FACTORIZATION HOMOLOGY OF ENRICHED ∞-CATEGORIES

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Abstract. For an arbitrary symmetric monoidal ∞-category $\mathcal{V}$, we define the factorization homology of $\mathcal{V}$-enriched ∞-categories over (possibly stratified) 1-manifolds and study its basic properties. In the case that $\mathcal{V}$ is *cartesian* symmetric monoidal, by considering the circle and its self-covering maps we obtain a notion of *unstable topological cyclic homology*, which we endow with an *unstable cyclotomic trace map*. As we show in [AMGRa], these induce their stable counterparts through linearization (in the sense of Goodwillie calculus).

Contents

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0. Introduction

0.1. **Categorified integration in quantum field theory.** A fundamental link between manifolds and higher algebraic structures is ***factorization homology***: in its most primitive form, this takes a framed n-manifold M and an $\mathbb{E}_n$-algebra A – in chain complexes, say – and returns a chain complex

$$\int_M A$$

obtained by "integrating" the algebra A over configurations of n-disks in M [Lura, AF15].[1] Besides being intimately related to the study of mapping spaces and proximate notions in manifold topology [Sal01, Kal01, Böd87, McD75, May72, Seg73, Lura, AF15], factorization homology is a close cousin of Beilinson–Drinfeld's algebro-geometric theory of *chiral homology* [BD04], which computes conformal blocks in conformal field theory. These ideas have since spawned much subsequent activity in mathematical physics – notably Costello–Gwilliam's work [CG17, CG] on perturbative quantum field theory, wherein they recover global observables from local ones via factorization homology.

In order to accommodate higher-codimensional defects, in [AFRa] Ayala–Francis–Rozenblyum generalized the algebraic input of factorization homology from $\mathbb{E}_n$-algebras to (∞, n)-*categories*:[2] for a framed n-manifold M and an (∞, n)-category $\mathcal{C}$, they defined the factorization homology

$$\int_M \mathcal{C}$$

of $\mathcal{C}$ over M as the space of labelings by $\mathcal{C}$ of *disk-stratifications* of M. This construction allows for field theories that are not necessarily determined by their point-local observables.

However, this construction is still one step removed from the production of TQFTs of physical interest, which are *linear* in nature: where the framework of [AFRa] yields a space, one would like to obtain a vector space or chain complex. In this paper, we provide a blueprint for the appropriate generalization: we construct the factorization homology of *enriched* $(\infty, 1)$-categories. We expect that our construction contains all the essential features of a full theory of factorization homology of enriched (∞, n)-categories. The key idea that drives our approach can be summarized as follows.

Slogan 0.1. ***Enriched*** *factorization homology arises from* ***categorified*** *factorization homology.*

We will explain Slogan 0.1 in §0.3.

0.2. **Hochschild homology, cyclic homology, and the cyclotomic trace.** In fact, our primary motivation for constructing enriched factorization homology comes from a different direction, namely topological Hochschild homology and its connection with algebraic K-theory.

Recall that the *Hochschild homology* of an associative ring A is the value

$$\mathsf{HH}(A) \in \mathbf{D}(\mathsf{Mod}_{\mathbb{Z}})$$

[1] One can weaken the condition that M be framed in exchange for requiring further structure on A [AF15].

[2] Just as an algebra determines a one-object category, an $\mathbb{E}_n$-algebra determines an (∞, n)-category with a unique k-morphism for all $k < n$.

of the left derived functor of the coinvariants functor

$$\begin{array}{ccc} \mathsf{BiMod}_{(A,A)} & \longrightarrow & \mathsf{Mod}_{\mathbb{Z}} \\ \Psi & & \Psi \\ M & \longmapsto & M/[A,M] \end{array}$$

at the (A,A)-bimodule A. This can be identified with the factorization homology

$$\mathsf{HH}(A) \simeq \int_{S^1} A$$

of A, considered as an $\mathbb{E}_1$-algebra in the ∞-category $\mathsf{Mod}_{H\mathbb{Z}}$ of $H\mathbb{Z}$-modules (i.e. chain complexes localized at the quasi-isomorphisms) [Lura]. In fact, this equivalence lifts to a direct identification of the cyclic bar construction as computing factorization homology over S^1 (see Remark 1.28).

From our perspective, a crucial advantage of the definition of Hochschild homology via factorization homology is that

factorization homology manifestly encodes the inherent symmetries of Hochschild homology.

For example, its natural action of the circle group $\mathbb{T}$ arises simply from the functoriality of factorization homology for automorphisms of framed manifolds. By contrast, this action only appears through the machinery of cyclic sets (or mixed complexes) when Hochschild homology is defined through simplicial (or homological) methods [Lod98]. This passage from handicrafted functoriality to manifest functoriality is analogous to the passage from cellular homology to singular homology.

This circle action on Hochschild homology is fundamental in the study of the ***algebraic K-theory*** of A. Namely, the latter admits a homotopy $\mathbb{T}$-invariant *Dennis trace* map to $\mathsf{HH}(A)$, and Goodwillie proved [Goo86] that the resulting *cyclic trace* map

$$\begin{tikzcd} \mathsf{K}(A) \arrow[rr] \arrow[dr, dashed] & & \mathsf{HH}(A) \\ & \mathsf{HC}^-(A) := \mathsf{HH}(A)^{\mathsf{h}\mathbb{T}} \arrow[ur] & \end{tikzcd}$$

to *negative cyclic homology* is "locally constant" after rationalization.

Seeing that his theorem could not hold integrally, Goodwillie conjectured the existence of a refinement of HH obtained by replacing the ground ring $\mathbb{Z}$ with the sphere spectrum $\mathbb{S}$. This was cleverly constructed by Bökstedt [Bök], despite an utter lack of sufficient foundations (namely a good model category of spectra): for an arbitrary associative (i.e. $\mathbb{E}_1$-)ring spectrum A, he defined its ***topological Hochschild homology*** spectrum

$$\mathsf{THH}(A) \in \mathcal{S}\mathsf{p} \ .$$

He moreover endowed this with a homotopy $\mathbb{T}$-action and a homotopy $\mathbb{T}$-invariant ("topological") Dennis trace map

$$\mathsf{K}(A) \longrightarrow \mathsf{THH}(A) \ ,$$

which factors the Dennis trace map to $\mathsf{HH}(A)$ in the case that A is an ordinary ring.

But the homotopy $\mathbb{T}$-fixedpoints of THH proved to be still insufficient to integrally approximate algebraic K-theory. In their celebrated paper [BHM93], Bökstedt–Hsiang–Madsen introduced a further refinement

$$\mathsf{TC}(A) := \mathsf{THH}(A)^{\mathsf{hCyc}}$$

of THH, which they called ***topological cyclic homology***: the $\mathbb{T}$-action on $\mathsf{THH}(A)$ admits an enhancement to a ***cyclotomic structure***, and $\mathsf{TC}(A)$ is by definition its homotopy invariants. Soon after, Dundas–Goodwillie–McCarthy proved that the resulting factorization

$$\begin{tikzcd} \mathsf{K}(A)a \arrow[rr] \arrow[dr, dashed] & & \mathsf{HH}(A) \\ & \mathsf{TC}(A) := \mathsf{THH}(A)^{\mathsf{hCyc}} \arrow[ur] & \end{tikzcd}$$

of the Dennis trace, called the ***cyclotomic trace***, is indeed integrally "locally constant" [Goo86, McC97, Dun97, DM94, DGM13]. Following these breakthroughs, the cyclotomic trace became a central tool: the vast majority of known computations of algebraic K-theory result from this infinitesimal behavior [HM97b, HM97a, HM03, HM04, AGH09, AGHL14, KR97, Rog03, BM].

However, TC and the cyclotomic trace remain mysterious from a conceptual standpoint, and in particular from an algebro-geometric point of view. This is largely due to the fact that cyclotomic spectra are defined through *genuine equivariant homotopy theory*, which likewise remains mysterious from an algebro-geometric point of view.

This paper is part of a trilogy, whose overarching purpose is to provide a precise conceptual description of the cyclotomic trace at the level of derived algebraic geometry: this is explained in [AMGRa, §0]. The trilogy begins with the paper [AMGRb], in which we reidentify the ∞-category of cyclotomic spectra in terms of *naive* equivariant homotopy theory.

In order to explain the role that this paper plays within the trilogy, let us first note that in the context of these various theories (K, THH, and TC), algebro-geometric objects such as schemes and stacks are incarnated through their stable ∞-categories of vector bundles (i.e. perfect complexes).[3] In fact, as we will see, it is possible to define THH – that is, factorization homology over the circle – of an arbitrary spectrally-enriched (e.g. stable) ∞-category, or more generally of a $\mathcal{V}$-enriched ∞-category $\mathcal{C}$ for an arbitrary (cocomplete) symmetric monoidal ∞-category $(\mathcal{V}, \boxtimes)$; let us simply write

$$\mathsf{THH}_{\mathcal{V}}(\mathcal{C}) := \left(\int_{S^1} \mathcal{C} \right) \in \mathcal{V}$$

for this latter construction, which we will describe in more detail in §0.3.

Now, the key output of this paper (vis-à-vis the trilogy) is a result of the fact that

> *if* $(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$ *is a* cartesian *symmetric monoidal* ∞*-category, then* $\mathcal{V}$*-enriched factorization homology is contravariantly functorial for fiber bundles.*

Specifically, taking $\mathcal{V} = \mathcal{S}$ to be the ∞-category of spaces, considering the collection of all fiber bundles among (framed) circles and the 0-disk $\mathbb{D}^0$ yields the following result.

Theorem A (The unstable cyclotomic trace (assembled in §5))**.** *Let* $\mathcal{C}$ *be an* ∞*-category.*

(1) The factorization homology space

$$\mathsf{THH}_{\mathcal{S}}(\mathcal{C}) := \int_{S^1} \mathcal{C}$$

[3] Indeed, these theories are all Morita invariant – K-theory by definition, and the rest because THH is so as a functor to cyclotomic spectra (see [AMGRa, Remarks 0.11, 0.12, and 0.14]) – so that we also lose nothing by passing from associative ring spectra to their stable ∞-categories of perfect modules.

admits an ***unstable cyclotomic structure****: a left action of the monoid* $\mathbb{W} := \mathbb{T} \rtimes \mathbb{N}^\times$.

(2) The resulting unstable topological cyclic homology space admits an ***unstable cyclotomic trace*** *map*

$$\mathcal{C}^{\simeq} \simeq \left(\int_{\mathbb{D}^0} \mathcal{C}\right) \longrightarrow \left(\int_{S^1} \mathcal{C}\right)^{\mathsf{h}\mathbb{W}} =: \mathsf{THH}_{\mathcal{S}}(\mathcal{C})^{\mathsf{h}\mathbb{W}} =: \mathsf{TC}_{\mathcal{S}}(\mathcal{C})$$

from the underlying ∞*-groupoid of* $\mathcal{C}$*.*

In turn, this is the basis for the following result.

Theorem B (The cyclotomic trace (proved in [AMGRa]))**.** *Linearization (in the sense of Goodwillie calculus) of the unstable data of Theorem A induces their stable analogs:*

(1) the ***cyclotomic structure*** *on*

$$\mathsf{THH} := \mathsf{THH}_{\mathcal{S}\mathsf{p}} ,$$

and

(2) the ***cyclotomic trace*** *map*

$$\mathsf{K} \longrightarrow \mathsf{TC} := \mathsf{THH}^{\mathsf{hCyc}}$$

from algebraic K-theory to topological cyclic homology (the homotopy invariants of the cyclotomic structure on THH*).* □

Together, Theorems A and B provide a novel conceptual framework for TC and the cyclotomic trace: these both ultimately arise from the *linearization* of more primitive analogs, whose construction is the primary purpose of this paper.

0.3. **Factorization homology of enriched ∞-categories.** We now outline the primary features of our construction of enriched factorization homology, with the goal of explaining Slogan 0.1.

We base our work in the ∞-category of *compact vari-framed stratified 1-manifolds* of [AFRa]; for simplicity, we just refer to this as the ∞-category of ***stratified 1-manifolds*** and denote it by $\boldsymbol{\mathcal{M}}$. An object of $\boldsymbol{\mathcal{M}}$ is a finite disjoint union of framed circles and finite directed connected graphs; in particular, note that the 0-disk $\mathbb{D}^0$ and the framed circle S^1 both define objects of $\boldsymbol{\mathcal{M}}$. We will not describe the morphisms in $\boldsymbol{\mathcal{M}}$ here, but let us note that they include the opposites of (vari-framed) constructible bundles. In particular, there is a unique morphism

$$\mathbb{D}^0 \longrightarrow S^1$$

in $\boldsymbol{\mathcal{M}}$. This morphism induces the unstable Dennis trace

$$\mathcal{C}^{\simeq} \longrightarrow \mathsf{THH}_{\mathcal{S}}(\mathcal{C}) ,$$

and the space of endomorphisms of $S^1 \in \boldsymbol{\mathcal{M}}$ assembles into the monoid $\mathbb{W}$ appearing in Theorem A.

Now, any object $M \in \boldsymbol{\mathcal{M}}$ admits a category $\boldsymbol{\mathcal{D}}(M)$ of ***disk-refinements*** of M.[4] A disk-refinement of M consists of a configuration of points in the 1-dimensional strata of M, such that no circles remain unstratified (i.e. it is a stratification of M by points and open intervals). Note that a

[4] A priori $\boldsymbol{\mathcal{D}}(M)$ is an ∞-category, but it is not hard to see that its hom-spaces are always discrete, i.e. that it is actually an ordinary category (see Observation 1.20).

disk-refinement inherits the structure of a finite directed graph, so that it is itself an object of $\mathbf{\mathcal{M}}$, namely one which is *disk-stratified*: it contains no unstratified circles. The morphisms in $\mathfrak{D}(M)$ are generated by

- disappearances of points,
- anticollisions of points, and
- isotopies of configurations of points.

We note here that when M is itself disk-stratified, then it defines a terminal object in $\mathfrak{D}(M)$.

We will define the factorization homology over M of a $\mathcal{V}$-enriched ∞-category $\mathcal{C}$ as the colimit

$$\int_M \mathcal{C} := \underset{R \in \mathfrak{D}(M)}{\mathsf{colim}} \left(\underset{R^{(0)} \xrightarrow{\lambda} \mathcal{C}^{\simeq}}{\mathsf{colim}} \left(\boxtimes_{e \in R^{(1)}} \underline{\mathsf{hom}}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e))) \right) \right) . \tag{1}$$

In words, this will be the colimit

- indexed over
  - disk-refinements R of M and
  - labelings λ of the set of vertices $R^{(0)}$ of R by objects of $\mathcal{C}$
- of the monoidal product in $\mathcal{V}$
  - indexed over the set of edges $e \in R^{(1)}$ of R
  - of the hom-object of $\mathcal{C}$
    * from the object $\lambda(s(e)) \in \mathcal{C}$ labeling the source of e
    * to the object $\lambda(t(e)) \in \mathcal{C}$ labeling the target of e.

The structure maps as $R \in \mathfrak{D}(M)$ varies will be generated by

- composition in $\mathcal{C}$ (for disappearances of points),
- insertion of identity maps in $\mathcal{C}$ (for anticollisions of points), and
- equivalences (for isotopies of configurations of points).

This definition will simplify considerably when M is itself disk-stratified: the colimit can then be identified as the value

$$\int_M \mathcal{C} \simeq \underset{M^{(0)} \xrightarrow{\lambda} \mathcal{C}^{\simeq}}{\mathsf{colim}} \left(\boxtimes_{e \in M^{(1)}} \underline{\mathsf{hom}}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e))) \right)$$

of the inner expression in formula (1) at the terminal object $R = M \in \mathfrak{D}(M)$.

In order to make this definition rigorously, let us recall the formalism of ***$\mathcal{V}$-enriched ∞-categories*** of [GH15]. We first recall two definitions.

- A monoidal ∞-category $(\mathcal{V}, \boxtimes)$ is specified by its *monoidal deloop*

$$\mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathfrak{B}\mathcal{V}} \mathsf{Cat} ,$$

i.e. its bar construction. In fact, this is a *category object* in Cat, i.e. it satisfies the Segal condition (though not necessarily the completeness condition).

- For a space $X \in \mathcal{S}$, the *codiscrete category object* on X is the Segal space

$$\mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{cd}(X)} \mathcal{S}$$

whose space of objects in X and whose hom-spaces are all contractible: in simplicial degree n, this is the space $X^{\times(n+1)}$.

Now, a $\mathcal{V}$-enriched ∞-category $\mathcal{C}$ is specified by the following two pieces of data:

- its ***underlying ∞-groupoid***, an object $\mathcal{C}^{\simeq} \in \mathcal{S}$, and
- its ***enriched hom functor***, a right-lax functor

$$\mathsf{cd}(\mathcal{C}^{\simeq}) \xrightarrow{\mathsf{hom}_{\mathcal{C}}} \mathfrak{B}\mathcal{V} \tag{2}$$

of category objects in Cat.

Let us unwind the functor (2). In simplicial degree n, it is given by

$$\begin{array}{ccc} (\mathcal{C}^{\simeq})^{\times(n+1)} & \longrightarrow & \mathcal{V}^{\times n} \\ \Psi & & \Psi \\ (C_0, \ldots, C_n) & \longmapsto & (\underline{\mathsf{hom}}_{\mathcal{C}}(C_0, C_1), \ldots, \underline{\mathsf{hom}}_{\mathcal{C}}(C_{n-1}, C_n)) \end{array},$$

and the right-laxness determines the categorical structure maps; for instance, restricting to the indicated morphisms in $\mathbf{\Delta}^{\mathsf{op}}$, this specifies the diagram

$$\begin{tikzcd} {[2]^\circ} \arrow[d, "\delta_1"'] & (\mathcal{C}^{\simeq})^{\times 3} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \mathcal{V}^{\times 2} \arrow[d] \\ {[1]^\circ} & (\mathcal{C}^{\simeq})^{\times 2} \arrow[r] & \mathcal{V} \\ {[0]^\circ} \arrow[u, "\sigma_0"] & \mathcal{C}^{\simeq} \arrow[r] \arrow[u] \arrow[ur, phantom, "\Nwarrow"] & \mathsf{pt} \arrow[u] \\ \mathbf{\Delta}^{\mathsf{op}} & \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r, "\mathsf{hom}_{\mathcal{C}}"] & \mathfrak{B}\mathcal{V} \end{tikzcd},$$

in which the upper square selects the composition maps

$$\underline{\mathsf{hom}}_{\mathcal{C}}(C_0, C_1) \boxtimes \underline{\mathsf{hom}}_{\mathcal{C}}(C_1, C_2) \longrightarrow \underline{\mathsf{hom}}_{\mathcal{C}}(C_0, C_2) \tag{3}$$

while the lower square selects the unit maps

$$\mathbb{1}_{\mathcal{V}} \longrightarrow \underline{\mathsf{hom}}_{\mathcal{C}}(C_0, C_0) \ . \tag{4}$$

And now appears ***categorified factorization homology***. Namely, we can take the factorization homology *of category objects* in Cat: this follows exactly the same prescription given above, only there is now an ∞-*category* (instead of just an ∞-groupoid) of objects with which to label the vertices of the disk-refinement. In particular, over a disk-stratified 1-manifold $R \in \mathbf{\mathcal{M}}$, this is easy to describe since $R \in \mathbf{\mathfrak{D}}(M)$ is a terminal object, and it is made easier still because Cat is *cartesian* symmetric monoidal. Namely, the ∞-category

$$\int_R \mathcal{Y} \in \mathsf{Cat}$$

is just a limit of copies of $\mathcal{Y}_{|[0]^\circ}$ and $\mathcal{Y}_{|[1]^\circ}$: one copy of $\mathcal{Y}_{|[0]^\circ}$ for each vertex, one copy of $\mathcal{Y}_{|[1]^\circ}$ for each edge, and with structure maps $s = \delta_1$ and $t = \delta_0$ determined by the incidence data of the directed graph R.

Now, for an arbitrary stratified 1-manifold $M \in \mathbf{\mathfrak{M}}$ and a category object $\mathcal{Y} \in \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat})$, restricting to the category $\mathbf{\mathfrak{D}}(M)$ defines a composite functor

$$\mathbf{\mathfrak{D}}(M) \longrightarrow \mathbf{\mathfrak{M}} \xrightarrow{\int_{(-)} \mathcal{Y}} \mathsf{Cat} ,$$

or equivalently a cocartesian fibration

$$\begin{tikzcd} \int_{|\mathbf{\mathfrak{D}}(M)} \mathcal{Y} \arrow[d] \\ \mathbf{\mathfrak{D}}(M) \end{tikzcd} . \tag{5}$$

This construction is suitably functorial for right-lax functors of category objects, so that a $\mathcal{V}$-enriched ∞-category $\mathcal{C}$ – that is, its enriched hom functor (2) – determines a functor

$$\int_{|\mathbf{\mathfrak{D}}(M)} \mathsf{cd}(\mathcal{C}^{\simeq}) \xrightarrow{\int_{|\mathbf{\mathfrak{D}}(M)} \underline{\mathsf{hom}}_{\mathcal{C}}} \int_{|\mathbf{\mathfrak{D}}(M)} \mathfrak{B}\mathcal{V} . \tag{6}$$

Over a disk-refinement $R \in \mathbf{\mathfrak{D}}(M)$, an object of its source is simply given by a labeling of its set of vertices $R^{(0)}$ by objects of the ∞-groupoid

$$\mathcal{C}^{\simeq} =: \mathsf{cd}(\mathcal{C}^{\simeq})_{|[0]^\circ}$$

(since the morphism-data in $\mathsf{cd}(\mathcal{C}^{\simeq})$ are canonically determined), while an object of its target is simply given by a labeling of its set of edges $R^{(1)}$ by objects of the ∞-category

$$\mathcal{V} =: \mathfrak{B}\mathcal{V}_{|[1]^\circ}$$

(since the object-data in $\mathfrak{B}\mathcal{V}$ are canonically determined). In other words, over this object of $\mathbf{\mathfrak{D}}(M)$ the map (6) restricts to a map

$$(\mathcal{C}^{\simeq})^{\times R^{(0)}} \longrightarrow \mathcal{V}^{\times R^{(1)}} .$$

Of course, this is given by nothing other than the enriched hom functor of $\mathcal{C}$.

Finally, when $\mathcal{V}$ is additionally *symmetric* monoidal, there exists a "tensor everything together" functor

$$\int_{|\mathbf{\mathfrak{D}}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} .$$

Then, we define the ***enriched factorization homology*** of $\mathcal{C}$ over M by the formula

$$\int_M \mathcal{C} := \mathsf{colim}\left(\int_{|\mathbf{\mathfrak{D}}(M)} \mathsf{cd}(\mathcal{C}^{\simeq}) \xrightarrow{\int_{|\mathbf{\mathfrak{D}}(M)} \underline{\mathsf{hom}}_{\mathcal{C}}} \int_{|\mathbf{\mathfrak{D}}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \right) .$$

Note that this indeed precisely rigorizes the heuristic definition of formula (1). First of all, this colimit is indexed over the data of a disk-refinement $R \in \mathbf{\mathfrak{D}}(M)$ and an object of

$$\int_R \mathsf{cd}(\mathcal{C}^{\simeq}) ,$$

i.e. a labeling λ of its vertices by points of $\mathcal{C}^{\simeq}$, and its value on such an object is precisely the monoidal product

$$\bigboxtimes_{e \in R^{(1)}} \underline{\mathsf{hom}}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e)))$$

in $\mathcal{V}$. Moreover, its structure maps are given either by disappearances of points, which are taken to composition maps (as map (3)), or by anticollisions of points, which are taken to unit maps (as map (4)); the isotopies of configurations of points are encoded implicitly as equivalences in $\mathfrak{D}(M)$.

