial \varepsilon_1}\mathcal{J}(\varepsilon_1,\varepsilon_2)\Big|_{(0,0)} = \delta J(y,h_1)$$

$$= \int_a^b \frac{d}{d\varepsilon_1}\Big[p(x)\,[y'+\varepsilon_1 h_1'(x)]^2 + q(x)\,[y+\varepsilon_1 h_1(x)]^2\Big]dx\Big|_{\varepsilon_1=0}$$

$$= \int_a^b 2p(x)[y'(x)]h_1'(x) + 2q(x)[y(x)]h_1(x)\,dx$$

Similarly,

$$\frac{\partial \mathcal{J}}{\partial \varepsilon_1}(0,0) = \delta J(y,h_2) = 2\int_a^b p(x)y'(x)h_2'(x) + q(x)y(x)h_2(x)\,dx$$

Finally,

$$\frac{\partial \mathcal{W}}{\partial \varepsilon_1}(0,0) = \delta W(y,h_1) = 2\int_a^b r(x)y(x)h_1(x)\,dx$$

and

$$\frac{\partial \mathcal{W}}{\partial \varepsilon_2}(0,0) = \delta W(y,h_2) = 2\int_a^b r(x)\,y(x)h_2(x)\,dx.$$

Then applying (1) & (2), we have

(1a) $$\int_a^b p(x)y'h_1' + q(x)yh_1 + \lambda r(x)yh_1\,dx = 0 \quad \forall (h_1,0)\in H$$

(2a) $$\int_a^b p(x)y'h_2' + q(x)yh_2 + \lambda r(x)yh_2\,dx = 0 \quad \forall (0,h_2)\in H.$$

Apply I.B.P to (1a) & use B.C.s satisfied by $h_1(x)$ yields

$$\int_a^b \left[(p(x)y')' + (q(x) + \lambda r(x))y \right] h_1(x)\, dx = 0 \quad \forall\, h_1 \in C^2[a,b]$$

$$h_1(a) = h_1(b) = 0$$

By F.L.C.V. this yields the SLP.

$$-(p(x)y')' + q(x)y + \lambda r(x)y = 0$$

OR

$$-(p(x)y')' + q(x)y = -\lambda r(x)y$$

$$y(a) = y(b) = 0.$$

Note: (2a) yields the same eqn, and the local minimizer must also satisfy

$$\int_a^b r(x)[y(x)]^2\, dx = 1$$

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Midterm Exam - #2

(2) $Lu = u_t - (k(x,t)u_x)_x + d(x,t)u$
is a Linear Operator

Let u, w be functions, and Let c be any constant.

(i) $L(u+w) = (u+w)_t - (k(x,t)(u+w)_x)_x + d(x,t)(u+w)$

$= u_t + w_t - [k(x,t)[u_x + w_x]]_x + d(x,t)u + c(x,t)w$

$= u_t + w_t - [k(x,t)u_x + k(x,t)w_x]_x + d(x,t)u + d(x,t)w$

$= u_t + w_t - (k(x,t)u_x)_x - (k(x,t)w_x)_x + d(x,t)u + d(x,t)w$

$= \left[u_t - (k(x,t)u_x)_x + d(x,t)u\right] + \left[w_t - (k(x,t)w_x)_x + d(x,t)w\right]$

$= L(u) + L(w)$

(ii) $L(cu) = (cu)_t - (k(x,t)(cu)_x)_x + d(x,t)(cu)$

$= cu_t - (c \cdot k(x,t)u_x)_x + c\,d(x,t)u$

$= cu_t - c\,(k(x,t)u_x)_x + c\,d(x,t)u$

$= c\left[u_t - (k(x,t)u_x)_x + d(x,t)u\right] = cL(u)$

So, L is a Linear Operator.

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Midterm Exam - # 3

(3) Determine whether the PDE is Linear or Nonlinear

$$u_{tx} + u^2 = \sin(x)$$

Let $L(u) = u_{tx} + u^2$, and NOTE that L is a Nonlinear operator. That is, if c is a constant and u is a function, then

$$L(cu) = (cu)_{tx} + (cu)^2$$

$$= c\,u_{tx} + c^2u^2$$

$$= c\,[u_{tx} + cu^2] \neq c\,L(u) \quad \text{for all } c \neq 1.$$

Since L is a Nonlinear Operator, then the P.D.E. is Nonlinear.

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Midterm Exam - #4

(4) Determine regions in the xt-plane where the following eqns are Hyperbolic, Parabolic OR Elliptic.