This definition of enriched factorization homology has the key feature of being optimized for the isolation of the various specific properties of the enriching ∞-category $\mathcal{V}$: all natural operations thereon have only to do with the factorization homology of $\mathfrak{B}\mathcal{V}$, while the factorization homology of $\mathsf{cd}(\mathcal{C}^{\simeq})$ simply comes along for the ride.

0.4. **Miscellaneous remarks.**

Remark 0.2. Over the course of this work, we clarify the manifold-theoretic origins of a number of categories of lasting interest: the paracyclic category $\mathbf{\Delta}^{\mathsf{op}}_{\circlearrowleft}$, the cyclic category $\mathbf{\Lambda}^{\mathsf{op}}$, and the epicyclic category $\widetilde{\mathbf{\Lambda}}^{\mathsf{op}}$ (see Observation 1.26). Though these admit combinatorial descriptions (through which they were originally defined), we find them easiest to manipulate when incarnated through manifolds: then, the geometry keeps track of the combinatorics.

Remark 0.3. In [AFRa], it is shown that (∞, n)-categories are actually *characterized* by their factorization homology. We do not expect an analogous result to hold for enriched factorization homology: this encodes (∞, n)-categories through Segal-type limit conditions, which are only valid in the case that the enrichment is *cartesian* symmetric monoidal.

Remark 0.4. There are multiple notions of "enriched (∞, n)-categories"; among these, we expect that TQFTs should be most directly connected to those that are "only enriched in dimension n". This is the notion we employ here, in the case that $n = 1$.

Remark 0.5. A key technical result in this paper (Proposition 4.1) furnishes an extension of the ∞-operad $\mathfrak{B}^{\Sigma}\mathcal{V}^{\times} \downarrow \mathsf{Fin}_*$ corresponding to a cartesian symmetric monoidal ∞-category $(\mathcal{V}, \times)$: we produce a cocartesian fibration $\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} \downarrow \mathsf{Corr}(\mathsf{Fin})$ over the ∞-category of correspondences of finite sets, which sits in a canonical pullback square

$$\begin{tikzcd} \mathfrak{B}^{\Sigma}\mathcal{V}^{\times} \arrow[r] \arrow[d] & \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} \arrow[d] \\ \mathsf{Fin}_* \arrow[r] & \mathsf{Corr}(\mathsf{Fin}) \end{tikzcd} .$$

This extension encodes diagonal maps in $\mathcal{V}$, as well as their interaction with products. We find this result to be of independent interest, and we expect that it will find use elsewhere.

0.5. **Outline.** This paper is organized as follows.

- We begin in §1 with our foundations regarding the ∞-category of stratified 1-manifolds, as introduced in [AFRa].
- Then, in §2 we review the theory of flagged enriched ∞-categories (as defined in [GH15]) and study their categorified factorization homology.

- Next, in §3 we define enriched factorization homology over an arbitrary stratified 1-manifold, and endow it with its functoriality for automorphisms of stratified 1-manifolds.
- Thereafter, in §4 we specialize to the case that our enriching ∞-category is *cartesian* symmetric monoidal, and deduce vast additional functoriality of enriched factorization homology via the resulting diagonal maps.
- Finally, in §5 we lay out the theory of the unstable cyclotomic trace (and in particular establish Theorem A) in parallel with its stable analog, and study a few examples.
- There are then two appendices containing some technical results that we need; in §A we establish some key foundational results regarding categorified factorization homology, and in §B we prove a result about factorization systems.

0.6. **Notation and conventions.**

(1) We work within the context of ∞-categories, taking [Lur09] and [Lura] as our standard references. We work model-independently (for instance, we make no reference to the simplices of a quasicategory), and we omit all technical uses of the word "essentially" (for instance, we shorten the term "essentially surjective" to "surjective"). We also make some light use of the theory of $(\infty, 2)$-categories, which is developed in the appendix of [GR17].

(2) We use the following decorations for our functors.

- An arrow $\hookrightarrow$ denotes a monomorphism, i.e. the inclusion of a subcategory: a functor which is fully faithful on equivalences and induces inclusions of path components (i.e. monomorphisms) on all hom-spaces.
- An arrow $\xhookrightarrow{\mathsf{f.f.}}$ denotes a fully faithful functor.
- An arrow $\twoheadrightarrow$ denotes a surjection.
- An arrow $\downarrow$ denotes a functor considered as an object in the overcategory of its target (which will often be some sort of fibration).

More generally, we use the notation $\downarrow$ to denote a morphism in any ∞-category that we consider as defining an object in an overcategory.

(3) Given some datum in an ∞-category (such as an object or morphism), for clarity we may use the superscript $(-)^\circ$ to denote the corresponding datum in the opposite ∞-category.

(4) Given a functor F, we write F^* for pullback along it, and $F_! \dashv F^* \dashv F_*$ for left and right Kan extensions along it.

(5) We write Cat for the ∞-category of ∞-categories, $\mathcal{S}$ for the ∞-category of spaces, and $\mathcal{S}\mathsf{p}$ for the ∞-category of spectra. These are related by the various adjoint functors

$$\begin{tikzcd} \mathsf{Cat} \arrow[r, bend left, "(-)^{\mathsf{gpd}}"] \arrow[r, bend right, "(-)^{\simeq}"'] & \mathcal{S} \arrow[l, hook'] \arrow[r, shift left, "\Sigma^\infty_+"] & \mathcal{S}\mathsf{p} \arrow[l, shift left, "\Omega^\infty"] \end{tikzcd} .$$

(6) Given an ∞-category $\mathcal{C}$, we write

$$\mathcal{C}^{\mathsf{name}} \hookrightarrow\!\!\!\!\rightarrow \mathcal{C}$$

for a surjective monomorphism from a subcategory whose morphisms are precisely those of type "name". Moreover, given a subcategory $\mathcal{C}_0 \subset \mathcal{C}$, we write

$$\mathsf{Ar}^{\mathcal{C}_0}(\mathcal{C}) \xhookrightarrow{\mathsf{f.f.}} \mathsf{Ar}(\mathcal{C})$$

for the full subcategory on those objects $([1] \to \mathcal{C}) \in \mathsf{Ar}(\mathcal{C})$ which admit a factorization

$$\begin{tikzcd} {[1]} \arrow[r] \arrow[dr, dashed] & \mathcal{C} \\ & \mathcal{C}_0 \arrow[u, hook] \end{tikzcd} .$$

Combining these two conventions, we simply write

$$\mathsf{Ar}^{\mathsf{name}}(\mathcal{C}) := \mathsf{Ar}^{\mathcal{C}^{\mathsf{name}}}(\mathcal{C}) \ ,$$

for convenience.

(7) For a base ∞-category $\mathcal{B}$, we define the commutative diagrams of monomorphisms among ∞-categories

$$\begin{tikzcd} \mathsf{coCart}_{\mathcal{B}} \arrow[r, hook, two heads] \arrow[d, hook, "\mathsf{f.f.}"'] & \mathsf{Cat}_{\mathsf{cocart}/\mathcal{B}} \arrow[d, hook, "\mathsf{f.f.}"] \\ \mathsf{loc.coCart}_{\mathcal{B}} \arrow[r, hook, two heads] & \mathsf{Cat}_{\mathsf{loc.cocart}/\mathcal{B}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{Cart}_{\mathcal{B}} \arrow[r, hook, two heads] \arrow[d, hook, "\mathsf{f.f.}"'] & \mathsf{Cat}_{\mathsf{cart}/\mathcal{B}} \arrow[d, hook, "\mathsf{f.f.}"] \\ \mathsf{loc.Cart}_{\mathcal{B}} \arrow[r, hook, two heads] & \mathsf{Cat}_{\mathsf{loc.cart}/\mathcal{B}} \end{tikzcd}$$

as follows:

- objects in the upper rows are co/cartesian fibrations over $\mathcal{B}$,
- objects in the lower rows are locally co/cartesian fibrations over $\mathcal{B}$,
- morphisms in the left columns are functors over $\mathcal{B}$ which preserve co/cartesian morphisms, and
- morphisms in the right columns are arbitrary functors over $\mathcal{B}$.

We also write

$$\begin{tikzcd} \mathsf{coCart}_{\mathcal{B}} \simeq \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}) \simeq \mathsf{Cart}_{\mathcal{B}^{\mathsf{op}}} \arrow[from=1-1, to=1-1, bend left, "(-)^{\vee}_{\mathsf{cocart}}"] \end{tikzcd}$$

$$\mathsf{coCart}_{\mathcal{B}} \underset{(-)^{\vee}_{\mathsf{cart}}}{\overset{(-)^{\vee}_{\mathsf{cocart}}}{\rightleftarrows}} \mathsf{Cart}_{\mathcal{B}^{\mathsf{op}}}, \qquad \mathsf{coCart}_{\mathcal{B}} \simeq \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}) \simeq \mathsf{Cart}_{\mathcal{B}^{\mathsf{op}}}$$

for the composite equivalences, and refer to them as the *cocartesian dual* and *cartesian dual* functors (named for their respective sources).

(8) We make use of the theory of *exponentiable fibrations* of [AF] (see also [AFRa, §5]), an ∞-categorical analog of the "Conduché fibrations" of [Gir64, Con72]: these are the objects $(\mathcal{E} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$ satisfying the condition that there exists a right adjoint

$$\begin{tikzcd} \mathsf{Cat}_{/\mathcal{B}} \arrow[r, shift left, "- \underset{\mathcal{B}}{\times} \mathcal{E}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_{/\mathcal{B}} \arrow[l, shift left, dashed, "\mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E},-)"] \end{tikzcd}$$

to the pullback; by the adjoint functor theorem, these can be equivalently characterized as those objects for which the proposed left adjoint preserves colimits. We refer to this right adjoint as the ***relative functor** ∞**-category*** construction; it is analogous to the internal hom of presheaves. Thus, for any target object $(\mathcal{F} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$ and any test object $(\mathcal{K} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$, a lift

$$\begin{tikzcd} & \mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F}) \arrow[d] \\ \mathcal{K} \arrow[ur, dashed] \arrow[r] & \mathcal{B} \end{tikzcd}$$

is equivalent data to a functor

$$\begin{tikzcd} \mathcal{E}_{|\mathcal{K}} \arrow[rr, dashed] \arrow[dr] & & \mathcal{F}_{|\mathcal{K}} \arrow[dl] \\ & \mathcal{B} & \end{tikzcd}$$

between pullbacks over $\mathcal{K}$. In particular, for

$$\mathcal{F} := \mathcal{G} \times \mathcal{B} \longrightarrow \mathcal{B}$$

a projection from a product we simply write

$$\mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E}, \underline{\mathcal{G}}) := \mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F})$$

(thinking of $\mathcal{F}$ as the "constant presheaf" at $\mathcal{G}$). This special case – which in fact will be the only case that we ever use – participates in the composite adjunction

$$\begin{tikzcd} \mathsf{Cat}_{/\mathcal{B}} \arrow[r, shift left, "- \underset{\mathcal{B}}{\times} \mathcal{E}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, shift left, dashed, "\mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E},\underline{(-)})"] \end{tikzcd} .$$

We write

$$\mathsf{EFib}_{\mathcal{B}} \xhookrightarrow{\mathsf{f.f.}} \mathsf{Cat}_{/\mathcal{B}}$$

for the full subcategory on the exponentiable fibrations, and we note once and for all that cocartesian fibrations and cartesian fibrations are exponentiable.

(9) In order not to overburden our language, at times we are slightly cavalier about exactly which limits and colimits we require to exist the ∞-category $\mathcal{V}$. This should not cause concern, however, as ultimately we will be specializing to the case that $\mathcal{V}$ is one of the ∞-categories $\mathcal{S}$ or $\mathcal{S}\mathsf{p}$.

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1. Stratified 1-manifolds

In this section, we establish our foundations regarding the ∞-category $\mathcal{M}$ of stratified 1-manifolds. This is the ∞-category $\mathsf{c}\mathcal{M}\mathsf{fd}_1^{\mathsf{vfr}}$ of *compact vari-framed stratified 1-manifolds*, originally introduced as as [AFRa, Definition 2.48]. We begin in §1.1 with the basic definitions. Then, in §1.2 we study categories of disk-refinements. Finally, in §1.3 we realize the simplicial indexing category $\mathbf{\Delta}^{\mathsf{op}}$ as a full subcategory of $\mathcal{M}$; this is used in the definition of categorified factorization homology in §2.

1.1. Stratified 1-manifolds.

Definition 1.1. The ∞-category $\mathcal{M}$ of (***compact***) (***vari-framed***) ***stratified 1-manifolds*** is defined as follows.[5] An object of $\mathcal{M}$ is a finite disjoint union of framed circles and finite directed connected graphs. A morphism in $\mathcal{M}$ from M_0 to M_1 is given by a commutative diagram

$$\begin{tikzcd} M_0 \arrow[r, hook] \arrow[d] & M \arrow[d] & M_1 \arrow[l, hook'] \arrow[d] \\ \Delta^{\{0\}} \arrow[r, hook] & \Delta^1 & \Delta^{\{1\}} \arrow[l, hook'] \end{tikzcd} \tag{7}$$

of stratified spaces, in which

- the two squares are pullbacks,
- $\Delta^1 := (\Delta^{\{0\}} \subset \Delta^1)$ denotes the standardly stratified 1-simplex, and
- the middle vertical map is a proper constructible bundle,

such that the middle vertical map is equipped with a fiberwise vari-framing that is compatible with the vari-framings of M_0 and M_1.[6]

More generally, a string of n composable morphisms is determined by a fiberwise vari-framed proper constructible bundle over the standardly stratified n-simplex Δ^n (whose fibers over the vertices $\Delta^{\{i\}} \subset \Delta^n$ all lie in $\mathcal{M}$).

Remark 1.2. It is nontrivial that the prescription of Definition 1.1 indeed defines an ∞-category (i.e. a complete Segal space). In Remark 1.9, we explain (among other things) how this follows from existing work.

Notation 1.3. We denote certain distinguished objects of $\mathcal{M}$ as follows:

- $\mathbb{D}^0$ denotes the 0-disk (i.e. the directed graph with one vertex and no edges),
- $\mathbb{D}^1$ denotes the closed 1-disk (i.e. the connected directed graph with two vertices and one edge),
- S^1_* denotes the pointed circle (i.e. the directed graph with one vertex and one edge), and

[5] We will generally just use the adjective "stratified", only referring to vari-framings when we specifically mean to discuss them. But the adjectives "compact" and "vari-framed" should be understood as always being implicit in the terminology.

[6] As a topological space, Δ^1 is isomorphic to the closed interval $[0, 1] \subset \mathbb{R}$. Through this isomorphism, its stratification to the two-element poset $[1] = \{0 < 1\}$ then becomes identified with the continuous function $f : [0, 1] \to [1]$ with $f^{-1}(0) = \{0\}$ and $f^{-1}(1) = (0, 1]$.

- S^1 denotes the framed circle.

Notation 1.4. For any stratified 1-manifold $M \in \mathcal{M}$, we write $M^{(0)}$ and $M^{(1)}$ for its 0- and 1-dimensional strata, which we'll often identify with their underlying spaces.

There is an evident and more pedestrian notion of morphism between stratified 1-manifolds, namely a continuous function of stratified topological spaces which is smooth on 1-dimensional strata.[7] Certain of these give rise to morphisms in $\mathcal{M}$, as follows.

Definition 1.5. Given a proper constructible bundle $\varphi : M_1 \to M_0$ between stratified 1-manifolds that respects framings on 1-dimensional strata,[8] its *reversed cylinder* is the stratified space

$$\mathsf{Cylr}(M_0 \longleftarrow M_1) := M_0 \coprod_{M_1 \times \Delta^{\{0\}}} M_1 \times \Delta^1 .$$

This comes equipped with a map to Δ^1, through which it determines a morphism in $\mathcal{M}$ from M_0 to M_1, called a ***closed-creation morphism***. This is called a ***closed morphism*** if φ is injective, and a ***creation morphism*** if φ is surjective.

Definition 1.6. A *refinement* of stratified 1-manifolds is a continuous map $M_0 \to M_1$ which is a homeomorphism on underlying topological spaces, restricts to each stratum of M_0 as an embedding, and respects framings on 1-dimensional strata. In this case, its *open cylinder* is the stratified space

$$\mathsf{Cylo}(M_0 \longrightarrow M_1) := M_0 \times \Delta^1 \coprod_{M_0 \times \left(\Delta^1 \backslash \Delta^{\{0\}}\right)} M_1 \times \left(\Delta^1 \backslash \Delta^{\{0\}}\right) .$$

This comes equipped with a map to Δ^1, through which it determines a ***refinement morphism*** in $\mathcal{M}$ from M_0 to M_1.

Definition 1.7. A morphism (7) in $\mathcal{M}$ is called ***active*** if for every point of M_0, there exists an extension (in the category of stratified spaces)

$$\begin{tikzcd} M_0 \arrow[r, hook] \arrow[d] & M \arrow[d] \\ \Delta^{\{0\}} \arrow[r, hook] \arrow[u, bend left] & \Delta^1 \arrow[u, bend left, dashed] \end{tikzcd}$$

of the section over $\Delta^{\{0\}}$ which it defines.

Notation 1.8. We use the symbols cls, act, cr, ref, and $\mathsf{cls.cr}$ to respectively refer to the subcategories of $\mathcal{M}$ consisting of closed, active, creation, refinement, and closed-creation morphisms.

Remark 1.9. The precise relationship between our ∞-category $\mathcal{M}$ and these more pedestrian notions of morphisms (such as proper constructible bundles and refinement morphisms) is as follows.

First of all, there is a category Strat of [AFT17], whose objects are stratified spaces and whose morphisms are continuous functions of stratified topological spaces satisfying a certain regularity

[7] Following the terminology of [AFT17], we use the term "stratified space" to denote a "stratified topological space" equipped with certain regularity data.

[8] More precisely, we require that the restriction $(M_1)_{|M_0^{(1)}} \to M_0^{(1)}$ of φ to the 1-dimensional stratum of M_0 is a finite-sheeted cover that respects framings.

condition. This is enhanced to a category enriched in Kan complexes in [AFRb], which by definition presents the ∞-category $\mathcal{S}\mathsf{trat}$.[9]

Later in [AFRb], this is used to construct an ∞-category $\mathcal{B}\mathsf{un}$, whose objects are those of $\mathcal{S}\mathsf{trat}$ but whose morphisms classify constructible bundles over Δ^1, as in Definition 1.1. By [AFRb, Theorem 6.6.15], the cylinder constructions of Definitions 1.5 and 1.6 determine surjective monomorphisms

$$\left(\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}}\right)^{\mathsf{op}} \xrightarrow{\mathsf{Cylr}} \mathcal{B}\mathsf{un} \xleftarrow{\mathsf{Cylo}} \mathcal{S}\mathsf{trat}^{\mathsf{ref}} ;$$

more generally, the restriction of Cylr to either the injective or surjective proper constructible bundles likewise defines a surjective monomorphism.

Finally, in [AFRa] is defined an enhancement of $\mathcal{B}\mathsf{un}$ which incorporates vari-framings. Our ∞-category $\boldsymbol{\mathcal{M}}$ is then defined by passing to the full subcategory on those objects which are compact and have dimension at most 1. In particular, we have pullback diagrams

$$\begin{tikzcd} \boldsymbol{\mathcal{M}}^{\mathsf{cls.cr}} \arrow[r] \arrow[d, hook, two heads] & \left(\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}}\right)^{\mathsf{op}} \arrow[d, hook, two heads] \\ \boldsymbol{\mathcal{M}} \arrow[r] & \mathcal{B}\mathsf{un} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \boldsymbol{\mathcal{M}}^{\mathsf{ref}} \arrow[r] \arrow[d, hook, two heads] & \mathcal{S}\mathsf{trat}^{\mathsf{ref}} \arrow[d, hook, two heads] \\ \boldsymbol{\mathcal{M}} \arrow[r] & \mathcal{B}\mathsf{un} \end{tikzcd} .$$

Observation 1.10. By [AFRb, Theorem 6.5.6], the pair $[\boldsymbol{\mathcal{M}}^{\mathsf{cls}}; \boldsymbol{\mathcal{M}}^{\mathsf{act}}]$ forms a factorization system on $\boldsymbol{\mathcal{M}}$: that is, every morphism in $\boldsymbol{\mathcal{M}}$ admits a unique factorization as the composite

$$\bullet \xrightarrow{\mathsf{cls}} \bullet \xrightarrow{\mathsf{act}} \bullet$$

of a *closed* morphism followed by an *active* morphism. On the other hand, any active morphism in $\boldsymbol{\mathcal{M}}$ can be non-uniquely factored as the composite

$$\bullet \xrightarrow{\mathsf{cr}} \bullet \xrightarrow{\mathsf{ref}} \bullet$$

of a creation morphism followed by a refinement morphism [AFRb, §6.2].

Definition 1.11. For emphasis, we may refer to an object of $\boldsymbol{\mathcal{M}}$ as ***smooth*** if its 0-dimensional stratum is empty, i.e. if it is a finite disjoint union of framed circles; in particular, we may refer to the object $S^1 \in \boldsymbol{\mathcal{M}}$ as the ***smooth circle***.

Notation 1.12. We write

$$\mathsf{B}\mathbb{T} \hookrightarrow \mathsf{B}\mathbb{W} \hookrightarrow \boldsymbol{\mathcal{M}}$$

respectively for the full subgroupoid and the full subcategory on the object $S^1 \in \boldsymbol{\mathcal{M}}$.

Observation 1.13. All endomorphisms of the object $S^1 \in \boldsymbol{\mathcal{M}}$ are creation morphisms. In particular, $\mathbb{T}$ is indeed the circle group as it is usually defined, and more generally the degree homomorphism

$$\mathbb{W} \longrightarrow \mathbb{N}^{\times}$$

[9] By [AFRb, Theorem 2.4.5], the canonical functor $\mathsf{Strat} \to \mathcal{S}\mathsf{trat}$ is the (∞-categorical) localization with respect to the stratified homotopy equivalences.

defines a right fibration on deloopings, with fiber

$$\begin{tikzcd} \mathsf{B}\mathbb{T} \arrow[r] \arrow[d] & \mathsf{B}\mathbb{W} \arrow[d] \\ \mathsf{pt} \arrow[r] & \mathsf{B}\mathbb{N}^{\times} \end{tikzcd} .$$

In other words, we have an identification

$$\mathbb{W} \simeq \mathbb{T} \rtimes \mathbb{N}^{\times}$$

with the semidirect product classified by the right action of $\mathbb{N}^{\times}$ on $\mathbb{T}$ given by positive-degree group endomorphisms.[10]

1.2. **Disk-refinements.**

Definition 1.14. The full subcategory $\mathcal{D} \subset \mathcal{M}$ of ***disk-stratified 1-manifolds*** consists of those stratified 1-manifolds that are finite directed graphs (i.e. those that do not contain any framed (and unstratified) circles among their connected components). We write

$$\mathcal{D} \overset{\delta}{\hookrightarrow} \mathcal{M}$$

for the inclusion.

Warning 1.15. Definition 1.14 does not quite agree with [AFRa, Definition 3.23]: for instance, here we consider S^1_* to be disk-stratified, whereas this does not satisfy the criteria given there. However, our definitions of factorization homology nevertheless agree (see Remarks 2.3 and 4.14).

Observation 1.16. In fact, $\mathcal{D}$ is an ordinary category. Aside from the discrepancy of Warning 1.15 this is proved as [AFRa, Lemma 3.26], and it is not hard to check that the proof given there also applies in our more general case.

Notation 1.17. We write

$$\mathcal{D}/^{\mathsf{ref}}\mathcal{M} := \lim \left(\begin{tikzcd} & \mathsf{Ar}^{\mathsf{ref}}(\mathcal{M}) \arrow[d, "s"] \\ \mathcal{D} \arrow[r, hook] & \mathcal{M} \end{tikzcd} \right)$$

for the full subcategory of the arrow ∞-category $\mathsf{Ar}(\mathcal{M})$ on those refinement morphisms that have disk-stratified source.

Definition 1.18. Let $M \in \mathcal{M}$ be a stratified 1-manifold. We define the category of ***disk-refinements*** of M to be

$$\mathcal{D}(M) := \lim \left(\begin{tikzcd} & \mathcal{D}/^{\mathsf{ref}}\mathcal{M} \arrow[d, "t"] \\ \{M\} \arrow[r] & \mathcal{M} \end{tikzcd} \right) \simeq \lim \left(\begin{tikzcd} & & \mathcal{D} \arrow[d, hook] \\ & \mathsf{Ar}^{\mathsf{ref}}(\mathcal{M}) \arrow[r, "s"] \arrow[d, "t"] & \mathcal{M} \\ \{M\} \arrow[r] & \mathcal{M} & \end{tikzcd} \right) .^{11}$$

[10] The notation $\mathbb{W}$ stems from the fact that this will keep track of Frobenius and Verschiebung operators, along the lines of [HM97b].

[11] A priori, $\mathcal{D}(M)$ is an ∞-category, but in fact it is always a 1-category (see Observation 1.20).

Definition 1.19. We write

$$\mathbf{\Delta}^{\mathsf{op}}_{\circlearrowleft} := \boldsymbol{\mathcal{D}}(S^1)$$

for the category of disk-refinements of the framed circle, and refer to it as the ***paracyclic category***.[12]

Observation 1.20. Though a priori $\boldsymbol{\mathcal{D}}(M)$ is an ∞-category, it is not hard to see that it is actually always a 1-category.[13] Indeed, consider the *total blowup*

$$M \xrightarrow{\mathsf{cr}} \mathsf{Bl}(M) \ ,$$

the terminal creation morphism from M among those whose corresponding proper constructible bundle defines an isomorphism on 1-dimensional strata; heuristically, this "splits apart" all the edges of M while leaving the isolated circles and vertices untouched. Pullback of disk-refinements (see Remark 1.23) evidently induces an equivalence

$$\boldsymbol{\mathcal{D}}(M) \xrightarrow{\sim} \boldsymbol{\mathcal{D}}(\mathsf{Bl}(M)) \ ,$$

and moreover we have a decomposition

$$\boldsymbol{\mathcal{D}}(\mathsf{Bl}(M)) \simeq \prod_i \boldsymbol{\mathcal{D}}(\mathsf{Bl}(M)_i)$$

indexed by the path components $\mathsf{Bl}(M)_i$ of $\mathsf{Bl}(M)$; together, these observations reduce us to the case that $M \in \{\mathbb{D}^0, \mathbb{D}^1, S^1\}$, which is handled by the evident equivalences

$$\boldsymbol{\mathcal{D}}(\mathbb{D}^0) \simeq \mathsf{pt} \ , \qquad \boldsymbol{\mathcal{D}}(\mathbb{D}^1) \simeq \mathbf{\Delta} \ , \qquad \text{and} \qquad \boldsymbol{\mathcal{D}}(S^1) \simeq \mathbf{\Delta}^{\mathsf{op}}_{\circlearrowleft} \ .$$

Remark 1.21. As described in §0.3, for a $\mathcal{V}$-enriched ∞-category $\mathcal{C}$, the $\mathcal{V}$-enriched factorization homology of $\mathcal{C}$ over M will be given by the colimit of a diagram

$$\boldsymbol{\mathcal{D}}(M) \longrightarrow \mathcal{V} \ ;$$

this diagram will take a disk-refinement $(R \to M) \in \boldsymbol{\mathcal{D}}(M)$ to a colimit in $\mathcal{V}$, indexed over all possible labelings

$$R^{(0)} \xrightarrow{\lambda} \mathcal{C}^{\simeq}$$

of the vertices of R by objects of $\mathcal{C}$, of the tensor product

$$\bigboxtimes_{e \in R^{(1)}} \underline{\mathsf{hom}}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e)))$$

of hom-objects of $\mathcal{C}$ indexed by the edges of R.

Observation 1.22. Let us examine the image of the composite functor

$$\boldsymbol{\mathcal{D}}(M) \longrightarrow \boldsymbol{\mathcal{D}}/^{\mathsf{ref}}\boldsymbol{\mathcal{M}} \xrightarrow{s} \boldsymbol{\mathcal{D}} \ ;$$

[12] It is straightforward to see that this agrees with the combinatorially-defined category originally introduced implicitly in [FL91, §1.5, Example 3].

[13] Note that this does not quite follow from Observation 1.16: there is no a priori reason that the map

$$\mathsf{hom}_{\boldsymbol{\mathcal{D}}(M)}\left(\left(M \xrightarrow{\mathsf{ref}} R_1\right), \left(M \xrightarrow{\mathsf{ref}} R_2\right)\right) \longrightarrow \mathsf{hom}_{\boldsymbol{\mathcal{D}}}(R_1, R_2)$$

must be a monomorphism.

that is, let us examine the possible horizontal morphisms in a commutative triangle

$$\begin{tikzcd} R_0 \arrow[rr] \arrow[dr, "\mathsf{ref}"'] & & R_1 \arrow[dl, "\mathsf{ref}"] \\ & M & \end{tikzcd} .$$

- We can't get a nonidentity closed morphism, because the refinement morphisms to M must be homeomorphisms on underlying topological spaces.
- If we get a creation morphism, it must be determined by a surjective proper constructible bundle $R_1 \to R_0$ with the following property:

  (*) the restriction $(R_1)_{|R_0^{(1)}} \to R_0^{(1)}$ to the 1-dimensional stratum of R_0 is an isomorphism.

  (Note that edges in M_1 can project to vertices in M_0.)
- Any refinement morphism can appear in this way.

Remark 1.23. The functoriality of the construction $\mathfrak{D}(-)$ is somewhat subtle. Given a morphism

$$M_0 \longrightarrow M_1 \tag{8}$$

in $\mathfrak{M}$, we would like for it to induce a functor

$$\mathfrak{D}(M_0) \longrightarrow \mathfrak{D}(M_1) \ ; \tag{9}$$

ideally, this would be encoded by the assertion that the pullback

$$\begin{tikzcd} \left(\mathfrak{D}/^{\mathsf{ref}}\mathfrak{M}\right)_{|(M_0 \to M_1)} \arrow[r] \arrow[d] & \mathfrak{D}/^{\mathsf{ref}}\mathfrak{M} \arrow[d, "t"] \\ {[1]} \arrow[r, "(M_0 \to M_1)"'] & \mathfrak{M} \end{tikzcd}$$

defines a cocartesian fibration

$$\begin{tikzcd} \left(\mathfrak{D}/^{\mathsf{ref}}\mathfrak{M}\right)_{|(M_0 \to M_1)} \arrow[d] \\ {[1]} \end{tikzcd} . \tag{10}$$

Indeed, this is the case when the morphism (8) is a refinement: the functor (9) is then simply given by postcomposition. It is also the case when the morphism (8) is closed: the functor (9) is then given by pullback along the corresponding injective proper constructible bundle. Complications arises when the morphism (8) is a creation. There are a number of issues.

First of all, we would like to obtain a functor (9) by pullback along the corresponding surjective proper constructible bundle $\varphi : M_1 \to M_0$, but for this to take a disk-refinement of M_0 to a disk-refinement of M_1, the morphism φ must satisfy the following condition:

(*) restricted to the path components of M_1 consisting of smooth circles, φ defines a covering map onto its image.

(The alternative is that φ collapses a smooth circle in M_1 to a vertex in M_0.)

However, even in this case, the functor (10) will not generally be a cocartesian fibration. The minimal example illustrating the problem occurs when we take the morphism (8) to be the creation $\mathbb{D}^0 \to \mathbb{D}^1$. Namely, the corresponding functor

$$\mathsf{pt} \simeq \mathcal{D}(\mathbb{D}^0) \longrightarrow \mathcal{D}(\mathbb{D}^1) \simeq \mathbf{\Delta} \tag{11}$$

selects the *terminal* object (which is not initial), whereas the functor

$$\begin{array}{ccc} \mathcal{D}(\mathbb{D}^1) & \longrightarrow & \mathcal{S} \\ \Psi & & \Psi \\ (R \xrightarrow{\mathsf{ref}} \mathbb{D}^1) & \longmapsto & \mathsf{hom}_{(\mathcal{D}/^{\mathsf{ref}}\mathcal{M})_{|(\mathbb{D}^0 \to \mathbb{D}^1)}}\left((\mathbb{D}^0 \xrightarrow{\mathsf{ref}} \mathbb{D}^0), (R \xrightarrow{\mathsf{ref}} \mathbb{D}^1)\right) \end{array}$$

is constant at the terminal space $\mathsf{pt} \in \mathcal{S}$. Said differently, we have an equivalence

$$(\mathcal{D}/^{\mathsf{ref}}\mathcal{M})_{|(\mathbb{D}^0 \to \mathbb{D}^1)} \simeq \mathcal{D}(\mathbb{D}^1)^{\triangleleft}$$

in $\mathsf{Cat}_{/[1]}$, and the functor to $[1]$ is not a cocartesian fibration (let alone the cocartesian unstraightening over $[1]$ of the functor (11)). Of course, this issue persists whenever the surjective proper constructible bundle φ fails to satisfy the following condition:

(**) the preimage $(M_1)_{|M_0^{(1)}}$ under φ of the 1-dimensional stratum of M_0 contains the entire 1-dimensional stratum of M_1.

(The alternative is that φ collapses an edge in M_1 to a vertex in M_0.)

On the other hand, the failure of a creation morphism to satisfy conditions (*) and (**) of Remark 1.23 is the only obstruction to the functor (10) being a cocartesian fibration. In particular, we have the following.

Observation 1.24. Write

$$\mathcal{M}^{\mathsf{cov}} \subset \mathcal{M}^{\mathsf{cls.cr}}$$

for the subcategory on those closed-creation morphisms which are opposite to *stratified covering spaces* (i.e. proper constructible bundles that are locally trivial (not just after restricting to strata)). Then, the restriction

$$\begin{tikzcd} (\mathcal{D}/^{\mathsf{ref}}\mathcal{M})_{|\mathcal{M}^{\mathsf{cov}}} \arrow[r] \arrow[d] & \mathcal{D}/^{\mathsf{ref}}\mathcal{M} \arrow[d, "t"] \\ \mathcal{M}^{\mathsf{cov}} \arrow[r, hook] & \mathcal{M} \end{tikzcd}$$

is a cocartesian fibration, with cocartesian pushforward functors given by pullback of disk-refinements.

Definition 1.25. We write

$$\begin{tikzcd} \mathbf{\Lambda}^{\mathsf{op}} \arrow[r] \arrow[d] & \mathcal{D}/^{\mathsf{ref}}\mathcal{M} \arrow[d, "t"] \\ \mathsf{B}\mathbb{T} \arrow[r] & \mathcal{M} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{\mathbf{\Lambda}}^{\mathsf{op}} \arrow[r] \arrow[d] & \mathcal{D}/^{\mathsf{ref}}\mathcal{M} \arrow[d, "t"] \\ \mathsf{B}\mathbb{W} \arrow[r] & \mathcal{M} \end{tikzcd}$$

respectively for the pullbacks, and refer to them respectively as the ***cyclic indexing category*** and ***epicyclic indexing category***.[14]

[14]It is straightforward to see that $\mathbf{\Lambda}^{\mathsf{op}}$ agrees with the combinatorially-defined category originally defined in [Con83]. On the other hand, in the small handful of places that it has appeared in the literature, $\widetilde{\mathbf{\Lambda}}^{\mathsf{op}}$ is actually

Observation 1.26. We have a diagram

$$
\begin{tikzcd}
\mathbf{\Delta}^{\mathsf{op}}_{\circlearrowleft} \arrow[r, two heads] \arrow[d] & \mathbf{\Lambda}^{\mathsf{op}} \arrow[r, hook, two heads] \arrow[d] & \widetilde{\mathbf{\Lambda}}^{\mathsf{op}} \arrow[r, hook, "\mathsf{f.f.}"] \arrow[d] & \mathfrak{D}/^{\mathsf{ref}}\mathfrak{M} \arrow[d, "t"] \\
\{S^1\} \arrow[r, two heads] & \mathsf{B}\mathbb{T} \arrow[r, hook, two heads] & \mathsf{B}\mathbb{W} \arrow[r, hook, "\mathsf{f.f.}"'] & \mathfrak{M}
\end{tikzcd}
\tag{12}
$$

of pullback squares, in which all vertical maps besides the rightmost one are cocartesian fibrations by Observation 1.24. In particular, we have a left action

$$\mathbb{W} \curvearrowright \mathbf{\Delta}^{\mathsf{op}}_{\circlearrowleft}$$

on the paracyclic category; through diagram (12), we can identify its left-lax homotopy quotient

$$\left(\mathbf{\Delta}^{\mathsf{op}}_{\circlearrowleft}\right)_{\mathsf{h}^{\mathsf{l.lax}}\mathbb{W}} \simeq \widetilde{\mathbf{\Lambda}}^{\mathsf{op}}$$

with the epicyclic category, and we can identify the (strict) homotopy quotient

$$\left(\mathbf{\Delta}^{\mathsf{op}}_{\circlearrowleft}\right)_{\mathsf{h}\mathbb{T}} \simeq \mathbf{\Lambda}^{\mathsf{op}}$$

of the resulting left $\mathbb{T}$-action with the cyclic category.

Remark 1.27. Poincaré duality of disk-refinements induces an equivalence

$$\mathbf{\Delta}^{\mathsf{op}}_{\circlearrowleft} \xrightarrow{\sim} (\mathbf{\Delta}^{\mathsf{op}}_{\circlearrowleft})^{\mathsf{op}}$$

between the paracyclic category and its opposite, which (being $\mathbb{T}$-equivariant) descends to the classical equivalence [Con83]

$$\mathbf{\Lambda}^{\mathsf{op}} \xrightarrow{\sim} (\mathbf{\Lambda}^{\mathsf{op}})^{\mathsf{op}}$$

between the cyclic category and its opposite.

Remark 1.28. For a stratified 1-manifold $M \in \mathfrak{M}$, a refinement

$$R \xrightarrow{\mathsf{ref}} M$$

may be thought of as an *excision site* for factorization homology over M. This choice of refinement determines an ∞-category

$$
\mathfrak{D}(M,R) := \mathfrak{D}^{R/^{\mathsf{cr}}}_{/^{\mathsf{ref}}M} := \lim \left(
\begin{tikzcd}
 & & \mathfrak{D}_{/^{\mathsf{ref}}M} \arrow[d, hook] \\
 & \mathfrak{M}^{R/^{\mathsf{cr}}}_{/^{\mathsf{ref}}M} \arrow[r] \arrow[d] & \mathfrak{M}_{/^{\mathsf{ref}}M} \\
\mathfrak{D}^{R/^{\mathsf{cr}}} \arrow[r, hook] & \mathfrak{M}^{R/^{\mathsf{cr}}} &
\end{tikzcd}
\right)
$$

of *relative* disk-refinements, namely factorizations

$$
\begin{tikzcd}
R \arrow[rr, "\mathsf{ref}"] \arrow[dr, dashed, "\mathsf{cr}"'] & & M \\
 & R' \arrow[ur, dashed, "\mathsf{ref}"'] &
\end{tikzcd}
$$

defined as the opposite of what we have indicated here. (See e.g. [BFG94] for a foundational survey, which attributes the definition to an unpublished letter from Goodwillie to Waldhausen dating back to 1987.) We have chosen our convention in the interest of uniformity.

(where $R' \in \mathfrak{D}$). By definition, this admits a functor

$$\mathfrak{D}(M,R) \longrightarrow \mathfrak{D}(M) \ , \tag{13}$$

and it is not hard to see that this is final: in other words, that enriched factorization homology satisfies *excision*.[15] Indeed, all nontrivial cases reduce to the special case of the refinement morphism

$$\left(R \xrightarrow{\mathsf{ref}} M\right) = \left(S^1_* \xrightarrow{\mathsf{ref}} S^1\right) \ .$$

In this case, the functor (13) can be identified as the canonical (some might even say defining) functor

$$\mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathbf{\Delta}^{\mathsf{op}}_{\circlearrowleft} \ ,$$

which is final [Lurb, Proposition 4.2.8]. Given a $\mathcal{V}$-enriched ∞-category $\mathcal{C}$, the resulting simplicial object in $\mathcal{V}$ whose colimit therefore computes

$$\int_{S^1} \mathcal{C}$$

is precisely its *cyclic bar construction*.

1.3. The standard simplicial disk-stratified 1-manifold.

Notation 1.29. There is an evident fully faithful functor

$$\mathbf{\Delta}^{\mathsf{op}} \xhookrightarrow{\langle - \rangle} \mathfrak{D} \ ,$$

which takes the object $[0]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$ to $\mathbb{D}^0 \in \mathfrak{D}$ and, for any $n \geq 1$, takes the object $[n]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$ to the closed interval stratified by its endpoints as well as $(n-1)$ internal vertices. To ease notation, we simply write $\langle n \rangle \in \mathfrak{D}$ for the image of the object $[n]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$.[16]

Remark 1.30. The functor $\langle - \rangle$ of Notation 1.29 is the case $n = 1$ of a more general functor, running from Joyal's category Θ_n [Joy, Rez10, BSP] to an n-dimensional version of $\mathfrak{D}$. The core technical result of [AFRa] is that this functor is fully faithful, which implies that (∞, n)-categories can actually be *characterized* by their factorization homology over (compact vari-framed) n-manifolds.

2. Categorified factorization homology

In this section, we define categorified factorization homology – a special instance of *cartesian* factorization homology – and deduce its functoriality that we need in order to define enriched factorization homology in §3. We begin in §2.1 by defining the categorified factorization homology of an arbitrary category object in Cat. We then recall the definition of flagged enriched ∞-categories in §2.2. This definition involves a right-lax functor of category obects in Cat. We will need to know that categorified factorization homology is functorial for such morphisms; this is not immediate, but we show that it is indeed true in §2.3.

[15] At the level of field theories, this gives a local-to-global expression for the observables on M.

[16] This notation is essentially concordant with that of [AFRa, Lemma 3.44], of which it is a special case.

2.1. **Categorified factorization homology.**

Definition 2.1. For a bicomplete cartesian symmetric monoidal ∞-category $\mathcal{X}$, we define the (***cartesian***) ***factorization homology*** functor to be the composite

$$\int_{(=)} (-) : \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{X}) \xrightarrow{\langle - \rangle_*} \mathsf{Fun}(\mathfrak{D}, \mathcal{X}) \xrightarrow{\delta_!} \mathsf{Fun}(\mathfrak{M}, \mathcal{X})$$

of right Kan extension along $\langle - \rangle$ followed by left Kan extension along δ. So given $\mathcal{Y} \in \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{X})$ and $M \in \mathfrak{M}$, we write

$$\int_M \mathcal{Y} \in \mathcal{X}$$

for the value of the functor

$$\int_{(-)} \mathcal{Y} : \mathfrak{M} \longrightarrow \mathcal{X}$$

on the object $M \in \mathfrak{M}$. When $\mathcal{X} = \mathsf{Cat}$, we refer to this as the (***categorified***) ***factorization homology*** functor.

Remark 2.2. In fact, the cartesian factorization homology functor is defined as soon as $\mathcal{X}$ admits finite limits and sifted colimits.

Remark 2.3. Taking $\mathcal{X} = \mathcal{S}$ to be the ∞-category of spaces, Definition 2.1 essentially recovers [AFRa, Definition 4.13] (in the case that $n = 1$), or rather it does so when restricted to the subcategory

$$\mathsf{Cat} \simeq \mathsf{CSS} \subset \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S})$$

of complete Segal spaces: the definitions do not quite agree (recall Warning 1.15), but the values are nevertheless equivalent by a straightforward verification of finality.

Remark 2.4. In this paper, we will only concern ourselves with the case that $\mathcal{X} = \mathsf{Cat}$ is the ∞-category of ∞-categories; moreover, we will only ever take the categorified factorization homology of *category objects* in Cat, and then only over objects of $\mathfrak{D} \subset \mathfrak{M}$.

Observation 2.5. Suppose that $\mathcal{Y} \in \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{X})$ is a category object, and choose any $R \in \mathfrak{D} \subset \mathfrak{M}$. By Proposition A.7, we can describe

$$\int_R \mathcal{Y}$$

quite explicitly, as indicated in §0.3: it is given by a limit, with one copy of $\mathcal{Y}_{|[1]^\circ}$ for each edge, one copy of $\mathcal{Y}_{|[0]^\circ}$ for each vertex, and "source" and "target" maps that from the copies of $\mathcal{Y}_{|[1]^\circ}$ to the copies of $\mathcal{Y}_{|[0]^\circ}$.[17]

Remark 2.6. As explained in Remark 2.3, cartesian factorization homology agrees with enriched factorization homology.

---

[17]As the inclusion $\mathfrak{D} \overset{\delta}{\hookrightarrow} \mathfrak{M}$ is fully faithful, the left Kan extension $\delta_!$ leaves the values on objects of $\mathfrak{D} \subset \mathfrak{M}$ unchanged.