(a) $tu_{tt} + u_{xx} = 0$

$a(x,t) = t$, $b(x,t) = 0$, $c(x,t) = 1$
$b^2 - 4ac = 0 - 4(t) = -4t$

Eqn is: Hyperbolic for $t < 0$ and for all x
Parabolic for $t = 0$ " "
Elliptic for $t > 0$ " "

(b) $u_{tt} + (1+x^2)u_x - u_t = e^{tx}$

$a(x,t) = 1$, $b(x,t) = 0$, $c(x,t) = 0$
$b^2 - 4ac = 0$ for all x,t

Eqn is Parabolic for all x,t.

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Midterm Exam - #5

5. Find extremals of the functional

$$J(y) = \int_a^b y^2 + (y')^2 + 2ye^x \, dx$$

- We can use the ELDE here

$L_y = 2y + 2e^x \quad , \quad L_{y'} = 2y'$

$$2y + 2e^x - \frac{d}{dx}\left[2y'\right] = 0$$
$$y + e^x - y'' = 0$$
$$-y'' + y = -e^x$$

(ELDE) $$y'' - y = e^x$$

<u>Char. EQN:</u> $r^2 - 1 = 0$

$r = \pm 1$

$y_h(x) = c_1 e^{-x} + c_2 e^x$

<u>MUDC's</u>: $Y(x) = Axe^x$

$Y'(x) = Ae^x + Axe^x \quad = Ae^x(x+1)$

$Y''(x) = Ae^x + Ae^x + Axe^x = Ae^x(x+2)$

$$Y'' - Y = e^x$$
$$Ae^x(x+2) - Axe^x = e^x$$
$$Axe^x + 2Ae^x - Axe^x = e^x$$
$$2A = 1$$
$$\boxed{A = \frac{1}{2}}$$

$\Rightarrow$ Extremals of J have the form
$y(x) = c_1 e^{-x} + c_2 e^x + \frac{1}{2} x e^x$

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Midterm Exam - #6
Page 184, Number 1d.

(6) Find extremal for
$$J(y) = \int_0^1 yy' + (y'')^2 \, dx$$
with $y \in C^4[0,1]$, $y(0)=0$, $y'(0)=1$, $y(1)=2$, $y'(1)=4$

We can use ELDE. Note that $L = L(x,y,y',y'')$
$$L_y = y' \quad , \quad L_{y'} = y \quad , \quad L_{y''} = 2y''$$

$$y' - \frac{d}{dx}[y] + \frac{d^2}{dx^2}[2y''] = 0 \qquad \text{(See (3.28) in text)}$$

$$y' - y' + 2y'''' = 0$$

(ELDE)
$$\begin{aligned} y'''' &= 0 \\ y''' &= C_3 \\ y'' &= C_3 x + C_2 \\ y' &= \tfrac{1}{2}C_3 x^2 + C_2 x + C_1 \\ y(x) &= \tfrac{1}{6}C_3 x^3 + \tfrac{1}{2}C_2 x^2 + C_1 x + C_0 \end{aligned}$$

<u>Rename Constants</u>: $y(x) = Ax^3 + Bx^2 + Cx + D$
$y'(x) = 3Ax^2 + 2Bx + C$

$y(0)=0 \Rightarrow D=0$
$y'(0)=1 \Rightarrow C=1$

So, $y(x) = Ax^3 + Bx^2 + x$
Now, we use the other two Bdry Cond's $\Rightarrow$

$y(1) = 2 \Rightarrow A + B + 1 = 2$

$y'(1) = 4 \Rightarrow 3A + 2B + 1 = 4$

<u>EQNs are</u>:

$$\left.\begin{aligned} A + B &= 1 \\ 3A + 2B &= 3 \end{aligned}\right\} \Rightarrow \begin{aligned} 3A + 3B &= 3 \\ 3A + 2B &= 3 \end{aligned}$$

Subtracting yields $\boxed{B = 0}$

and this gives $\boxed{A = 1}$

Hence, $y(x) = x^3 + x$ is the extremal.

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Midterm Exam - #7
Page 346, Problem 5

⑦ Show that the general soln of

$$yu_x - xu_y = 0$$

has the general solution $u(x,y) = \Psi(x^2+y^2)$, where $\Psi : \mathbb{R} \to \mathbb{R}$ is any smooth fctn. Do so by introducing polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$.

$$u = u(x,y) = u(x(r,\theta), y(r,\theta)) = u(r\cos\theta, r\sin\theta)$$

① $u_r = u_x\cos\theta + u_y\sin\theta$ (By Chain Rule)

② $u_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$ (By Chain Rule)

$\Rightarrow$

$$u_x = \frac{u_r - u_y\sin\theta}{\cos\theta}$$

& subst. into ② gives

$$u_\theta = -r\sin\theta\left[\frac{u_r - u_y\sin\theta}{\cos\theta}\right] + u_y r\cos\theta$$