2.2. **Flagged enriched ∞-categories.**

Notation 2.7. For a monoidal ∞-category $(\mathcal{V}, \boxtimes)$, we write

$$\begin{tikzcd} \mathfrak{B}\mathcal{V}^{\boxtimes} \arrow[d] \\ \mathbf{\Delta}^{\mathsf{op}} \end{tikzcd}$$

for its *monoidal deloop*, i.e. its corresponding nonsymmetric ∞-operad. When the monoidal structure is understood, we simply write

$$\mathfrak{B}\mathcal{V} := \mathfrak{B}\mathcal{V}^{\boxtimes} \ .$$

The underlying ∞-groupoid of a flagged $\mathcal{V}$-enriched ∞-category participates in its definition via the following.

Definition 2.8. The ***codiscrete category object*** functor is the right Kan extension

$$\begin{tikzcd} \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S}) \arrow[r, shift left, "(-)_{|[0]^{\circ}}"] \arrow[r, phantom, "\perp"] & \mathcal{S} \arrow[l, shift left, hook, dashed, "\mathsf{cd}"] \end{tikzcd}$$

along the fully faithful inclusion of the initial object $[0]^{\circ} \in \mathbf{\Delta}^{\mathsf{op}}$.

Observation 2.9. The functor cd takes values in the full subcategory of $\mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S})$ on the category objects in $\mathcal{S}$ (as its name suggests). Hence, it does too when considered as landing in $\mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat})$ since the inclusion $\mathcal{S} \subset \mathsf{Cat}$ preserves fiber products.

Remark 2.10. Category objects in $\mathcal{S}$ (i.e. Segal spaces) are equivalent to *flagged ∞-categories*: a Segal space is equivalently specified by an ∞-category equipped with a surjective functor from an ∞-groupoid, and it is *complete* just when this functor is the inclusion of its underlying ∞-groupoid. For a space $V \in \mathcal{S}$, the Segal space $\mathsf{cd}(V)$ corresponds to the flagged ∞-category

$$V \longrightarrow \tau_{\leq(-1)} V \ ,$$

the canonical map from V to its (-1)-truncation. In other words, the underlying ∞-category is pt when V is nonempty and $\varnothing$ when V is empty.

We now proceed to define right-lax functors of category objects in Cat. This requires a few preliminaries.

Notation 2.11. Through the composite fully faithful embedding

$$\mathbf{\Delta}^{\mathsf{op}} \xhookrightarrow{\langle - \rangle} \boldsymbol{\mathcal{D}} \xhookrightarrow{\delta} \boldsymbol{\mathcal{M}} \ ,$$

the category $\mathbf{\Delta}^{\mathsf{op}}$ inherits a factorization system, which we denote by

$$[\mathbf{\Delta}^{\mathsf{op,cls}}; \mathbf{\Delta}^{\mathsf{op,act}}] \ .$$

Remark 2.12. The factorization system

$$[\mathbf{\Delta}^{\mathsf{op,cls}}; \mathbf{\Delta}^{\mathsf{op,act}}]$$

on $\mathbf{\Delta}^{\mathsf{op}}$ is opposite to a factorization system

$$[\mathbf{\Delta}^{\mathsf{act}}; \mathbf{\Delta}^{\mathsf{cls}}]$$

on $\mathbf{\Delta}$, which can be described explicitly as follows:

- the *active* morphisms in $\mathbf{\Delta}$ are those that are surjective on endpoints, and
- the *closed* morphisms in $\mathbf{\Delta}$ are the interval inclusions, i.e. those morphisms of the form $i \mapsto i + k$ for some fixed k.

Remark 2.13. The morphisms in $\mathbf{\Delta}^{\mathsf{op}}$ that we call "closed" are elsewhere called "inert" (e.g. in [Lura]). We choose our terminology to remain consistent with that surrounding the ∞-category $\mathcal{M}$.

Notation 2.14. Given a pair $\mathcal{B} \supset \mathcal{B}_0$ of an ∞-category and a subcategory thereof, we write

$$\begin{tikzcd} \mathsf{Cat}^{\mathcal{B}_0}_{\mathsf{cocart}/\mathcal{B}} \arrow[r, hook, two heads] \arrow[d] & \mathsf{Cat}_{\mathsf{cocart}/\mathcal{B}} \arrow[d] \\ \mathsf{coCart}_{\mathcal{B}_0} \arrow[r, hook, two heads] & \mathsf{Cat}_{\mathsf{cocart}/\mathcal{B}_0} \end{tikzcd}$$

for the pullback, where the right vertical functor is given by pullback along the inclusion and the bottom functor is the surjective monomorphism. In the special case of the pair $\mathbf{\Delta}^{\mathsf{op}} \supset \mathbf{\Delta}^{\mathsf{op},\mathsf{cls}}$ we simply write

$$\mathsf{Cat}^{\mathsf{cls}}_{\mathsf{cocart}/\mathbf{\Delta}^{\mathsf{op}}} := \mathsf{Cat}^{\mathbf{\Delta}^{\mathsf{op},\mathsf{cls}}}_{\mathsf{cocart}/\mathbf{\Delta}^{\mathsf{op}}} .$$

Remark 2.15. The ∞-category $\mathsf{Cat}^{\mathcal{B}_0}_{\mathsf{cocart}/\mathcal{B}}$ can be informally described as follows:

- its objects are the cocartesian fibrations over $\mathcal{B}$, and
- its morphisms are those functors over $\mathcal{B}$ which preserve cocartesian lifts of morphisms in $\mathcal{B}_0 \subset \mathcal{B}$.

Definition 2.16. A (***strict***) ***functor*** of category objects in Cat is a morphism in $\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$ between the cocartesian fibrations that they classify, and a ***right-lax functor*** of category objects in Cat is a morphism in $\mathsf{Cat}^{\mathsf{cls}}_{\mathsf{cocart}/\mathbf{\Delta}^{\mathsf{op}}}$ between the cocartesian fibrations that they classify.

In particular, we have the following.

Definition 2.17. A ***flagged $\mathcal{V}$-enriched ∞-category*** $\mathcal{C}$ is the data of an ∞-groupoid $\mathcal{C}^{\simeq} \in \mathcal{S}$, called its ***underlying ∞-groupoid***, equipped with a right-lax functor

$$\begin{tikzcd} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[rr, "\underline{\mathsf{hom}}_{\mathcal{C}}"] \arrow[dr] & & \mathfrak{B}\mathcal{V} \arrow[dl] \\ & \mathbf{\Delta}^{\mathsf{op}} & \end{tikzcd}$$

of category objects in Cat. These form an evident ∞-category, which we denote by $\mathsf{fCat}(\mathcal{V})$.

2.3. **Categorified factorization homology of right-lax functors.** Fix a flagged $\mathcal{V}$-enriched ∞-category $\mathcal{C}$. In order to define its enriched factorization homology, we will want to construct a morphism

$$\begin{tikzcd} \int_{|\mathfrak{D}} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r, dashed, "\int_{|\mathfrak{D}} \underline{\mathsf{hom}}_{\mathcal{C}}"] \arrow[d] & \int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V} \arrow[dl] \\ \mathfrak{D} & \end{tikzcd}$$

in $\mathsf{Cat}_{\mathsf{cocart}/\mathfrak{D}}$. However, this does not come for free: a priori, the right Kan extension

$$\mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat}) \xrightarrow{\langle - \rangle_*} \mathsf{Fun}(\mathfrak{D}, \mathsf{Cat})$$

of Definition 2.1 is only functorial for *strict* (and not right-lax) functors of category objects. To obtain it, we will exploit the closed-active factorization system on $\mathbf{\Delta}^{\mathsf{op}}$ via the following result.

Proposition 2.18. *There exists a left adjoint*

$$\begin{tikzcd} \mathsf{Cat}^{\mathsf{cls}}_{\mathsf{cocart}/\mathbf{\Delta}^{\mathsf{op}}} \arrow[r, dashed, shift left, "\mathsf{F}^{\mathsf{act}}"] \arrow[r, phantom, "\perp"] & \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}} \arrow[l, two heads, hook, shift left] \end{tikzcd} \tag{14}$$

to the surjective monomorphism, which takes an object $(\mathcal{E} \to \mathbf{\Delta}^{\mathsf{op}}) \in \mathsf{Cat}^{\mathsf{cls}}_{\mathsf{cocart}/\mathbf{\Delta}^{\mathsf{op}}}$ *to the horizontal composite in the diagram*

$$\begin{tikzcd} \mathcal{E} \underset{\mathbf{\Delta}^{\mathsf{op}}}{\times} \mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}) \arrow[r] \arrow[d] & \mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}) \arrow[r, "\mathsf{ev}_t"] \arrow[d, "\mathsf{ev}_s"] & \mathbf{\Delta}^{\mathsf{op}} \\ \mathcal{E} \arrow[r] & \mathbf{\Delta}^{\mathsf{op}} & \end{tikzcd} .$$

Proof. This is the special case of Proposition B.1 in which we take $\mathcal{B} = \mathbf{\Delta}^{\mathsf{op}}$ and $[\mathcal{B}_0; \mathcal{B}_1] = [\mathbf{\Delta}^{\mathsf{op},\mathsf{cls}}; \mathbf{\Delta}^{\mathsf{op},\mathsf{act}}]$. $\square$

Remark 2.19. As suggested by the notation, the left adjoint in Proposition 2.18 can be informally described as freely adjoining cocartesian lifts for active morphisms in $\mathbf{\Delta}^{\mathsf{op}}$; this generalizes the "free cocartesian fibration" construction of [GHN].

Notation 2.20. For a space $V \in \mathcal{S}$, we just write

$$\mathsf{F}^{\mathsf{act}}(V) := \mathsf{F}^{\mathsf{act}}(\mathsf{cd}(V)) ,$$

for simplicity.

Observation 2.21. The object $(\mathsf{F}^{\mathsf{act}}(V) \downarrow \mathbf{\Delta}^{\mathsf{op}}) \in \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$ can be heuristically described as follows: an object in the fiber over $[n]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$ is given by the data of an active morphism $[m]^\circ \to [n]^\circ$ and a labeling $[m] \to V$ (of the elements of the poset $[m]$ by points in V). In particular, it defines (i.e. straightens to) a category object in Cat,

- whose ∞-category of objects is $V \in \mathcal{S} \subset \mathsf{Cat}$,
- in which a morphism from $v_0 \in V$ to $v_1 \in V$ is given by the data of
  - a natural number $m \geq 1$ (corresponding to the unique active morphism $[m]^\circ \to [1]^\circ$) along with
  - a sequence
  $$v_0 = w_0 \ , \ w_1 \ , \ w_2 \ , \ \ldots \ , \ w_m = v_1$$
  of $m+1$ points in V, starting with v_0 and ending with v_1,

  and
- in which composition is given by concatenation of sequences.

Remark 2.22. For a space $V \in \mathcal{S} \subset \mathsf{Cat}$, the object $\mathsf{F}^{\mathsf{act}}(V) \in \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$ is a "generalized" monoidal ∞-category (in the sense of the generalized ∞-operads of [Lura, §2.3.2]): it satisfies the Segal condition, but its fiber over $[0]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$ isn't contractible. Modulo this detail, the unit map

$$\begin{tikzcd} \mathsf{cd}(V) \arrow[rr] \arrow[dr] & & \mathsf{F}^{\mathsf{act}}(V) \arrow[dl] \\ & \mathbf{\Delta}^{\mathsf{op}} & \end{tikzcd}$$

may be thought of as "the free enriched flagged ∞-category with underlying ∞-groupoid V" (with functoriality for *strict* monoidal functors of generalized monoidal ∞-categories).

Construction 2.23. For any flagged $\mathcal{V}$-enriched ∞-category $\mathcal{C}$, by Proposition 2.18 there exists a canonical factorization

$$\begin{tikzcd} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r] \arrow[dr] \arrow[rr, bend left, "\underline{\mathsf{hom}}_{\mathcal{C}}"] & \mathsf{F}^{\mathsf{act}}(\mathcal{C}^{\simeq}) \arrow[r, dashed, "\underline{\mathsf{hom}}^{\dagger}_{\mathcal{C}}"] \arrow[d] & \mathfrak{B}\mathcal{V} \arrow[dl] \\ & \mathbf{\Delta}^{\mathsf{op}} & \end{tikzcd}$$

in which

- the first map is the unit of the adjunction (14), and so lies in $\mathsf{Cat}^{\mathsf{cls}}_{\mathsf{cocart}/\mathbf{\Delta}^{\mathsf{op}}}$, and
- the second map lies in $\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$.

The latter map is taken by the right Kan extension

$$\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}} \simeq \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat}) \xrightarrow{\langle - \rangle_*} \mathsf{Fun}(\mathfrak{D}, \mathsf{Cat}) \simeq \mathsf{coCart}_{\mathfrak{D}}$$

to a map

$$\begin{tikzcd} \int_{|\mathfrak{D}} \mathsf{F}^{\mathsf{act}}(\mathcal{C}^{\simeq}) \arrow[r, "\int_{|\mathfrak{D}} \underline{\mathsf{hom}}^{\dagger}_{\mathcal{C}}"] \arrow[d] & \int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V} \arrow[dl] \\ \mathfrak{D} & \end{tikzcd} \tag{15}$$

in $\mathsf{coCart}_{\mathfrak{D}}$.

Observation 2.24. There is a canonical monomorphism

$$\begin{tikzcd} \int_{|\mathfrak{D}} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r, hook] \arrow[d] & \int_{|\mathfrak{D}} \mathsf{F}^{\mathsf{act}}(\mathcal{C}^{\simeq}) \arrow[dl] \\ \mathfrak{D} & \end{tikzcd} \tag{16}$$

in $\mathsf{Cat}_{\mathsf{cocart}/\mathfrak{D}}$, namely the identification of the source as the full subcategory of the target on those disk-stratified 1-manifolds whose framed intervals are labeled by morphisms in $\mathsf{F}^{\mathsf{act}}(\mathcal{C}^{\simeq})$ associated to the natural number 1 (corresponding to the unique active morphism $[1]^\circ \to [1]^\circ$ (recall Observation 2.21)) – that is, in which the sequence of points of $\mathcal{C}^{\simeq}$ has exactly two elements.

Construction 2.25. From Construction 2.23 and Observation 2.24, we obtain a canonical composite

$$\begin{tikzcd}
\int_{|\mathfrak{D}} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r, hook, "(16)"] \arrow[d] \arrow[rr, dashed, bend left, "\int_{|\mathfrak{D}} \underline{\mathsf{hom}}_{\mathcal{C}}"] & \int_{|\mathfrak{D}} \mathsf{F}^{\mathsf{act}}(\mathcal{C}^{\simeq}) \arrow[r, "\int_{|\mathfrak{D}} \underline{\mathsf{hom}}_{\mathcal{C}}^{\dagger}"] \arrow[dl] & \int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V} \arrow[dll, bend left] \\
\mathfrak{D} & &
\end{tikzcd} \tag{17}$$

in $\mathsf{Cat}_{\mathsf{cocart}/\mathfrak{D}}$.

Remark 2.26. The composite (17) takes an object

$$\Big(R \in \mathfrak{D}\ ,\ R^{(0)} \xrightarrow{\lambda} \mathcal{C}^{\simeq}\Big) \in \int_{|\mathfrak{D}} \mathsf{cd}(\mathcal{C}^{\simeq})$$

to the object

$$\Big(R \in \mathfrak{D}\ ,\ R^{(1)} \longrightarrow \mathcal{V}\Big) \in \int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V}$$

determined by the prescription

$$\begin{tikzcd}
R^{(1)} \arrow[r] & \mathcal{V} \\
\Cup & \Cup \\
e \arrow[r, mapsto] & \underline{\mathsf{hom}}_{\mathcal{C}}(\lambda(s(e)), \lambda(t(e)))
\end{tikzcd}.$$

The source is a left fibration over $\mathfrak{D}$, so every morphism therein is cartesian over $\mathfrak{D}$. In view of Observation 1.10, the action of the composite (17) is completely specified by the following facts.

- A closed morphism $R \to R'$ in $\mathfrak{D}$ determines a commutative diagram

$$\begin{tikzcd}
R^{(0)} & R'^{(0)} \arrow[l, hook'] \\
R^{(1)} \arrow[u, "s"] \arrow[d, "t"'] & R'^{(1)} \arrow[l, hook'] \arrow[u, "s"'] \arrow[d, "t"] \\
R^{(0)} & R'^{(0)} \arrow[l, hook']
\end{tikzcd}$$

  of finite sets. A (necessarily cocartesian) morphism in the source $\int_{|\mathfrak{D}} \mathsf{cd}(\mathcal{C}^{\simeq})$ over this morphism in $\mathfrak{D}$ must be determined by restriction along the map $R'^{(0)} \to R^{(0)}$: the labeling of the vertices of R' must be pulled back from those of R via the defining proper constructible bundle. The composite (17) then takes this to another cocartesian morphism in the target $\int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V}$, likewise given by restriction along the monomorphism $R'^{(1)} \to R^{(1)}$.

- A creation morphism $R \to R'$ in $\mathfrak{D}$ determines a commutative diagram

$$\begin{tikzcd}
R^{(0)} & & R'^{(0)} \arrow[ll, two heads] \\
R^{(1)} \arrow[u, "s"] \arrow[d, "t"'] & \big(R'^{(1)}\big)_{|R^{(1)}} \arrow[l, two heads] \arrow[r, hook] & R'^{(1)} \arrow[u, "s"'] \arrow[d, "t"] \\
R^{(0)} & & R'^{(0)} \arrow[ll, two heads]
\end{tikzcd}$$

of finite sets, where the restriction refers to the corresponding proper constructible bundle. Specifically, the elements of

$$\left(R'^{(1)}\right)_{|R^{(0)}} \cong R'^{(1)} \setminus \left(R'^{(1)}\right)_{|R^{(1)}} \tag{18}$$

correspond to edges in R' that project to vertices in R. The cocartesian monodromy in $\int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V}$ over this creation morphism is given by the composite

$$\mathcal{V}^{\times R^{(1)}} \longrightarrow \mathcal{V}^{\times \left(R'^{(1)}\right)_{|R^{(1)}}} \longrightarrow \mathcal{V}^{\times R'^{(1)}}$$

of the pullback followed by the functor taking all elements of the set (18) to the unit object $\mathbb{1}_{\mathcal{V}} \in \mathcal{V}$. Then, a (necessarily cocartesian) morphism in the source $\int_{|\mathfrak{D}} \mathsf{cd}(\mathcal{C}^{\simeq})$ over this morphism in $\mathfrak{D}$ whose source is an object

$$R^{(0)} \xrightarrow{\lambda} \mathcal{V}$$

must have target the object

$$R'^{(0)} \xrightarrow{\varphi^{(0)}} R^{(0)} \xrightarrow{\lambda} \mathcal{V} .$$

Then, the composite (17) takes this to a morphism in $\int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V}$ which, after taking its cocartesian-active factorization, is given in the factor of $\mathcal{V}$ corresponding to an element $e \in R'^{(1)}$ by

- the identity map of the object

$$\underline{\mathsf{hom}}_{\mathcal{C}}\left(\lambda\left(\varphi^{(0)}(s(e))\right), \lambda\left(\varphi^{(0)}(t(e))\right)\right)$$

  if $e \in \left(R'^{(1)}\right)_{|R^{(1)}}$, and

- the unit map

$$\mathbb{1}_{\mathcal{V}} \longrightarrow \underline{\mathsf{hom}}_{\mathcal{C}}\left(\lambda\left(\varphi^{(0)}(s(e))\right), \lambda\left(\varphi^{(0)}(t(e))\right)\right)$$

  if $e \notin \left(R'^{(1)}\right)_{|R^{(1)}}$ (whose existence arises from the application of the map λ to the identification

$$\varphi^{(0)}(s(e)) \cong \varphi^{(0)}(t(e))$$

  in $R^{(0)}$).

- A refinement morphism $R \to R'$ in $\mathfrak{D}$ determines a commutative diagram

$$\begin{tikzcd} R^{(0)} & R'^{(0)} \arrow[l, hook'] \\ R^{(1)} \arrow[u, "s"] \arrow[d, "t"'] \arrow[r, two heads] & R'^{(1)} \arrow[u, "s"'] \arrow[d, "t"] \\ R^{(0)} & R'^{(0)} \arrow[l, hook'] \end{tikzcd}$$

of finite sets. Now, a (necessarily cocartesian) morphism in the source $\int_{|\mathfrak{D}} \mathsf{cd}(\mathcal{C}^{\simeq})$ over this morphism in $\mathfrak{D}$ is taken by the composite (17) to a morphism in $\int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V}$ determined by the composition maps of $\mathcal{C}$.

Observation 2.27. The composite morphism (17) (or more primitively, the morphism (16)) in $\mathsf{Cat}_{\mathsf{cocart}/\boldsymbol{\mathcal{D}}}$ actually preserves cocartesian morphisms over certain morphisms in $\boldsymbol{\mathcal{D}}$: the closed morphisms, as well as the creation maps corresponding contravariantly to *stratified covering spaces*.[18] Note in particular that the latter class includes (the images in $\boldsymbol{\mathcal{D}}$ of) the cocartesian morphisms in the cocartesian fibration $\widetilde{\boldsymbol{\Lambda}}^{\mathsf{op}} \downarrow \mathsf{B}\mathbb{W}$. Hence, when the composite morphism (17) in $\mathsf{Cat}_{\mathsf{cocart}/\boldsymbol{\mathcal{D}}}$ is pulled back along the composite

$$\widetilde{\boldsymbol{\Lambda}}^{\mathsf{op}} \longrightarrow \boldsymbol{\mathcal{D}}/^{\mathsf{ref}}\boldsymbol{\mathcal{M}} \xrightarrow{s} \boldsymbol{\mathcal{D}} ,$$

it defines a morphism in $\mathsf{coCart}_{\mathsf{B}\mathbb{W}} \subset \mathsf{Cat}_{\mathsf{cocart}/\mathsf{B}\mathbb{W}}$.

3. Enriched factorization homology

In this section, we study enriched factorization homology. We begin in §3.1 with some preliminaries on symmetric monoidal ∞-categories as well as a study of the functoriality of 1-dimensional strata on the ∞-category $\boldsymbol{\mathcal{D}}$. Then, in §3.2 we construct enriched factorization homology over a fixed object $M \in \boldsymbol{\mathcal{M}}$ (as described in §0.3) and equip it with a canonical action of its automorphism group.

3.1. Preliminaries on symmetric monoidal ∞-categories.

Notation 3.1. For a symmetric monoidal ∞-category $(\mathcal{V}, \boxtimes)$, we write

$$\begin{array}{c} \mathfrak{B}^{\Sigma}\mathcal{V}^{\boxtimes} \\ \big\downarrow \\ \mathsf{Fin}_* \end{array} \tag{19}$$

for its *symmetric monoidal deloop*, i.e. its corresponding ∞-operad. When the symmetric monoidal structure is understood, we simply write

$$\mathfrak{B}^{\Sigma}\mathcal{V} := \mathfrak{B}^{\Sigma}\mathcal{V}^{\boxtimes} .$$

Notation 3.2. Recall that the functor (19) restricts to a cocartesian fibration when pulled back along the surjective monomorphism

$$\mathsf{Fin} \hookrightarrow\!\!\!\!\rightarrow \mathsf{Fin}_*$$

from the subcategory of *active* morphisms. Hence, the terminal maps in Fin collectively induce a "tensor everything together" functor to the fiber $\mathfrak{B}^{\Sigma}\mathcal{V}_{|\mathsf{pt}} \simeq \mathcal{V}$, which we denote by

$$\mathfrak{B}^{\Sigma}\mathcal{V}_{|\mathsf{Fin}} \xrightarrow{\boxtimes} \mathcal{V} . \tag{20}$$

---

[18] Such a morphism $R \to R'$ in $\boldsymbol{\mathcal{D}}$ determines a diagram

$$\begin{tikzcd} R^{(0)} & R'^{(0)} \arrow[l] \\ R^{(1)} \arrow[u, "s"] \arrow[d, "t"'] & R'^{(1)} \arrow[l] \arrow[u, "s"'] \arrow[d, "t"] \\ R^{(0)} & R'^{(0)} \arrow[l] \end{tikzcd}$$

of finite sets. A (necessarily cocartesian) morphism in $\int_{|\boldsymbol{\mathcal{D}}} \mathsf{cd}(\mathcal{C}^{\simeq})$ over this morphism in $\boldsymbol{\mathcal{D}}$ must start at an object $R^{(0)} \to \mathcal{C}^{\simeq}$ and end at the composite $R'^{(0)} \to R^{(0)} \to \mathcal{C}^{\simeq}$, and its image under the composite (17) will start at an object $R^{(1)} \to \mathcal{V}$ and likewise end at the composite $R'^{(1)} \to R^{(1)} \to \mathcal{V}$.

Whereas we can only generally take indexed tensor products in $\mathcal{V}$ over morphisms in Fin, the morphisms in $\mathfrak{D}$ can have more exotic behavior on 1-dimensional strata. This behavior is encoded as follows.

Definition 3.3. We write

$$\mathsf{Corr}(\mathsf{Fin})$$

for the ∞-category of ***correspondences of finite sets***: its objects are finite sets, a morphism from S to T is given by a span

$$\begin{tikzcd} & U \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ S & & T \end{tikzcd} \tag{21}$$

of finite sets, and composition is given by pullback.[19,20]

Observation 3.4. There is a canonical composite

$$\mathsf{Fin} \hookrightarrow\!\!\!\!\rightarrow \mathsf{Fin}_* \hookrightarrow\!\!\!\!\rightarrow \mathsf{Corr}(\mathsf{Fin})$$

of surjective monomorphisms, in which the first functor adds a disjoint basepoint and the second functor takes a map $S_+ \xrightarrow{\varphi} T_+$ to the span

$$\begin{tikzcd} & \varphi^{-1}(T) \arrow[dl] \arrow[dr] & \\ S & & T \end{tikzcd}$$

given by the preimage of the subset $T \subset T_+$.

Remark 3.5. Whereas a morphism in Fin_* might be thought of as a "partially-defined function" (on the objects of Fin obtained by removing basepoints), a morphism in $\mathsf{Corr}(\mathsf{Fin})$ might be thought of as a "partially-defined multi-valued function" (which allows for the repetition of values).

Notation 3.6. A morphism (7) in $\mathfrak{D}$ determines a span

$$M_0^{(1)} \simeq \Gamma\left(\begin{array}{c} M_0^{(1)} \\ \downarrow \\ \Delta^{\{0\}} \end{array}\right) \longleftarrow \Gamma'\left(\begin{array}{c} M \\ \downarrow \\ \Delta^{1} \end{array}\right) \longrightarrow \Gamma\left(\begin{array}{c} M_1^{(1)} \\ \downarrow \\ \Delta^{\{1\}} \end{array}\right) \simeq M_1^{(1)}$$

of finite sets, where

- we write Γ for the space of sections (among stratified spaces), and
- we write Γ' for the space of sections (also among stratified spaces) that take values in the 1-dimensional stratum of each fiber.

By [AFRa, Lemma 3.32], this assembles into a functor

$$\mathfrak{D} \xrightarrow{(-)^{(1)}} \mathsf{Corr}(\mathsf{Fin}) \ . \tag{22}$$

[19] This can be easily constructed e.g. as a complete Segal space, using the fact that Fin admits finite limits.

[20] In fact, $\mathsf{Corr}(\mathsf{Fin})$ is a $(2,1)$-category: for any finite sets $S, T \in \mathsf{Fin}$ we have an equivalence $\hom_{\mathsf{Corr}(\mathsf{Fin})}(S,T) \simeq (\mathsf{Fin}_{/(S\times T)})^{\simeq}$ of 1-groupoids.

Observation 3.7. The action of the functor $(-)^{(1)}$ on the various classes of morphisms in $\mathfrak{D}$ is as follows.

- It takes closed morphisms to backwards monomorphisms.
- It takes refinement morphisms to forwards surjections.
- A creation morphism $R \to R'$ corresponds contravariantly to a proper constructible bundle, which determines a "partially-defined multi-valued function" from $R^{(1)}$ to $R'^{(1)}$ by taking an edge of R to its preimage in R'.[21]

Absent any further assumptions on $(\mathcal{V}, \boxtimes)$, the pullback

$$\begin{tikzcd} \mathfrak{D}_{|\mathsf{Fin}} \arrow[r] \arrow[d, hook, two heads] & \mathsf{Fin} \arrow[d, hook, two heads] \\ \mathfrak{D} \arrow[r] & \mathsf{Fin}_* \end{tikzcd}$$

dictates the functoriality of enriched factorization homology: it is the largest source on which we can expect a "tensor everything together" functor. This is encoded by the following result.

Lemma 3.8. *For any symmetric monoidal ∞-category $\mathcal{V}$, there is a canonical pullback square*

$$\begin{tikzcd} \int_{|\mathfrak{D}_{|\mathsf{Fin}_*}} \mathfrak{B}\mathcal{V} \arrow[r] \arrow[d] & \mathfrak{B}^{\Sigma}\mathcal{V} \arrow[d] \\ \mathfrak{D}_{|\mathsf{Fin}_*} \arrow[r] & \mathsf{Fin}_* \end{tikzcd} .$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathbf{\Delta}^{\mathsf{op}} \arrow[dr, dashed, "\varphi"] \arrow[drr, "\Delta^1/\partial\Delta^1"] \arrow[ddr, "\langle - \rangle"'] & & \\ & \mathfrak{D}_{|\mathsf{Fin}_*} \arrow[r] \arrow[d, hook, two heads, "\psi"] & \mathsf{Fin}_* \arrow[d, hook, two heads] \\ & \mathfrak{D} \arrow[r, "(-)^{(1)}"'] & \mathsf{Corr}(\mathsf{Fin}) \end{tikzcd} , \qquad (23)$$

in which the factorization arises from the universal property of the pullback. Recall that by definition of the underlying monoidal ∞-category of a symmetric monoidal ∞-category, we have a pullback square

$$\begin{tikzcd} \mathfrak{B}\mathcal{V} \arrow[r] \arrow[d] & \mathfrak{B}^{\Sigma}\mathcal{V} \arrow[d] \\ \mathbf{\Delta}^{\mathsf{op}} \arrow[r, "\Delta^1/\partial\Delta^1"'] & \mathsf{Fin}_* \end{tikzcd} .$$

[21] Note that edges in R' can project to vertices in R, so that the forwards map in this span need not be surjective.

From diagram (23), we therefore obtain a cospan

$$\int_{|\boldsymbol{\mathfrak{D}}_{|\mathsf{Fin}_*}} \mathfrak{B}\mathcal{V} := \psi^* \langle - \rangle_* \mathfrak{B}\mathcal{V} \longrightarrow \varphi_* \mathfrak{B}\mathcal{V} \simeq \varphi_* \varphi^* \left(\mathfrak{B}^{\Sigma}\mathcal{V} \right)_{|\boldsymbol{\mathfrak{D}}_{|\mathsf{Fin}_*}} \longleftarrow \left(\mathfrak{B}^{\Sigma}\mathcal{V} \right)_{|\boldsymbol{\mathfrak{D}}_{|\mathsf{Fin}_*}} \tag{24}$$

in $\mathsf{coCart}_{\boldsymbol{\mathfrak{D}}_{|\mathsf{Fin}_*}}$, where the rightwards morphism arises from the universal property of the right Kan extension φ_* while the leftwards morphism is the unit of the adjunction $\varphi^* \dashv \varphi_*$.

We claim that cospan (24) consists of equivalences. Since it lies in $\mathsf{coCart}_{\boldsymbol{\mathfrak{D}}_{|\mathsf{Fin}_*}}$, it suffices to check this on fibers over an arbitrary object $R \in \boldsymbol{\mathfrak{D}}_{|\mathsf{Fin}_*}$, where we obtain a cospan

$$\int_R \mathfrak{B}\mathcal{V} \longrightarrow (\varphi_* \mathfrak{B}\mathcal{V})_{|R} \longleftarrow \mathcal{V}^{\times R^{(1)}} . \tag{25}$$

The fact that the rightwards functor in cospan (25) is an equivalence follows by combining Propositions A.7 and A.14. Moreover, the equivalence between the outer two terms in cospan (25) of Observation 2.5 (which appealed to Proposition A.7) is clearly compatible with the cospan.[22] So cospan (25) consists of equivalences, which proves the claim. $\square$

Notation 3.9. For any functor $\mathcal{K} \to \boldsymbol{\mathfrak{D}}_{|\mathsf{Fin}}$, we write

$$\begin{tikzcd}
\int_{|\mathcal{K}} \mathfrak{B}\mathcal{V} \arrow[r] \arrow[d] \arrow[rr, bend left, dashed, "\boxtimes"] & \int_{|\boldsymbol{\mathfrak{D}}_{|\mathsf{Fin}}} \mathfrak{B}\mathcal{V} \arrow[r, "\boxtimes"] \arrow[d] & \mathcal{V} \\
\mathcal{K} \arrow[r] & \boldsymbol{\mathfrak{D}}_{|\mathsf{Fin}} &
\end{tikzcd}$$

for the "tensor everything together" functor that results from Lemma 3.8.

Remark 3.10. As we will show in §4.1, when $(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$ is *cartesian* symmetric monoidal, then there exists a "product everything together" functor indexed over all of $\mathsf{Corr}(\mathsf{Fin})$.[23] This induces a "product everything together" functor over all of $\boldsymbol{\mathfrak{D}}$, which as we will show in §4.2 extends the functoriality of enriched factorization homology in this case.

---

[22] In the notation of §A, this is just the assertion that the functor $R^{(1)} \to \mathsf{Enter}(R)$ induces an equivalence on limits with respect to the long composite in the commutative diagram

$$\begin{tikzcd}
 & (\boldsymbol{\Delta}^{\mathsf{op}})_{R/*} \arrow[dd, hook] \arrow[dr] & & \\
\mathsf{Enter}(R) \arrow[ur, "\mathsf{mc}_R"] \arrow[dr, "\mathsf{mc}_R"'] & & \boldsymbol{\Delta}^{\mathsf{op}} \arrow[r, "\mathfrak{B}\mathcal{V}"] & \mathsf{Cat} \\
 & (\boldsymbol{\Delta}^{\mathsf{op}})_{R/} \arrow[ur] & &
\end{tikzcd} ,$$

which is clear in light of the simplicity of the category $\mathsf{Enter}(R)$ and the functor $\boldsymbol{\Delta}^{\mathsf{op},\mathsf{cls}}_{\leq 1} \hookrightarrow \boldsymbol{\Delta}^{\mathsf{op}} \xrightarrow{\mathfrak{B}\mathcal{V}} \mathsf{Cat}$ through which this composite factors.

[23] In fact, definitionally, this hypothesis can be weakened (see Remark 4.2).

3.2. **Enriched factorization homology.**

Observation 3.11. For any stratified 1-manifold $M \in \mathcal{M}$, from Observation 1.22 and Observation 3.7 we see that the composite

$$\mathfrak{D}(M) \longrightarrow \mathfrak{D}/^{\mathsf{ref}}\mathcal{M} \xrightarrow{s} \mathfrak{D} \xrightarrow{(-)^{(1)}} \mathsf{Corr}(\mathsf{Fin})$$

factors through the subcategory $\mathsf{Fin} \subset \mathsf{Corr}(\mathsf{Fin})$; it follows that there exists a factorization

$$\begin{tikzcd} \mathfrak{D}(M) \arrow[r, dashed] \arrow[d] & \mathfrak{D}_{|\mathsf{Fin}} \arrow[d, hook, two heads] \\ \mathfrak{D}/^{\mathsf{ref}}\mathcal{M} \arrow[r, "s"'] & \mathfrak{D} \end{tikzcd} .$$

Definition 3.12. Let $M \in \mathcal{M}$ be a stratified 1-manifold, let $\mathcal{V}$ be a symmetric monoidal ∞-category, and let $\mathcal{C} \in \mathsf{fCat}(\mathcal{V})$ be a flagged $\mathcal{V}$-enriched ∞-category. Then, we define the (***enriched***) ***factorization homology*** of $\mathcal{C}$ over M to be the colimit

$$\int_M \mathcal{C} := \mathsf{colim}\left(\int_{|\mathfrak{D}(M)} \mathsf{cd}(\mathcal{C}^{\simeq}) \xrightarrow{\int_{|\mathfrak{D}(M)} \underline{\mathsf{hom}}_{\mathcal{C}}} \int_{|\mathfrak{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}\right) .$$

This assembles into a functor

$$\mathsf{fCat}(\mathcal{V}) \xrightarrow{\int_M(-)} \mathcal{V} . \tag{26}$$

Remark 3.13. Depending on the combinatorics of $\mathfrak{D}(M)$, it can be possible to weaken the hypothesis that $\mathcal{V}$ be symmetric monoidal while still obtaining a "tensor everything together" functor

$$\int_{|\mathfrak{D}(M)} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} .$$

For instance, if $M = S^1$ then it suffices (definitionally) for $\mathcal{V}$ to be merely *cyclically* monoidal.

Remark 3.14. There is a sense in which factorization homology is given by a colimit over factorization homology. Namely, by Proposition 4.13, for any simplicial object $\mathcal{Y} \in \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat})$ and any $M \in \mathcal{M}$, there is a canonical functor

$$\int_{|\mathfrak{D}(M)} \mathcal{Y} \longrightarrow \int_M \mathcal{Y}$$

which is a localization with respect to the cocartesian morphisms over $\mathfrak{D}(M)$. By inspection, we see that for any $\mathcal{C} \in \mathsf{fCat}(\mathcal{V})$ there exists an extension

$$\begin{tikzcd} \int_{|\mathfrak{D}(M)} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r, "\int_{|\mathfrak{D}(M)} \underline{\mathsf{hom}}_{\mathcal{C}}"] \arrow[d] & \int_{|\mathfrak{D}(M)} \mathfrak{B}\mathcal{V} \arrow[r, "\boxtimes"] & \mathcal{V} \\ \int_M \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[urr, dashed, bend right] & & \end{tikzcd}$$

along this localization, whose colimit therefore also computes the enriched factorization homology

$$\int_M \mathcal{C}$$

since localizations are final.[24]

It is easy to construct an action of the automorphism (∞-)group

$$\mathsf{Aut}(M) := \mathsf{Aut}_{\boldsymbol{\mathcal{M}}}(M)$$

of the object $M \in \boldsymbol{\mathcal{M}}$ on the factorization homology $\int_M \mathcal{C}$.

Observation 3.15. The pullback

$$\begin{tikzcd} \boldsymbol{\mathcal{D}}(M)_{\mathsf{hAut}(M)} \arrow[r, hook] \arrow[d] & \boldsymbol{\mathcal{D}}/^{\mathsf{ref}}\boldsymbol{\mathcal{M}} \arrow[d, "t"] \\ \mathsf{BAut}(M) \arrow[r, hook] & \boldsymbol{\mathcal{M}} \end{tikzcd} \tag{27}$$

is a cocartesian fibration: disk-refinements are functorial for automorphisms.

Remark 3.16. In the case that $M = S^1$ is the smooth circle, we can identify the pullback (27) with the canonical functor

$$\begin{tikzcd} \boldsymbol{\Lambda}^{\mathsf{op}} \arrow[d] \\ \mathsf{B}\mathbb{T} \end{tikzcd}$$

from the *cyclic category* (originally introduced by Connes [Con83]) to its groupoid completion, namely the classifying space of the circle group $\mathbb{T}$.

Observation 3.17. Extending Observation 3.11, for any stratified 1-manifold $M \in \boldsymbol{\mathcal{M}}$, there exists a factorization

$$\begin{tikzcd} \boldsymbol{\mathcal{D}}(M)_{\mathsf{hAut}(M)} \arrow[r, dashed] \arrow[d] & \boldsymbol{\mathcal{D}}_{|\mathsf{Fin}} \arrow[d, hook, two heads] \\ \boldsymbol{\mathcal{D}}/^{\mathsf{ref}}\boldsymbol{\mathcal{M}} \arrow[r, "s"'] & \boldsymbol{\mathcal{D}} \end{tikzcd} .$$

[24]On the other hand, there does not exist an extension

$$\begin{tikzcd} \int_{|\boldsymbol{\mathcal{D}}(M)} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r, "\int_{|\boldsymbol{\mathcal{D}}(M)} \underline{\mathsf{hom}}_{\mathcal{C}}"] \arrow[d] & \int_{|\boldsymbol{\mathcal{D}}(M)} \mathfrak{B}\mathcal{V} \arrow[d] \\ \int_M \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r, dotted, "\nexists"] & \int_M \mathfrak{B}\mathcal{V} \end{tikzcd}$$

before postcomposition with the "total tensor product" functor. In general cartesian factorization homology is only functorial for *strict* (as opposed to right-lax) functors of category objects, and in this case it is also easy to see directly that such an extension does not exist (except in the trivial case that $M \in \boldsymbol{\mathcal{M}}$ is a finite disjoint union of copies of the 0-disk $\mathbb{D}^0$).

Construction 3.18. The left Kan extension

$$\begin{tikzcd}
\int_{|\mathfrak{D}(M)_{\mathsf{hAut}(M)}} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r, "\int_{|\mathfrak{D}(M)_{\mathsf{hAut}(M)}} \underline{\mathsf{hom}}_{\mathcal{C}}"] \arrow[d] & \int_{|\mathfrak{D}(M)_{\mathsf{hAut}(M)}} \mathfrak{B}\mathcal{V} \arrow[r, "\boxtimes"] \arrow[dl] & \mathcal{V} \\
\mathfrak{D}(M)_{\mathsf{hAut}(M)} \arrow[d] & & \\
\mathsf{BAut}(M) \arrow[rruu, dashed, bend right, "\mathsf{Aut}(M) \curvearrowright \int_M \mathcal{C}"'] & &
\end{tikzcd} \tag{28}$$

can be computed as a fiberwise colimit since it is along a cocartesian fibration. As the fibers of the vertical composite are given by

$$\begin{tikzcd}
\int_{|\mathfrak{D}(M)} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r] \arrow[d] & \int_{|\mathfrak{D}(M)_{\mathsf{hAut}(M)}} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[d] \\
\mathfrak{D}(M) \arrow[r] \arrow[d] & \mathfrak{D}(M)_{\mathsf{hAut}(M)} \arrow[d] \\
\{M\} \arrow[r] & \mathsf{BAut}(M)
\end{tikzcd} ,$$

we see that the underlying object of the left Kan extension (28) is indeed $\int_M \mathcal{C}$, as indicated. This therefore constructs a canonical action of $\mathsf{Aut}(M)$ on $\int_M \mathcal{C}$, and thereafter a lift

$$\begin{tikzcd}
\mathsf{fCat}(\mathcal{V}) \arrow[r, dashed] \arrow[dr, "\int_M(-)"'] & \mathsf{Fun}(\mathsf{BAut}(M), \mathcal{V}) \arrow[d, "\mathsf{fgt}"] \\
& \mathcal{V}
\end{tikzcd}$$

of the factorization homology over M functor (26).

Definition 3.19. For any flagged $\mathcal{V}$-enriched ∞-category $\mathcal{C}$, we define its ***($\mathcal{V}$-enriched) topological Hochschild homology*** to be its enriched factorization homology over the smooth circle $S^1 \in \mathfrak{M}$; we denote this by

$$\mathsf{THH}_{\mathcal{V}}(\mathcal{C}) := \left(\int_{S^1} \mathcal{C} \right) \in \mathcal{V} ,$$

and the construction assembles into a functor

$$\mathsf{fCat}(\mathcal{V}) \xrightarrow{\mathsf{THH}_{\mathcal{V}}} \mathcal{V} .$$

Remark 3.20. As a particular case of Construction 3.18, for any $\mathcal{V}$-enriched ∞-category $\mathcal{C}$ we obtain a natural action

$$\mathbb{T} \curvearrowright \mathsf{THH}_{\mathcal{V}}(\mathcal{C})$$

of the circle group $\mathbb{T}$ on its $\mathcal{V}$-enriched topological Hochschild homology. In the special case that $(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$ is *cartesian* symmetric monoidal, in §5.1 (based on our work in §4) we will enhance this $\mathbb{T}$-action to an *unstable cyclotomic structure* on $\mathsf{THH}_{\mathcal{V}}(\mathcal{C})$.

Observation 3.21. There is a canonical identification

$$\int_{\mathbb{D}^0} \mathcal{C} \simeq \mathcal{C}^{\simeq} \odot \mathbb{1}_{\mathcal{V}}$$

of the enriched factorization homology of $\mathcal{C}$ over $\mathbb{D}^0$ with the tensoring of the unit object $\mathbb{1}_{\mathcal{V}} \in \mathcal{V}$ over the underlying ∞-groupoid of $\mathcal{C}$.

4. Cartesian enriched factorization homology

In this section, we study enriched factorization homology as defined in §3 in the special case that our enriching symmetric monoidal ∞-category $(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$ is cartesian. In this case, enriched factorization homology becomes functorial over all of $\mathbf{\mathcal{M}}$, as we show in §4.2. In order to prove this, we extend the cartesian symmetric monoidal deloop $\mathfrak{B}^{\Sigma}\mathcal{V}^{\times} \downarrow \mathsf{Fin}_*$ to a larger object $\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} \downarrow \mathsf{Corr}(\mathsf{Fin})$ in §4.1.

4.1. **Preliminaries on cartesian symmetric monoidal ∞-categories.** The following result codifies the additional structure present on *cartesian* symmetric monoidal ∞-categories.

Proposition 4.1. *Let $\mathcal{V}$ be an ∞-category admitting finite products.*

(1) There exists a canonical cocartesian fibration

$$\begin{tikzcd} \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} \arrow[d] \\ \mathsf{Corr}(\mathsf{Fin}) \end{tikzcd}$$

satisfying the following properties.

(a) Its pullback along the surjective monomorphism $\mathsf{Fin}_* \hookrightarrow\!\!\!\!\rightarrow \mathsf{Corr}(\mathsf{Fin})$ is the symmetric monoidal deloop

$$\begin{tikzcd} \mathfrak{B}^{\Sigma}\mathcal{V}^{\times} \arrow[r] \arrow[d] & \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} \arrow[d] \\ \mathsf{Fin}_* \arrow[r, hook, two heads] & \mathsf{Corr}(\mathsf{Fin}) \end{tikzcd}$$

of the cartesian symmetric monoidal ∞-category $(\mathcal{V}, \times)$; in particular, its fiber over an object $S \in \mathsf{Corr}(\mathsf{Fin})$ is canonically identified with the indexed product $\mathcal{V}^{\times S} \simeq \mathsf{Fun}(S, \mathcal{V})$.