$\Rightarrow$

$$u_y = \frac{1}{r}\cos\theta\, u_\theta + \sin\theta\, u_r$$

and

$$u_x = \cos\theta\, u_r - \frac{1}{r}\sin\theta\, u_\theta$$

Pluging these into the pde, we have

$$y u_x - x u_y = 0$$

$$r\sin\theta\left[\cos\theta\, u_r - \frac{1}{r}\sin\theta\, u_\theta\right] - r\cos\theta\left[\frac{1}{r}\cos\theta\, u_\theta + \sin\theta\, u_r\right] = 0$$

$$r\sin\theta\cos\theta\, u_r - \sin^2\theta\, u_\theta - \cos^2\theta\, u_\theta - r\sin\theta\cos\theta\, u_r = 0$$

$$-u_\theta = 0$$

$$u_\theta = 0 \qquad \text{for all } r,\theta$$

Hence, u must be a function of the independent variable r only. In particular, any function that is cont'sly diff'ble and that is only a function of r will satisfy this pde. Hence, if Ψ is any cont'sly diff'ble function $\Psi: \mathbb{R}^1 \to \mathbb{R}^1$, then we can define

$$u(x,y) = \Psi(x^2+y^2),$$

and u will be a soln to the pde.

Alternatives to the previous discussion:

Observe that

$$u_\theta = -r\sin\theta\, u_x + r\cos\theta\, u_y$$
$$= -y u_x + x u_y$$

So, the pde reduces

$$y u_x - x u_y = 0 \ \forall x,y \implies -u_\theta = 0 \quad \forall (r,\theta)$$

And $u_\theta = 0 \ \forall (r,\theta)$ implies that u is only a function of r; that is, $u = \Psi(x^2+y^2)$ where Ψ an arbitrary function is a soln. to the pde.

Yet another alternative is to begin with

$$u = \Psi(x^2+y^2) \qquad \text{(with } \Psi \text{ sufficiently diff'ble)}$$

and compute

$$u_x = \Psi'(x^2+y^2)\cdot 2x \qquad \text{and} \qquad u_y = \Psi'(x^2+y^2)\, 2y$$

Plugging these into the eqn yield

$$\begin{aligned} yu_x - xu_y &= y(\Psi'\cdot 2x) - x(\Psi'\cdot 2y) \\ &= 2xy\,\Psi' - 2xy\,\Psi' \\ &= 0 \end{aligned}$$

$\forall\ x,y$ where Ψ is sufficiently smooth

Other techniques used include writing

$$r = (x^2+y^2)^{1/2} \quad \text{and} \quad \theta = \tan^{-1}(y/x)$$

and computing

$$u_x = u_r\cdot\frac{dr}{dx} + u_\theta\cdot\frac{d\theta}{dx} = u_r\cdot\tfrac{1}{2}(x^2+y^2)^{-1/2}(2x) + u_\theta\cdot\frac{1}{1+(y/x)^2}\cdot{}^{-y}/_{x^2}$$

$$= u_r\cdot\frac{x}{r} + u_\theta\cdot\frac{-y}{x^2+y^2}$$

$$u_x = u_r\cdot\frac{x}{r} - u_\theta\cdot\frac{y}{r^2} = u_r\cdot\frac{r\cos\theta}{r} - u_\theta\cdot\frac{r\sin\theta}{r^2} = u_r\cos\theta - \frac{u_\theta\sin\theta}{r}$$

$$\boxed{u_x = u_r\cos\theta - u_\theta\,\frac{\sin\theta}{r}}$$

→

Similarly, (4)

$$u_y = u_r \cdot \frac{dr}{dy} + u_\theta \cdot \frac{d\theta}{dy}$$

$$= u_r \, \tfrac{1}{2}(x^2+y^2)^{-1/2}(2y) + u_\theta \cdot \left(\frac{1}{1+(y/x)^2} \cdot \frac{1}{x}\right)$$

$$= u_r \cdot \frac{y}{r} + u_\theta \cdot \left(\frac{1}{x+(y^2/x)}\right)$$

$$= u_r \cdot \frac{r\sin\theta}{r} + u_\theta\left(\frac{x}{x^2+y^2}\right)$$

$$= u_r \cdot \sin\theta + u_\theta \frac{r\cos\theta}{r^2}$$

$$\boxed{u_y = u_r \cdot \sin\theta + u_\theta \frac{\cos\theta}{r}}$$

Substituting these into $yu_x - xu_y = 0$ leads to

$$-u_\theta = 0 \qquad \forall (r,\theta)$$
$$u_\theta = 0 \qquad \forall (r,\theta)$$

i.e. any function that is only a function of the radius variable and is sufficiently smooth is a soln of the pde..