(b) Its cocartesian monodromy along a morphism (21) in $\mathsf{Corr}(\mathsf{Fin})$ is the composite

$$\begin{array}{ccccc} \mathcal{V}^{\times S} & \longrightarrow & \mathcal{V}^{\times U} & \longrightarrow & \mathcal{V}^{\times T} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (V_s)_{s \in S} & \longmapsto & \left(V_{\varphi(u)}\right)_{u \in U} & \longmapsto & \left(\prod_{u \in \psi^{-1}(t)} V_{\varphi(u)}\right)_{t \in T} \end{array} \tag{29}$$

of pullback along the map $S \xleftarrow{\varphi} U$ followed by the indexed product along the map $U \xrightarrow{\psi} T$.

(2) The "product everything together" functor (20) for $(\mathcal{V},\times)$ admits a canonical extension

$$\begin{tikzcd} \mathfrak{B}^{\Sigma}\mathcal{V}_{|\mathsf{Fin}} \arrow[r, hook, two heads] \arrow[rrr, bend left, "\prod"] & \mathfrak{B}^{\Sigma}\mathcal{V} \arrow[r, hook, two heads] & \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} \arrow[r, dashed, "\prod"] & \mathcal{V} \end{tikzcd} .$$

(3) There is a canonical pullback square

$$\begin{tikzcd} \int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V} \arrow[r] \arrow[d] & \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V} \arrow[d] \\ \mathfrak{D} \arrow[r, "(-)^{(1)}"'] & \mathsf{Corr}(\mathsf{Fin}) \end{tikzcd} .$$

Remark 4.2. For a general symmetric monoidal ∞-category $(\mathcal{V},\boxtimes)$, one might define an extension to a *bicommutative bialgebra* structure on $\mathcal{V}$ to be the data of a cocartesian fibration

$$\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\boxtimes} \downarrow \mathsf{Corr}(\mathsf{Fin})$$

which extends its symmetric monoidal deloop

$$\mathfrak{B}^{\Sigma}\mathcal{V}^{\boxtimes} \downarrow \mathsf{Fin}_*$$

as in Proposition 4.1 (but perhaps allowing for different and more exotic cocartesian monodromy than indexed monoidal products as in the composite (29)). An extended "tensor everything together" functor might then be furnished by an extension

$$\begin{tikzcd} \mathsf{Corr}(\mathsf{Fin}) \arrow[r, "\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\boxtimes}"] \arrow[d, hook, two heads] & \mathsf{Cat} \\ \underline{\mathsf{Corr}(\mathsf{Fin})}^{\text{2-op}} \arrow[ur, dashed] & \end{tikzcd}$$

of its unstraightening – which might be called an *augmentation* –, where $\underline{\mathsf{Corr}(\mathsf{Fin})}$ denotes the evident enhancement of $\mathsf{Corr}(\mathsf{Fin})$ to a $(2,2)$-category with

$$\underline{\mathsf{hom}}_{\underline{\mathsf{Corr}(\mathsf{Fin})}}(S,T) \simeq \mathsf{Fin}_{/(S\times T)} \ :$$

given a morphism (21) in $\mathsf{Corr}(\mathsf{Fin})$, this provides a natural transformation in the diagram

$$\begin{tikzcd} \mathcal{V}^{\times S} \arrow[r] \arrow[dr] & \mathcal{V}^{\times U} \arrow[r] \arrow[d, phantom, "\Rightarrow"] & \mathcal{V}^{\times T} \arrow[dl] \\ & \mathcal{V} & \end{tikzcd}$$

whose components are "diagonal" maps in $\mathcal{V}$. We expect this to suffice to extend the functoriality of $\mathcal{V}$-enriched factorization homology, as we achieve in §4.2 via Proposition 4.1 in the cartesian symmetric monoidal case.

The remainder of this subsection is devoted to the proof of Proposition 4.1.

Definition 4.3. We define the ***cartesian dual*** of an exponentiable fibration

$$(\mathcal{F} \downarrow \mathcal{B}^{\mathsf{op}}) \in \mathsf{EFib}_{\mathcal{B}^{\mathsf{op}}}$$

to be the object

$$\left(\mathcal{F}^{\overset{\mathsf{cart}}{\vee}} \downarrow \mathcal{B}\right) \in \mathsf{Cat}_{/\mathcal{B}}$$

satisfying the universal property that for any test object $(\mathcal{K} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$, the space of lifts

$$\begin{tikzcd} & \mathcal{F}^{\overset{\mathsf{cart}}{\vee}} \arrow[d] \\ \mathcal{K} \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

is equivalent to the space of lifts

$$\begin{tikzcd} \mathsf{TwAr}(\mathcal{K})^{\mathsf{op}} \arrow[r, dashed] \arrow[d, "t^{\mathsf{op}}"'] & \mathcal{F} \arrow[d] \\ \mathcal{K}^{\mathsf{op}} \arrow[r] & \mathcal{B}^{\mathsf{op}} \end{tikzcd}$$

which take all cartesian morphisms for the right fibration t^{op} to cartesian morphisms in $\mathcal{F}$ over $\mathcal{B}^{\mathsf{op}}$.

Remark 4.4. In the case that

$$(\mathcal{F} \downarrow \mathcal{B}^{\mathsf{op}}) \in \mathsf{Cart}_{\mathcal{B}^{\mathsf{op}}} \subset \mathsf{EFib}_{\mathcal{B}^{\mathsf{op}}}$$

is a *cartesian* fibration, by [BGN] its cartesian dual in the sense of Definition 4.3 is indeed its cartesian dual in the sense of item (7) of §0.6.

Observation 4.5. Let $(\mathcal{E}^{\mathsf{op}} \downarrow \mathcal{B}^{\mathsf{op}}) \in \mathsf{EFib}_{\mathcal{B}^{\mathsf{op}}}$ be an exponentiable fibration, and let $\mathcal{Z} \in \mathsf{Cat}$ be an arbitrary ∞-category. For any test object $(\mathcal{K} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$, we will describe an equivalent datum to a lift

$$\begin{tikzcd} & \mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}^{\mathsf{op}}}(\mathcal{E}^{\mathsf{op}}, \underline{\mathcal{Z}})^{\overset{\mathsf{cart}}{\vee}} \arrow[d] \\ \mathcal{K} \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd} \tag{30}$$

to the cartesian dual of the relative functor ∞-category. For any object $k \in \mathcal{K}$, consider the diagram

$$\begin{tikzcd} \left(\{k\} \underset{\mathcal{B}}{\times} \mathcal{E}\right)^{\mathsf{op}} \arrow[r] \arrow[d] & \left(\mathcal{K}_{k/} \underset{\mathcal{B}}{\times} \mathcal{E}\right)^{\mathsf{op}} \arrow[r] \arrow[d] & \left(\mathsf{TwAr}(\mathcal{K}) \underset{\mathcal{B}}{\times} \mathcal{E}\right)^{\mathsf{op}} \arrow[r] \arrow[d] & \left(\mathcal{K} \underset{\mathcal{B}}{\times} \mathcal{E}\right)^{\mathsf{op}} \arrow[r] \arrow[d] & \mathcal{E}^{\mathsf{op}} \arrow[d] \\ \{(k \to k)^{\circ}\} \arrow[r] & \left(\mathcal{K}_{k/}\right)^{\mathsf{op}} \arrow[r] \arrow[d] & \mathsf{TwAr}(\mathcal{K})^{\mathsf{op}} \arrow[r, "t^{\mathsf{op}}"'] \arrow[d, "s^{\mathsf{op}}"] & \mathcal{K}^{\mathsf{op}} \arrow[r] & \mathcal{B}^{\mathsf{op}} \\ & \{k\} \arrow[r] & \mathcal{K} & & \end{tikzcd}$$

in which all squares are pullbacks. Then, unwinding the definitions, we see that a lift (30) is equivalently given by a functor

$$\left(\mathsf{TwAr}(\mathcal{K}) \underset{\mathcal{B}}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \mathcal{Z}$$

such that for each $k \in \mathcal{K}$, the resulting commutative triangle

$$\begin{tikzcd}
\left(\{k\} \underset{\mathcal{B}}{\times} \mathcal{E}\right)^{\mathsf{op}} \arrow[r] \arrow[d] & \mathcal{Z} \\
\left(\mathcal{K}_{k/} \underset{\mathcal{B}}{\times} \mathcal{E}\right)^{\mathsf{op}} \arrow[ur]
\end{tikzcd}$$

is a right Kan extension. We will refer to this as the *right Kan extension condition* in the proof of Proposition 4.1(2).

Example 4.6. Let us study the special case of Observation 4.5 when $\mathcal{K} = [1]$. It suffices to study the universal case where the map $\mathcal{K} \xrightarrow{\sim} \mathcal{B}$ is an equivalence; however, for clarity regarding the general case (where $\mathcal{B}$ is arbitrary), we write $\mathcal{E}_{|[1]} := \mathcal{E}$ where appropriate.

First of all, note the identification

$$\mathsf{TwAr}([1])^{\mathsf{op}} \simeq \left(\begin{tikzcd} & 01 \arrow[dl] \arrow[dr] & \\ 00 & & 11 \end{tikzcd} \right) ,$$

under which the right fibration $\mathsf{TwAr}([1])^{\mathsf{op}} \xrightarrow{t^{\mathsf{op}}} [1]^{\mathsf{op}}$ is given by collapsing the rightwards morphism; the leftwards morphism is its unique nonidentity cartesian morphism. Then, an arbitrary lift

$$\begin{tikzcd}
\mathsf{TwAr}([1])^{\mathsf{op}} \arrow[r, dashed] \arrow[d, "t^{\mathsf{op}}"'] & \mathsf{Fun}^{\mathsf{rel}}_{/[1]^{\mathsf{op}}}(\mathcal{E}^{\mathsf{op}}, \underline{\mathcal{Z}}) \arrow[d] \\
{[1]^{\mathsf{op}}} \arrow[r, "\sim"'] & {[1]^{\mathsf{op}}}
\end{tikzcd} \tag{31}$$

is equivalent data to an arbitrary functor

$$(\mathcal{E}^{\mathsf{op}})_{|[1]^{\mathsf{op}}} \coprod_{(\mathcal{E}^{\mathsf{op}})_{|1^\circ}} \left((\mathcal{E}^{\mathsf{op}})_{|1^\circ} \times [1] \right) \longrightarrow \mathcal{Z} \ .$$

Moreover, the lift (31) takes the unique nonidentity cartesian morphism in $\mathsf{TwAr}([1])^{\mathsf{op}}$ to a cartesian morphism in $\mathsf{Fun}^{\mathsf{rel}}_{/[1]^{\mathsf{op}}}(\mathcal{E}^{\mathsf{op}}, \underline{\mathcal{Z}})$ precisely if in the commutative diagram

$$\begin{tikzcd}
(\mathcal{E}^{\mathsf{op}})_{|0^\circ} \arrow[d, hook] \arrow[drr, bend left] & & \\
(\mathcal{E}^{\mathsf{op}})_{|[1]^{\mathsf{op}}} \arrow[r, hook] & (\mathcal{E}^{\mathsf{op}})_{|[1]^{\mathsf{op}}} \coprod_{(\mathcal{E}^{\mathsf{op}})_{|1^\circ}} \left((\mathcal{E}^{\mathsf{op}})_{|1^\circ} \times [1] \right) \arrow[r] & \mathcal{Z} \\
(\mathcal{E}^{\mathsf{op}})_{|1^\circ} \arrow[u, hook] \arrow[urr, bend right] & &
\end{tikzcd}$$

the upper triangle is a right Kan extension. From here, it follows that a lift (31) which preserves cartesian morphisms, i.e. the datum of a lift

$$\begin{tikzcd} & \mathsf{Fun}^{\mathsf{rel}}_{/[1]^{\mathsf{op}}}(\mathcal{E}^{\mathsf{op}}, \underline{\mathcal{Z}})^{\mathsf{cart}}_{\vee} \arrow[d] \\ {[1]} \arrow[ur, dashed] \arrow[r, "\sim"'] & {[1]} \end{tikzcd} ,$$

can be equivalently specified by a diagram

$$\begin{tikzcd} (\mathcal{E}^{\mathsf{op}})_{|0^\circ} \arrow[d, hook, "i"'] \arrow[drr, "F"] & & \\ (\mathcal{E}^{\mathsf{op}})_{|[1]^{\mathsf{op}}} \arrow[rr, "i_* F"] & & \mathcal{Z} \\ (\mathcal{E}^{\mathsf{op}})_{|1^\circ} \arrow[u, hook, "j"] \arrow[urr, "j^* i_* F"] \arrow[urr, bend right, "G"'] \arrow[urr, phantom, bend right=15, "\Downarrow"] & & \end{tikzcd}$$

in which the upper and middle triangles commute and the upper triangle is a right Kan extension.[25]

Example 4.7. Building on Example 4.6, we immediately find that for an arbitrary exponentiable fibration $(\mathcal{E}^{\mathsf{op}} \downarrow \mathcal{B}^{\mathsf{op}}) \in \mathsf{EFib}_{\mathcal{B}^{\mathsf{op}}}$ and any $\mathcal{Z} \in \mathsf{Cat}$, the datum of a lift

$$\begin{tikzcd} & \mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}^{\mathsf{op}}}(\mathcal{E}^{\mathsf{op}}, \underline{\mathcal{Z}})^{\mathsf{cart}}_{\vee} \arrow[d] \\ {[n]} \arrow[ur, dashed] \arrow[r] & \mathcal{B} \end{tikzcd}$$

is equivalent to the data of a list

$$\left(F_0 \ , \ (j_{0,1})^*(i_{0,1})_* F_0 \longrightarrow F_1 \ , \ (j_{1,2})^*(i_{1,2})_* F_1 \longrightarrow F_2 \ , \ \ldots \ , \ (j_{n-1,n})^*(i_{n-1,n})_* F_{n-1} \longrightarrow F_n \right) \ ,$$

where for each $k \in [n]^{\mathsf{op}}$ we have chosen an arbitrary functor

$$(\mathcal{E}^{\mathsf{op}})_{|k^\circ} \xrightarrow{F_k} \mathcal{Z}$$

and for $1 \leq k \leq n$ we write

$$(\mathcal{E}^{\mathsf{op}})_{|(k-1)^\circ} \overset{i_{k-1,k}}{\hookrightarrow} (\mathcal{E}^{\mathsf{op}})_{|\{k-1<k\}^{\mathsf{op}}} \overset{j_{k-1,k}}{\hookleftarrow} (\mathcal{E}^{\mathsf{op}})_{|k^\circ}$$

for the inclusions of the fibers.

Notation 4.8. We write

$$\begin{tikzcd} \overline{\mathsf{Corr}}(\mathsf{Fin}) \arrow[d] \\ \mathsf{Corr}(\mathsf{Fin}) \end{tikzcd}$$

[25] As the functor i is fully faithful, if the upper triangle is a right Kan extension then it necessarily commutes.

for the universal exponentiable fibration whose sections are all finite sets.[26] Explicitly, the fiber over an object $S \in \mathsf{Corr}(\mathsf{Fin})$ is the set S, and a morphism in $\overline{\mathsf{Corr}}(\mathsf{Fin})$ from (S,s) to (T,t) is given by a span

$$\begin{tikzcd} & (U,u) \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ (S,s) & & (T,t) \end{tikzcd}$$

in Fin_*.

Definition 4.9. Let $\mathcal{V}$ be an ∞-category admitting finite products. Its ***extended cartesian symmetric monoidal deloop*** is the ∞-category

$$\begin{tikzcd} \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} := \mathsf{Fun}^{\mathsf{rel}}_{/\mathsf{Corr}(\mathsf{Fin})^{\mathsf{op}}}\left(\overline{\mathsf{Corr}}(\mathsf{Fin})^{\mathsf{op}}, \underline{\mathcal{V}}\right)^{\overset{\mathsf{cart}}{\vee}} \arrow[d] \\ \mathsf{Corr}(\mathsf{Fin}) \end{tikzcd}$$

over $\mathsf{Corr}(\mathsf{Fin})$.

We now prove Proposition 4.1; for clarity, we separate the proofs of its three parts.

Proof of Proposition 4.1(1). Consider the morphism (21) in $\mathsf{Corr}(\mathsf{Fin})$; this induces a pullback diagram

$$\begin{tikzcd} S \underset{U}{\star} T \arrow[r] \arrow[d] & \overline{\mathsf{Corr}}(\mathsf{Fin}) \arrow[d] \\ {[1]} \arrow[r] & \mathsf{Corr}(\mathsf{Fin}) \end{tikzcd} .$$

In view of Example 4.6, we see that the functor

$$\begin{tikzcd} \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} \arrow[d] \\ \mathsf{Corr}(\mathsf{Fin}) \end{tikzcd} \tag{32}$$

is a locally cocartesian fibration: its fiber over an object $S \in \mathsf{Corr}(\mathsf{Fin})$ is given by the functor ∞-category

$$\mathsf{Fun}(S^{\mathsf{op}}, \mathcal{V}) \simeq \mathsf{Fun}(S, \mathcal{V}) ,$$

and its cocartesian monodromy over the morphism (21) is given by right Kan extension followed by restriction in the diagram

$$\begin{tikzcd} S^{\mathsf{op}} \arrow[d, hook] \arrow[dr] & \\ \left(S \underset{U}{\star} T\right)^{\mathsf{op}} \arrow[r] & \mathcal{V} \\ T^{\mathsf{op}} \arrow[u, hook] \arrow[ur] & \end{tikzcd} .$$

[26] That is, $\mathsf{Corr}(\mathsf{Fin})$ classifies such exponentiable fibrations, with the equivalence given by pulling back $\overline{\mathsf{Corr}}(\mathsf{Fin})$.

For each element $t \in T^{\mathsf{op}}$ we can identify

$$S^{\mathsf{op}} \underset{\left(S \underset{U}{\star} T\right)^{\mathsf{op}}}{\times} \left(\left(S \underset{U}{\star} T\right)^{\mathsf{op}}\right)_{t/} \simeq \left(S \underset{\left(S \underset{U}{\star} T\right)}{\times} \left(S \underset{U}{\star} T\right)_{/t}\right)^{\mathsf{op}} \simeq \left(\psi^{-1}(t)\right)^{\mathsf{op}} ,$$

so that this cocartesian pushforward takes a functor

$$S^{\mathsf{op}} \xrightarrow{(V_s)_{s \in S^{\mathsf{op}}}} \mathcal{V}$$

to the functor

$$\begin{tikzcd} T^{\mathsf{op}} \arrow[r] & \mathcal{V} \\[-2em] \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-2em] t \arrow[r, mapsto] & \left(\prod_{u \in \psi^{-1}(t)} V_{\varphi(u)}\right) \end{tikzcd} .$$

These cocartesian monodromy functors evidently compose, so that the functor (32) is indeed a cocartesian fibration. Finally, we identify the pullback of the extended cartesian symmetric monoidal deloop $\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times}$ along $\mathsf{Fin}_* \hookrightarrow\!\!\!\!\rightarrow \mathsf{Corr}(\mathsf{Fin})$ as the cartesian symmetric monoidal deloop $\mathfrak{B}^{\Sigma}\mathcal{V}^{\times}$ by [Lura, Corollary 2.4.1.8]. □

Proof of Proposition 4.1(2). We construct the functor

$$\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} \xrightarrow{\prod} \mathcal{V}$$

as a natural transformation of complete Segal spaces. By definition, a functor $[n] \to \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times}$ is the data of an exponentiable fibration $\mathcal{E} \downarrow [n]$ along with a functor

$$\begin{tikzcd} \left(\mathsf{TwAr}([n]) \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \arrow[r] \arrow[d] \arrow[rrr, dashed, bend left] & \mathcal{E}^{\mathsf{op}} \arrow[r] \arrow[d] & \overline{\mathsf{Corr}}(\mathsf{Fin})^{\mathsf{op}} \arrow[d] & \mathcal{V} \\ \mathsf{TwAr}([n])^{\mathsf{op}} \arrow[r, "t^{\mathsf{op}}"'] & {[n]}^{\mathsf{op}} \arrow[r] & \mathsf{Corr}(\mathsf{Fin})^{\mathsf{op}} & \end{tikzcd} \tag{33}$$

from the interated pullback that satisfies the right Kan extension condition of Observation 4.5. To this $[n]$-point of $\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times}$, we assign the $[n]$-point of $\mathcal{V}$ given by the right Kan extension

$$\begin{tikzcd} \left(\mathsf{TwAr}([n]) \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \arrow[r] \arrow[d] & \mathcal{V} \\ \mathsf{TwAr}([n])^{\mathsf{op}} \arrow[d, "s^{\mathsf{op}}"'] & \\ {[n]} \arrow[uur, dashed] & \end{tikzcd} . \tag{34}$$

Observe that the vertical composite in diagram (34) is the opposite of the vertical composite in the diagram

$$\begin{tikzcd}
\mathsf{TwAr}([n]) \underset{[n]}{\times} \mathcal{E} \arrow[r] \arrow[d] & \mathsf{TwAr}([n]) \arrow[d, "{(s,t)}"] \\
{[n]^{\mathsf{op}} \times \mathcal{E}} \arrow[r] \arrow[d] & {[n]^{\mathsf{op}} \times [n]} \\
{[n]^{\mathsf{op}}} &
\end{tikzcd} ,$$

in which the square is a pullback and the lower vertical functor is the projection. This is the composite of two cocartesian fibrations, so it is again a cocartesian fibration. Thus, the vertical composite in diagram (34) is a cartesian fibration. It follows that that right Kan extension is given by fiberwise limit: it takes the object $i \in [n]$ to the limit

$$\mathsf{lim}\left(\left([n]_{i/} \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \left(\mathsf{TwAr}([n]) \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \mathcal{V}\right) .$$

But further, as by assumption our functor (33) satisfies the right Kan extension condition, since right Kan extensions compose it follows that the canonical morphism

$$\mathsf{lim}\left(\left([n]_{|i} \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \left(\mathsf{TwAr}([n]) \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \mathcal{V}\right) \xrightarrow{\sim} \mathsf{lim}\left(\left([n]_{i/} \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \left(\mathsf{TwAr}([n]) \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \mathcal{V}\right) \tag{35}$$

is an equivalence. Through this reduction, setting $n = 0$ we see that this right Kan extension does indeed act on the fiber over an object $S \in \mathsf{Corr}(\mathsf{Fin})$ as the functor

$$\left(\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times}\right)_{|S} \simeq \mathsf{Fun}(S^{\mathsf{op}}, \mathcal{V}) \xrightarrow{\Pi} \mathcal{V} ,$$

namely the limit over S^{op}.

We now show that this assignment respects the simplicial structure maps of the complete Segal spaces of $\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times}$ and $\mathcal{V}$. For this, suppose we are given a composite

$$[m] \xrightarrow{\rho} [n] \longrightarrow \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} .$$

From the universal property of right Kan extension, taking right Kan extensions as in diagram (34) yields a diagram

$$\begin{tikzcd}
{[m]} \arrow[d, "\rho"'] \arrow[dr] \arrow[d, phantom, shift left=4ex, "\Nearrow"] & \\
{[n]} \arrow[r] & \mathcal{V}
\end{tikzcd} ,$$

which it remains to show strictly commutes. This amounts to showing, for each $i \in [m]$, that its component

$$\mathsf{lim}\left(\left([m]_{i/} \underset{[m]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \left(\mathsf{TwAr}([m]) \underset{[m]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \mathcal{V}\right) \longrightarrow \mathsf{lim}\left(\left([n]_{\rho(i)/} \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \left(\mathsf{TwAr}([n]) \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \mathcal{V}\right)$$

is an equivalence. But through the equivalence (35) this is identified with the morphism

$$\mathsf{lim}\left(\left([m]_{|i} \underset{[m]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \left(\mathsf{TwAr}([m]) \underset{[m]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \mathcal{V}\right) \longrightarrow \mathsf{lim}\left(\left([n]_{|\rho(i)} \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \left(\mathsf{TwAr}([n]) \underset{[n]}{\times} \mathcal{E}\right)^{\mathsf{op}} \longrightarrow \mathcal{V}\right) ,$$

which is tautologically an equivalence: these are the limits of identical diagrams. Thus, we have indeed constructed a functor

$$\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}^{\times} \xrightarrow{\Pi} \mathcal{V} .$$

That this restricts to $\mathfrak{B}^{\Sigma}\mathcal{V}^{\times}$ appropriately follows from [Lura, Proposition 2.4.1.6]. $\square$

Proof of Proposition 4.1(3). Observe that we have a commutative diagram

$$\begin{tikzcd}
 & \mathfrak{B}^{\Sigma}\mathcal{V} \arrow[rr] \arrow[dd] & & \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V} \arrow[dd] \\
\mathfrak{B}\mathcal{V} \arrow[ur] \arrow[dd] & & & \\
 & \mathsf{Fin}_* \arrow[rr] & & \mathsf{Corr}(\mathsf{Fin}) \\
\mathbf{\Delta}^{\mathsf{op}} \arrow[ur, "\Delta^1/\partial\Delta^1"] \arrow[rr, hook] & & \mathfrak{D} \arrow[ur, "(-)^{(1)}"'] &
\end{tikzcd}$$

in which both upper squares are pullbacks – the left square by definition of the underlying nonsymmetric ∞-operad of an ∞-operad, and the right square by part (1)(a). The unit of the adjunction

$$\mathsf{coCart}_{\mathfrak{D}} \simeq \mathsf{Fun}(\mathfrak{D}, \mathsf{Cat}) \underset{\langle - \rangle_*}{\overset{\langle - \rangle^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat}) \simeq \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$$

furnishes a morphism

$$\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}_{|\mathfrak{D}} \longrightarrow \langle - \rangle_* \left(\langle - \rangle^* \left(\widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}_{|\mathfrak{D}} \right) \right) \simeq \langle - \rangle_* \mathfrak{B}\mathcal{V} =: \int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V}$$

in $\mathsf{coCart}_{\mathfrak{D}}$, which we will show is an equivalence. It suffices to check this on fibers over an object $R \in \mathfrak{D}$, i.e. that the functor

$$\mathcal{V}^{\times R^{(1)}} \simeq \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V}_{|R} \longrightarrow \int_R \mathfrak{B}\mathcal{V} \tag{36}$$

is an equivalence.

There is an evident enrichment of $\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$ over Cat, determined by the formula

$$\mathsf{hom}_{\mathsf{Cat}} \left([n], \underline{\mathsf{hom}}_{\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}}(\mathcal{Y}, \mathcal{Y}') \right) \simeq \mathsf{hom}_{\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}}([n] \times \mathcal{Y}, \mathcal{Y}')$$

(where we write $[n] \in \mathsf{coCart}_{\mathbf{\Delta}}^{\mathsf{op}} \simeq \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat})$ for the constant diagram at the object $[n] \in \mathsf{Cat}$). Using this enrichment, and writing

$$\mathfrak{C}(R) := \mathsf{hom}_{\mathfrak{D}}(R, \langle \bullet \rangle) \in \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat}) \simeq \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$$

for the value of the restricted (covariant) Yoneda embedding at the object $R \in \mathfrak{D}$, the end formula for right Kan extensions yields an equivalence

$$\int_R \mathfrak{B}\mathcal{V} \simeq \underline{\mathsf{hom}}_{\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}}(\mathfrak{C}(R), \mathfrak{B}\mathcal{V}) . \tag{37}$$

Note that the object $\mathfrak{C}(R) \in \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$ is evidently free: it's in the image of the left Kan extension

$$\begin{tikzcd}
\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}_{\leq 1}} \arrow[r, hook, dashed, shift left] \arrow[r, phantom, "\perp"] & \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}} \arrow[l, hook, shift left]
\end{tikzcd} \tag{38}$$

along the full inclusion $\mathbf{\Delta}^{\mathsf{op}}_{\leq 1} \hookrightarrow \mathbf{\Delta}^{\mathsf{op}}$. Hence, we obtain a composite equivalence

$$\int_R \mathfrak{B}\mathcal{V} \underset{(37)}{\simeq} \underline{\mathsf{hom}}_{\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}}(\mathfrak{C}(R), \mathfrak{B}\mathcal{V})$$

$$\simeq \underline{\mathsf{hom}}_{\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}_{\leq 1}}}\left(\mathfrak{C}(R)_{|\mathbf{\Delta}^{\mathsf{op}}_{\leq 1}}, \mathfrak{B}\mathcal{V}_{|\mathbf{\Delta}^{\mathsf{op}}_{\leq 1}}\right) \tag{39}$$

$$\simeq \mathsf{Fun}\left(\mathfrak{C}(R)_{|[1]^{\circ}}, \mathfrak{B}\mathcal{V}_{|[1]^{\circ}}\right) \tag{40}$$

$$\simeq \mathcal{V}^{\times R^{(1)}} ,$$

where equivalence (39) follows from the (evidently Cat-enriched) adjunction (38) and equivalence (40) follows from the fact that $\mathfrak{B}\mathcal{V}_{|[0]^{\circ}} \simeq \mathsf{pt}$. By construction, it is now clear that the functor (36) is indeed an equivalence: the identity functor on the ∞-category $\mathcal{V}^{\times R^{(1)}}$. $\square$

4.2. Factorization homology enriched in a cartesian symmetric monoidal ∞-category.

Construction 4.10. Let $\mathcal{V}$ be a cartesian symmetric monoidal ∞-category. By Proposition 4.1, there exists a canonical "product everything together" composite

$$\begin{tikzcd} \int_{|\mathfrak{D}} \mathfrak{B}\mathcal{V} \arrow[r] \arrow[rr, bend left, dashed, "\prod"] & \widetilde{\mathfrak{B}}^{\Sigma}\mathcal{V} \arrow[r, "\prod"] & \mathcal{V} \end{tikzcd} .$$

Notation 4.11. We write

$$\mathfrak{D}/\mathfrak{M} := \lim \left(\begin{tikzcd} & \mathsf{Ar}(\mathfrak{M}) \arrow[d, "t"] \\ \mathfrak{D} \arrow[r, hook] & \mathfrak{M} \end{tikzcd} \right)$$

for the pullback. This comes equipped with a functor

$$\mathfrak{D}/\mathfrak{M} \xrightarrow{s} \mathfrak{D}$$

through which we obtain the pulled back cocartesian fibration

$$\begin{tikzcd} \int_{|\mathfrak{D}/\mathfrak{M}} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r] \arrow[d] & \int_{|\mathfrak{D}} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[d] \\ \mathfrak{D}/\mathfrak{M} \arrow[r, "s"'] & \mathfrak{D} \end{tikzcd} ,$$

as well as a cocartesian fibration

$$\begin{tikzcd} \mathfrak{D}/\mathfrak{M} \arrow[d, "t"] \\ \mathfrak{M} \end{tikzcd}$$

whose cocartesian monodromy functors are given by postcomposition.

Definition 4.12. Suppose that $(\mathcal{V}, \times)$ is a cartesian symmetric monoidal ∞-category, and let $\mathcal{C} \in \mathsf{fCat}(\mathcal{V})$ be a flagged $\mathcal{V}$-enriched ∞-category. Then, the (***cartesian enriched***) ***factorization***

homology functor of $\mathcal{C}$ is the left Kan extension

$$\begin{tikzcd}
\int_{|\boldsymbol{\mathcal{D}}/\boldsymbol{\mathcal{M}}} \mathsf{cd}(\mathcal{C}^{\simeq}) \arrow[r, "\int_{|\boldsymbol{\mathcal{D}}/\boldsymbol{\mathcal{M}}} \mathsf{hom}_{\mathcal{C}}"] \arrow[d] & \int_{|\boldsymbol{\mathcal{D}}/\boldsymbol{\mathcal{M}}} \mathfrak{B}\mathcal{V}^{\times} \arrow[r, "\Pi"] & \mathcal{V} \\
\boldsymbol{\mathcal{D}}/\boldsymbol{\mathcal{M}} \arrow[d, "t"'] & & \\
\boldsymbol{\mathcal{M}} \arrow[urr, dashed, "\int_{(-)} \mathcal{C}"'] & &
\end{tikzcd} ,$$

a fiberwise colimit since it is along a cocartesian fibration; assembling over all $\mathcal{C} \in \mathsf{fCat}(\mathcal{V})$ determines a bifunctor

$$\int_{(-)} (-) : \boldsymbol{\mathcal{M}} \times \mathsf{fCat}(\mathcal{V}) \longrightarrow \mathcal{V} \ .$$

The notation $\int_{(-)} \mathcal{C}$ of Definition 4.12 is justified by the following result.

Proposition 4.13. *On fibers over any object $M \in \boldsymbol{\mathcal{M}}$, the monomorphism*

$$\begin{tikzcd}
\boldsymbol{\mathcal{D}}/^{\mathsf{ref}}\boldsymbol{\mathcal{M}} \arrow[rr, hook] \arrow[dr, "t"'] & & \boldsymbol{\mathcal{D}}/\boldsymbol{\mathcal{M}} \arrow[dl, "t"] \\
& \boldsymbol{\mathcal{M}} &
\end{tikzcd}$$

induces a final functor

$$\boldsymbol{\mathcal{D}}(M) \longrightarrow \boldsymbol{\mathcal{D}}_{/M} := \boldsymbol{\mathcal{D}} \underset{\boldsymbol{\mathcal{M}}}{\times} \boldsymbol{\mathcal{M}}_{/M} \ .$$

Proof. In Observation 1.20 it is shown that the construction $\boldsymbol{\mathcal{D}}(-)$ takes disjoint unions to products of categories, and moreover disjoint unions evidently compute products in $\boldsymbol{\mathcal{M}}$. So it suffices to consider the case that $M \in \boldsymbol{\mathcal{M}}$ is connected. In this case, either $M \in \boldsymbol{\mathcal{D}}$ or else $M = S^1$. In the former case, the identity morphism determines a final object of both $\boldsymbol{\mathcal{D}}(M)$ and $\boldsymbol{\mathcal{D}}_{/M}$, which is preserved by the inclusion. Thus, it remains to consider the case that $M = S^1$.

Now by Quillen's Theorem A, it suffices to show that for any object $(R \to S^1) \in \boldsymbol{\mathcal{D}}_{/S^1}$, the groupoid completion of the category

$$\boldsymbol{\mathcal{D}}(S^1) \underset{\boldsymbol{\mathcal{D}}_{/S^1}}{\times} (\boldsymbol{\mathcal{D}}_{/S^1})_{(R \to S^1)/} \tag{41}$$

of factorizations

$$\begin{tikzcd}
R \arrow[rr] \arrow[dr, dashed] & & S^1 \\
& R' \arrow[ur, dashed, "\mathsf{ref}"'] &
\end{tikzcd} \tag{42}$$

(where $R' \in \boldsymbol{\mathcal{D}}$) is contractible. Observe that in diagram (42), the closed-active factorization of the downwards morphism $R \to R'$ must compose to the closed-active factorization $R \xrightarrow{\mathsf{cls}} R_0 \xrightarrow{\mathsf{act}} S^1$ of the (chosen) horizontal morphism, since such factorizations are unique. So taking closed-active factorizations defines an equivalence between the category (41) and the category

$$\boldsymbol{\mathcal{D}}(S^1) \underset{\boldsymbol{\mathcal{D}}_{/S^1}}{\times} (\boldsymbol{\mathcal{D}}_{/S^1})_{(R_0 \to S^1)/^{\mathsf{act}}} \tag{43}$$

of factorizations

$$\begin{tikzcd}
R \arrow[rr] \arrow[d, "\mathsf{cls}"'] & & S^1 \\
R_0 \arrow[urr, "\mathsf{act}"] \arrow[r, dashed, "\mathsf{act}"'] & R' \arrow[ur, dashed, "\mathsf{ref}"'] &
\end{tikzcd}$$

(where $R' \in \boldsymbol{\mathfrak{D}}$). Thus, it suffices to show that the groupoid completion of the category (43) is contractible.

To check that the groupoid completion $(43)^{\mathsf{gpd}}$ is contractible, it suffices to check that its homotopy groups vanish, and for this it suffices to show that any map to it from a sphere is freely nullhomotopic. We check this using the theory of stratified spaces.

Observe first that any map

$$S^{d-1} \longrightarrow (43)^{\mathsf{gpd}} \tag{44}$$

of ∞-groupoids is represented by a functor

$$\mathsf{Exit}(S^{d-1}) \xrightarrow{\tilde{X}} (43)$$

of ∞-categories, where we abuse notation by also writing S^{d-1} for the $(d-1)$-sphere (thought of as a manifold) equipped with some stratification (e.g. a triangulation). There exists a unique extension of this stratification of S^{d-1} to a stratification of D^d in which the interior is a single stratum; we likewise abuse notation by simply denoting this again by D^d.

Now, observe that the stratified space $\mathsf{C}(D^d)$ – the cone on D^d – has the property that $\mathsf{Exit}(\mathsf{C}(D^d)) \simeq \mathsf{Exit}(S^{d-1})^{\triangleleft\triangleright}$: the cone point over D^d corresponds to the left cone point, while the interior of the disk corresponds to the right cone point. Hence, the functor $\tilde{X}$ is equivalent data to a functor

$$\mathsf{Exit}(\mathsf{C}(D^d)) \simeq \mathsf{Exit}(S^{d-1})^{\triangleleft\triangleright} \longrightarrow \boldsymbol{\mathfrak{M}} ,$$

equipped with certain additional structures and satisfying certain conditions, which amount to the following on the corresponding proper constructible bundle $X \downarrow \mathsf{C}(D^d)$:

- its fiber over the cone point is identified with R,
- its fiber over the interior of D^d is identified with S^1,
- its restriction to any exiting path starting at the cone point is an active morphism, and
- its restriction to any exiting path from S^{d-1} to the interior of D^d is a refinement morphism.

We will use this proper constructible bundle to construct a nullhomotopy of the original map (44).

So, consider the link $\mathsf{Link}_R(X)$ of R in X; this admits a composite proper constructible bundle map

$$\mathsf{Link}_R(X) \longrightarrow R \times X_{|D^d} \longrightarrow R \times D^d .$$

We then form the "D^d-parametrized reversed cylinder" of this map, namely the pushout

$$\begin{tikzcd}
R \times D^d \arrow[r] \arrow[d] & \mathsf{Cylr}\left(R \times D^d \longleftarrow \mathsf{Link}_R(X)\right) \arrow[d] \\
R \arrow[r] & Y
\end{tikzcd} \tag{45}$$

of stratified spaces, where the upper map is the injective constructible bundle given by the inclusion of the fiber over $\Delta^{\{0\}} \subset \Delta^1$ and the left map is the projection. By construction, the pushout square (45) maps to the defining pushout square

$$\begin{tikzcd} \Delta^{\{0\}} \times D^d \arrow[r] \arrow[d] & \Delta^1 \times D^d \arrow[d] \\ \Delta^{\{0\}} \arrow[r] & \mathsf{C}(D^d) \end{tikzcd}$$

by proper constructible bundles, and hence in particular we obtain a proper constructible bundle map $Y \downarrow \mathsf{C}(D^d)$. Also by construction, we see that Y comes equipped with a canonical morphism $Y \to X$ of proper constructible bundles over $\mathsf{C}(D^d)$; unwinding the definitions, we see that the natural transformation

$$\begin{tikzcd} \mathsf{Exit}(\mathsf{C}(D^d)) \arrow[r, bend left, "Y"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{\mathfrak{M}} \end{tikzcd}$$

that this classifies determines a natural transformation

$$\begin{tikzcd} \mathsf{Exit}(S^{d-1}) \arrow[r, bend left, "\tilde{Y}"] \arrow[r, bend right, "\tilde{X}"'] \arrow[r, phantom, "\Downarrow"] & (43) \end{tikzcd}$$

to our originally chosen functor $\tilde{X}$ representing our chosen map (44).

We complete the proof by constructing a nullhomotopy of the map $S^{d-1} \xrightarrow{\tilde{Y}^{\mathsf{gpd}}} (43)^{\mathsf{gpd}}$ of spaces. For this, observe first that the map $\mathsf{Link}_R(X) \to X_{|D^d}$ of stratified spaces is a refinement, so that the map $Y \to X$ is as well. We take the open cylinder $\mathsf{Cylo}(Y \longrightarrow X)$ of the latter, and then take the pushout

$$\begin{tikzcd} R \times \Delta^1 \arrow[r] \arrow[d] & \mathsf{Cylo}(Y \longrightarrow X) \arrow[d] \\ R \arrow[r] & Z \end{tikzcd} \tag{46}$$

of stratified spaces, where the upper map is the inclusion of the fiber over $\Delta^{\{0\}} \times \Delta^1$ (where $\Delta^{\{0\}} \subset \mathsf{C}(D^d)$ denotes the cone point) and the left map is the projection. By construction, the pushout square (46) maps to the pushout square

$$\begin{tikzcd} \Delta^{\{0\}} \times \Delta^1 \arrow[r] \arrow[d] & \mathsf{C}(D^d) \times \Delta^1 \arrow[d] \\ \Delta^{\{0\}} \arrow[r] & \mathsf{C}(D^d \times \Delta^1) \end{tikzcd}$$

by proper constructible bundles, and hence in particular we obtain a proper constructible bundle map $Z \downarrow \mathsf{C}(D^d \times \Delta^1)$, which is classified by a functor

$$\mathsf{Exit}(\mathsf{C}(D^d \times \Delta^1)) \longrightarrow \mathbf{\mathfrak{M}} \; . \tag{47}$$

We now construct three maps $\mathsf{C}(D^d) \to \mathsf{C}(D^d \times \Delta^1)$ of stratified spaces:

- we write i_0 for the inclusion of $\mathsf{C}(D^d \times \Delta^{\{0\}})$,

- we write i_1 for any map that carries the interior to the interior, and whose restriction to $\mathsf{C}(S^{d-1}) \subset \mathsf{C}(D^d)$ is the composite map

$$\mathsf{C}(S^{d-1}) \longrightarrow \mathsf{C}(\mathbb{D}^0) \xrightarrow{\mathsf{C}(p)} \mathsf{C}(D^d \times \Delta^1)$$

in which $\mathbb{D}^0 \xrightarrow{p} D^d \times \Delta^1$ selects an interior point, and

- we write i for any inclusion which extends the inclusion of $\mathsf{C}(S^{d-1} \times \Delta^{\{0\}})$ and takes the interior of D^d into the interior of $\mathsf{C}(D^d \times \Delta^1)$.

On exit-path ∞-categories, these evidently participate in a diagram

$$\begin{tikzcd} \mathsf{Exit}(\mathsf{C}(D^d)) \arrow[rr, bend left=60, "\mathsf{Exit}(i_1)"] \arrow[rr, "\mathsf{Exit}(i_0)"] \arrow[rr, bend right=60, "\mathsf{Exit}(i)"'] \arrow[rr, phantom, bend left=30, "\Uparrow"] \arrow[rr, phantom, bend right=30, "\Downarrow"] & & \mathsf{Exit}(\mathsf{C}(D^d \times \Delta^1)) \end{tikzcd}$$

of natural transformations; unwinding the definitions, we see that this precomposes with the functor (47) to determine a span of diagrams

$$\begin{tikzcd} \mathsf{Exit}(S^{d-1}) \arrow[rr, bend left=60] \arrow[rr, "\tilde{Y}"] \arrow[rr, bend right=60, "\tilde{X}"'] \arrow[rr, phantom, bend left=30, "\Uparrow"] \arrow[rr, phantom, bend right=30, "\Downarrow"] & & (43) \end{tikzcd}$$

in which the upper functor is constant. This completes the proof. □

Remark 4.14. Under certain mild hypotheses on the cartesian symmetric monoidal ∞-category $(\mathcal{V}, \times)$ (which in particular induce a canonical embedding $\mathcal{S} \hookrightarrow \mathcal{V}$), cartesian-enriched factorization homology (Definition 4.12) is equivalent to cartesian factorization homology (Definition 2.1); the proof amounts to an elaboration of the result that flagged $\mathcal{S}$-enriched ∞-categories are equivalent to Segal spaces [GH15, Theorem 4.4.7], or really its generalization [Hau15, Theorem 7.5].

Remark 4.15. As indicated in Remark 4.2, to obtain the functoriality of enriched factorization homology over all of $\boldsymbol{\mathcal{M}}$ it should suffice to assume that $\mathcal{V}$ is an augmented bicommutative bialgebra object in Cat. Moreover, echoing Remark 3.13, if we are only interested in extending the definition of enriched factorization homology over some subcategory of $\boldsymbol{\mathcal{M}}$, it can be possible to weaken this hypothesis further; for instance, if we are only interested extending the definition of enriched factorization homology over the full subcategory $\mathsf{B}\mathbb{W} \subset \boldsymbol{\mathcal{M}}$ on the smooth circle, then it should suffice to assume that $\mathcal{V}$ is an augmented cyclically monoidal and cyclically comonoidal bialgebra object in Cat.[27]

[27] To make this assertion true, it suffices to declare that it is true by definition.

5. The unstable cyclotomic trace

In this section, we summarize the output of this paper as it relates to the trilogy into which it fits. Namely, using the functoriality of cartesian enriched factorization homology established in §4, we lay out the theory of the unstable cyclotomic trace (and in particular establish Theorem A), in parallel with its stable analog: the cyclotomic trace. As we show in [AMGRa], the unstable cyclotomic trace produced here induces the (stable) cyclotomic trace through linearization (in the sense of Goodwillie calculus). First, in §5.1 we define unstable topological cyclic homology. Then, in §5.2 we obtain the unstable cyclotomic trace. Finally, we study some examples of the unstable cyclotomic trace in §5.3.

Notation 5.1. In this section, we fix a cartesian symmetric monoidal ∞-category $(\mathcal{V}, \times)$ as well as a flagged $\mathcal{V}$-enriched ∞-category $\mathcal{C} \in \mathsf{fCat}(\mathcal{V})$. We also write

$$\mathsf{THH}^{\times}_{\mathcal{V}}(\mathcal{C}) := \mathsf{THH}_{\mathcal{V}}(\mathcal{C})$$

to emphasize this assumption on the symmetric monoidal structure.

5.1. Unstable topological cyclic homology.

Definition 5.2. We define the ∞-category of ***unstable cyclotomic objects*** in $\mathcal{V}$ to be

$$\mathsf{Cyc}^{\mathsf{h}}(\mathcal{V}) := \mathsf{Fun}(\mathsf{B}\mathbb{W}, \mathcal{V}) \ .$$

We define the ***homotopy invariants*** (***of the unstable cyclotomic structure***) functor to be the homotopy $\mathbb{W}$-fixedpoints functor

$$\mathsf{Cyc}^{\mathsf{h}}(\mathcal{V}) \xrightarrow{(-)^{\mathsf{h}\mathbb{W}}} \mathcal{V} \ .$$

Remark 5.3. Unwinding the definitions, we see that an unstable cyclotomic structure on an object $V \in \mathcal{V}$ consists of a $\mathbb{T}$-action along with a system of $\mathbb{T}$-equivariant *unstable cyclotomic structure maps*

$$V \longrightarrow V^{\mathsf{hC}_r}$$

(where $\mathbb{T}$ acts on the target through the identification $\mathbb{T} \simeq (\mathbb{T}/\mathsf{C}_r)$) that are compatible as $r \in \mathbb{N}^{\times}$ varies.

Remark 5.4. The simplest example of an unstable cyclotomic object is a *free loop object*, i.e. the cotensoring

$$S^1 \pitchfork V$$

of the circle into an arbitrary object $V \in \mathcal{V}$ (e.g. the free loopspace of a space); the unstable cyclotomic structure arises from the identifications $S^1 \simeq (S^1)_{\mathsf{hC}_r}$ (compatibly for all $r \in \mathbb{N}^{\times}$), so that its $\mathbb{T}$-equivariant structure maps

$$S^1 \pitchfork V \longrightarrow (S^1 \pitchfork V)^{\mathsf{hC}_r} \simeq (S^1)_{\mathsf{hC}_r} \pitchfork V$$

are actually *equivalences*. When $\mathcal{V} = \mathcal{S}$, this is nothing but $\mathsf{THH}^{\times}_{\mathcal{S}}(V)$, the spatially-enriched factorization homology of the ∞-groupoid $V \in \mathcal{S} \subset \mathsf{Cat}$ over the circle.

Observation 5.5. The composite

$$\mathsf{B}\mathbb{W} \xhookrightarrow{\mathsf{f.f.}} \mathcal{M} \xrightarrow{\int_{(-)} \mathcal{C}} \mathcal{V}$$

of the defining inclusion with the cartesian enriched factorization homology functor defines an unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{V}}(\mathcal{C})$; this construction assembles into a lift

$$\begin{tikzcd} \mathsf{fCat}(\mathcal{V}) \arrow[r, dashed] \arrow[dr, "\mathsf{THH}^{\times}_{\mathcal{V}}"'] & \mathsf{Cyc}^{\mathsf{h}}(\mathcal{V}) \arrow[d, "\mathsf{fgt}"] \\ & \mathcal{V} \end{tikzcd} .$$

Definition 5.6. We define the ***unstable topological cyclic homology*** of $\mathcal{C}$ to be the homotopy invariants

$$\mathsf{TC}^{\times}_{\mathcal{V}}(\mathcal{C}) := \mathsf{THH}_{\mathcal{V}}(\mathcal{C})^{\mathsf{h}\mathbb{W}}$$

of its unstable cyclotomic structure; this construction assembles as a composite functor

$$\mathsf{TC}^{\times}_{\mathcal{V}} : \mathsf{fCat}(\mathcal{V}) \xrightarrow{\mathsf{THH}_{\mathcal{V}}} \mathsf{Cyc}^{\mathsf{h}}(\mathcal{V}) \xrightarrow{(-)^{\mathsf{h}\mathbb{W}}} \mathcal{V} .$$

Observation 5.7. As homotopy fixedpoints compose, we can compute unstable topological cyclic homology as the composite

$$\mathsf{TC}^{\times}_{\mathcal{V}}(\mathcal{C}) := \mathsf{THH}^{\times}_{\mathcal{V}}(\mathcal{C})^{\mathsf{h}\mathbb{W}} \simeq \left(\mathsf{THH}^{\times}_{\mathcal{V}}(\mathcal{C})^{\mathsf{h}\mathbb{T}}\right)^{\mathsf{h}\mathbb{N}^{\times}} .$$ [28]

5.2. **The unstable cyclotomic trace.**

Definition 5.8. We define the ***unstable algebraic K-theory*** of $\mathcal{C}$ to be the tensoring

$$\mathsf{K}^{\times}_{\mathcal{V}}(\mathcal{C}) := \mathcal{C}^{\simeq} \odot \mathbb{1}_{\mathcal{V}} \simeq \int_{\mathbb{D}^0} \mathcal{C}$$

of the unit object $\mathbb{1}_{\mathcal{V}} \in \mathcal{V}$ over the underlying ∞-groupoid of $\mathcal{C}$ (where the equivalence comes from Observation 3.21).

Definition 5.9. We define the ***unstable cyclotomic trace*** of $\mathcal{C}$ to be the morphism

$$\mathsf{K}^{\times}_{\mathcal{V}}(\mathcal{C}) \simeq \int_{\mathbb{D}^0} \mathcal{C} \longrightarrow \left(\int_{S^1} \mathcal{C}\right)^{\mathsf{h}\mathbb{W}} =: \mathsf{THH}^{\times}_{\mathcal{V}}(\mathcal{C})^{\mathsf{h}\mathbb{W}} =: \mathsf{TC}^{\times}_{\mathcal{V}}(\mathcal{C})$$

classified by the composite

$$\mathsf{B}\mathbb{W}^{\triangleleft} \hookrightarrow \boldsymbol{\mathcal{M}} \xrightarrow{\int_{(-)} \mathcal{C}} \mathcal{V}$$

of the inclusion of the full subcategory on the objects $\mathbb{D}^0$ and S^1 followed by the cartesian enriched factorization homology functor; this assembles into a natural transformation

$$\begin{tikzcd} \mathsf{fCat}(\mathcal{V}) \arrow[r, bend left, "\mathsf{K}^{\times}_{\mathcal{V}}"] \arrow[r, bend right, "\mathsf{TC}^{\times}_{\mathcal{V}}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{V} \end{tikzcd} .$$

[28] In fact, this composite construction is closely analogous to the construction of (*stable*) topological cyclic homology from $\mathsf{THH} := \mathsf{THH}_{\mathcal{S}\mathsf{p}}$, as described in [AMGRa].

5.3. **Examples of the unstable cyclotomic trace.** In this subsection, we study three examples of the unstable cyclotomic trace on ($\mathcal{S}$-enriched) ∞-categories.[29] The three corresponding classes of ∞-categories are somewhat "orthogonal" in nature, and we expect that in general the unstable cyclotomic trace should combine the phenomena that appear here.

Example 5.10. Consider an ∞-groupoid

$$X \in \mathcal{S} \subset \mathsf{Cat} \simeq \mathsf{Cat}(\mathcal{S})$$

as an ∞-category. Then, we have an identification

$$\mathsf{THH}^{\times}_{\mathcal{S}}(X) \simeq \mathsf{hom}_{\mathcal{S}}(S^1, X) =: \mathscr{L}X$$

of its $\mathcal{S}$-enriched topological Hochschild homology with its *free loopspace*; in fact, the paracyclic space whose colimit computes $\mathsf{THH}^{\times}_{\mathcal{S}}(X)$ this is already constant. Of course, the unstable cyclotomic structure – a left $\mathbb{W}$-action – is given by precomposing with its action on the space $S^1 \in \mathcal{S}$.[30,31]

Now, note that $(\mathsf{B}\mathbb{N}^{\times})^{\mathsf{gpd}} \simeq \mathsf{B}\mathbb{Q}^{\times}_{>0}$, where $\mathbb{Q}^{\times}_{>0}$ denotes the commutative monoid of positive rational numbers under multiplication.[32] Hence, we find that

$$\begin{aligned}
\mathsf{TC}^{\times}_{\mathcal{S}}(X) &:= \mathsf{THH}^{\times}_{\mathcal{S}}(X)^{\mathsf{h}\mathbb{W}} \\
&\simeq \left(\mathsf{THH}^{\times}_{\mathcal{S}}(X)^{\mathsf{h}\mathbb{T}}\right)^{\mathsf{h}\mathbb{N}^{\times}} \\
&\simeq \left(\mathsf{hom}_{\mathcal{S}}((S^1)_{\mathsf{h}\mathbb{T}}, X)\right)^{\mathsf{h}\mathbb{N}^{\times}} \\
&\simeq X^{\mathsf{h}\mathbb{N}^{\times}} \\
&\simeq \mathsf{Fun}(\mathsf{B}\mathbb{N}^{\times}, X) \qquad (48) \\
&\simeq \mathsf{hom}_{\mathcal{S}}(\mathsf{B}\mathbb{Q}^{\times}_{>0}, X) ~,
\end{aligned}$$

where the equivalence (48) arises from the fact that we are taking homotopy $\mathbb{N}^{\times}$-fixedpoints with respect to the trivial action on X. Thus, in this case the unstable cyclotomic trace

$$X \simeq X^{\simeq} =: \mathsf{K}^{\times}_{\mathcal{S}}(X) \longrightarrow \mathsf{TC}^{\times}_{\mathcal{S}}(X) \simeq \mathsf{hom}_{\mathcal{S}}(\mathsf{B}\mathbb{Q}^{\times}_{>0}, X)$$

is the map induced by precomposition with the terminal map

$$\mathsf{pt} \longleftarrow \mathsf{B}\mathbb{Q}^{\times}_{>0} ~.$$

Example 5.11. Consider the ∞-category

$$\mathsf{Idem} \in \mathsf{Cat} \simeq \mathsf{Cat}(\mathcal{S}) ~,$$

[29] Due to our definition of unstable algebraic K-theory, it makes little sense to study the unstable cyclotomic trace for more general *flagged* ($\mathcal{S}$-enriched) ∞-categories.

[30] As the full subcategory $\mathsf{B}\mathbb{W} \subset \boldsymbol{\mathcal{M}}$ on the object $S^1 \in \boldsymbol{\mathcal{M}}$ lies in $\boldsymbol{\mathcal{M}}^{\mathsf{cls.cr}}$, it determines a functor $\mathsf{B}\mathbb{W}^{\mathsf{op}} \to \mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}}$ (recall Remark 1.9), which determines a functor $\mathsf{B}\mathbb{W}^{\mathsf{op}} \to \mathcal{S}$ on underlying spaces. This records a right $\mathbb{W}$-action on the object $S^1 \in \mathcal{S}$, which induces a left $\mathbb{W}$-action on the functor $\mathsf{hom}_{\mathcal{S}}(S^1, -)$.

[31] The unstable cyclotomic space $\mathsf{THH}^{\times}_{\mathcal{S}}(X) \simeq \mathscr{L}X$ satisfies the additional property that the $\mathbb{T}$-equivariant structure maps

$$\mathsf{THH}^{\times}_{\mathcal{S}}(X) \longrightarrow \mathsf{THH}^{\times}_{\mathcal{S}}(X)^{\mathsf{hC}_r}$$

are actually *equivalences*. (This need not be true of $\mathsf{THH}_{\mathcal{S}}(\mathcal{C})$ for an arbitrary ∞-category $\mathcal{C}$.)

[32] For a (discrete) commutative monoid M and its (discrete) group-completion G, it follows from Quillen's Theorem A that the induced functor $\mathsf{B}M \to \mathsf{B}G$ induces an equivalence on (∞-)groupoid completions, so that G is also the ∞-group completion of M.

the walking idempotent: this has a single object, which has a single non-identity endomorphism φ, which is equipped with a canonical equivalence $\varphi^{\circ n} \simeq \varphi$ for all $n \geq 1$. Then, we have an identification

$$\mathsf{THH}^{\times}_{\mathcal{S}}(\mathsf{Idem}) \simeq S^0 ,$$

where one point is the image of all cocycles in which all intervals are labeled by the identity endomorphism and the other point is the image of all cocycles in which at least one interval is labeled by φ. Of course, the $\mathbb{W}$-action is the trivial one, and we find that

$$\mathsf{TC}^{\times}_{\mathcal{S}}(\mathsf{Idem}) := \mathsf{THH}^{\times}_{\mathcal{S}}(\mathsf{Idem})^{\mathsf{h}\mathbb{W}} \simeq (S^0)^{\mathsf{h}\mathbb{W}} \simeq \mathsf{hom}_{\mathsf{Cat}}(\mathsf{B}\mathbb{W}, S^0) \simeq S^0 .$$

In this case, the unstable cyclotomic trace is the map

$$\mathsf{pt} \simeq \mathsf{Idem}^{\simeq} =: \mathsf{K}^{\times}_{\mathcal{S}}(\mathsf{Idem}) \longrightarrow \mathsf{TC}^{\times}_{\mathcal{S}}(\mathsf{Idem}) \simeq S^0$$

selecting the component corresponding to the cocycles in which all intervals are labeled by the identity endomorphism.

Example 5.12. Given a space X, consider the free associative monoid space

$$A \simeq \coprod_{n \geq 0} X^{\times n}$$

thereon and the resulting $\mathcal{S}$-enriched ∞-category

$$\mathfrak{B}A \in \mathsf{Cat}(\mathcal{S}) \subset \mathsf{fCat}(\mathcal{S}) \ :$$

this has a single object, its space of endomorphisms is A, and composition is given by concatenation (so that indeed the monoid of automorphisms is contractible). It is not hard to see that we have a canonical decomposition

$$\mathsf{THH}^{\times}_{\mathcal{S}}(\mathfrak{B}A) \simeq \coprod_{n \geq 0} \mathsf{Conf}_n(S^1; X)$$

of $\mathsf{THH}_{\mathcal{S}}(\mathfrak{B}A)$ as a coproduct of configuration spaces of n distinct X-labeled points in the manifold S^1.[33]

Now, in order to understand

$$\mathsf{TC}^{\times}_{\mathcal{S}}(\mathfrak{B}A) := \mathsf{THH}^{\times}_{\mathcal{S}}(\mathfrak{B}A)^{\mathsf{h}\mathbb{W}} \simeq \left(\mathsf{THH}^{\times}_{\mathcal{S}}(\mathfrak{B}A)^{\mathsf{h}\mathbb{T}}\right)^{\mathsf{h}\mathbb{N}^{\times}} ,$$

we first examine

$$\mathsf{THH}^{\times}_{\mathcal{S}}(\mathfrak{B}A)^{\mathsf{h}\mathbb{T}} \simeq \left(\coprod_{n \geq 0} \mathsf{Conf}_n(S^1; X)\right)^{\mathsf{h}\mathbb{T}} \simeq \coprod_{n \geq 0} \mathsf{Conf}_n(S^1; X)^{\mathsf{h}\mathbb{T}} ,$$

where the second equivalence comes from the fact that coproducts commute with connected limits (and the fact that *any* $\mathbb{T}$-action decomposes over coproducts).

Let us first consider the case that $X = \mathsf{pt}$. For any $n \geq 1$, it is not hard to see that we have an equivalence

$$\mathsf{Conf}_n(S^1; \mathsf{pt}) \simeq (\mathbb{T}/\mathsf{C}_n)$$

of $\mathbb{T}$-modules, so that in this case

$$\mathsf{Conf}_n(S^1; \mathsf{pt})^{\mathsf{h}\mathbb{T}} \simeq \varnothing .$$

[33] To adhere more closely to the definition of factorization homology, these might be more properly called "configuration spaces of framed open intervals in the framed circle", although the spaces themselves are equivalent.

But we always have a $\mathbb{T}$-equivariant map

$$\mathsf{Conf}_n(S^1; X) \longrightarrow \mathsf{Conf}_n(S^1; \mathsf{pt})$$

by functoriality (i.e. by pushforward of labels along the terminal map $X \to \mathsf{pt}$), so that

$$\mathsf{Conf}_n(S^1; X)^{\mathsf{h}\mathbb{T}} \simeq \varnothing$$

for $n \geq 1$ as well. As $\mathsf{Conf}_0(S^1; X) \simeq \mathsf{pt}$, we thus conclude that

$$\mathsf{THH}^{\times}_{\mathcal{S}}(\mathfrak{B}A)^{\mathsf{h}\mathbb{T}} \simeq \mathsf{pt}^{\mathsf{h}\mathbb{T}} \simeq \mathsf{pt}$$

is contractible. It follows that

$$\mathsf{TC}^{\times}_{\mathcal{S}}(\mathfrak{B}A) \simeq \mathsf{pt}^{\mathsf{h}\mathbb{N}^{\times}} \simeq \mathsf{pt}$$

is contractible as well. So in this case, the unstable cyclotomic trace

$$\mathsf{pt} \simeq (\mathfrak{B}A)^{\simeq} \simeq \mathsf{K}^{\times}_{\mathcal{S}}(\mathfrak{B}A) \longrightarrow \mathsf{TC}^{\times}_{\mathcal{S}}(\mathfrak{B}A) \simeq \mathsf{pt}$$

is the unique endomorphism of the contractible space pt.

Appendix A. Cartesian factorization homology over disk-stratified 1-manifolds

Notation A.1. For any stratified space M, we write

$$\mathsf{Enter}(M)$$

for the ∞-category of *entering paths* in M: this is the opposite of the ∞-category $\mathsf{Exit}(M)$ of [AFRb, Definition 3.3.1 and Corollary 3.3.6], whose space of n composable morphisms is the space of maps of stratified spaces from Δ^n (the standardly stratified n-simplex) to M.

Remark A.2. The space of objects of $\mathsf{Enter}(M)$ is the disjoint union of its strata. Its morphisms – that is, entering paths in M – may either stay within a stratum (in which case they are equivalences) or pass from a higher-dimensional stratum to a lower-dimensional stratum. So for $M \in \mathcal{M}$ a stratified 1-manifold, the only nonequivalence morphisms in $\mathsf{Enter}(M)$ run from its 1-dimensional strata to its 0-dimensional strata, and for each vertex $v \in M^{(0)}$ we have an equivalence

$$\mathsf{Enter}(M)_{/v} \simeq \mathsf{Link}_v(M)^{\triangleright}$$

of its overcategory with the right cone on its link (which in this case is just a set). In particular, for a disk-stratified 1-manifold $R \in \mathcal{D}$, we see that $\mathsf{Enter}(R)$ is a category of depth at most 1; in fact, this is a poset so long as R has no edges that start and end at the same vertex, which situation gives rise to two parallel morphisms.

Definition A.3. The ***maximal*** (***closed-creation***) ***cover*** of a disk-stratified 1-manifold $R \in \mathcal{D}$ is the diagram

$$\mathsf{Enter}(R) \xrightarrow{\mathsf{mc}_R} \mathcal{D}_{R/} \hookrightarrow \mathcal{M}_{R/}$$

defined as follows:

- each 0-dimensional stratum of R is taken to the closed morphism

  $$R \longrightarrow \mathbb{D}^0$$

  corresponding contravariantly to its inclusion;

- each 1-dimensional stratum of R is taken to the evident closed-creation morphism

$$R \longrightarrow \mathbb{D}^1$$

which it defines;[34]

- each nonidentity morphism is taken to the unique morphism making the resulting triangle

$$\begin{tikzcd} & \mathbb{D}^1 \arrow[dd, dashed] \\ R \arrow[ur] \arrow[dr] & \\ & \mathbb{D}^0 \end{tikzcd}$$

commute.

Remark A.4. As $\boldsymbol{\mathcal{D}}$ is a 1-category, there are no issues of homotopy coherence in Definition A.3. Indeed, the functor $\mathsf{Enter}(R) \xrightarrow{\mathsf{mc}_R} \boldsymbol{\mathcal{D}}_{R/}$ is a monomorphism, and so it can be *defined* simply by specifying its image.[35]

Observation A.5. The maximal cover of a disk-stratified 1-manifold $R \in \boldsymbol{\mathcal{D}}$ admits a factorization

$$\begin{tikzcd} \mathsf{Enter}(R) \arrow[rr, hook, "\mathsf{mc}_R"] \arrow[dr, hook, dashed, "\mathsf{mc}_R"'] & & \boldsymbol{\mathcal{D}}_{R/} \\ & \left(\boldsymbol{\Delta}^{\mathsf{op,cls}}_{\leq 1}\right)_{R/^{\mathsf{cls.cr}}} := \left(\boldsymbol{\Delta}^{\mathsf{op,cls}}_{\leq 1}\right) \underset{\boldsymbol{\mathcal{D}}}{\times} \boldsymbol{\mathcal{D}}_{R/} \underset{\mathsf{Ar}(\boldsymbol{\mathcal{D}})}{\times} \mathsf{Ar}^{\mathsf{cls.cr}}(\boldsymbol{\mathcal{D}}) \arrow[ur, hook] & \end{tikzcd} , \tag{49}$$

yielding a commutative triangle

$$\begin{tikzcd} & \mathsf{Enter}(R) \arrow[dl, "\mathsf{mc}_R"'] \arrow[dr, "\mathsf{mc}_R"] & \\ \boldsymbol{\Delta}^{\mathsf{op}} \underset{\boldsymbol{\mathcal{D}}}{\times} \boldsymbol{\mathcal{D}}_{R/} \underset{\mathsf{Ar}(\boldsymbol{\mathcal{D}})}{\times} \mathsf{Ar}^{\mathsf{cls.cr}}(\boldsymbol{\mathcal{D}}) =: (\boldsymbol{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr}}} \arrow[rr] & & (\boldsymbol{\Delta}^{\mathsf{op}})_{R/} := \boldsymbol{\Delta}^{\mathsf{op}} \underset{\boldsymbol{\mathcal{D}}}{\times} \boldsymbol{\mathcal{D}}_{R/} \end{tikzcd} \tag{50}$$

in $\mathsf{Cat}_{/\boldsymbol{\Delta}^{\mathsf{op}}}$.

Definition A.6. A morphism

$$\begin{tikzcd} \mathcal{K}_0 \arrow[rr] \arrow[dr] & & \mathcal{K}_1 \arrow[dl] \\ & \boldsymbol{\Delta}^{\mathsf{op}} & \end{tikzcd}$$

in $\mathsf{Cat}_{/\boldsymbol{\Delta}^{\mathsf{op}}}$ is called ***Segal-initial*** if for every complete ∞-category $\mathcal{X}$ and every category object (i.e. Segal object)

$$\mathcal{Y} \in \mathsf{Fun}(\boldsymbol{\Delta}^{\mathsf{op}}, \mathcal{X}) ,$$

[34] This will be a closed morphism unless the endpoints of the 1-dimensional stratum agree, in which case the corresponding proper constructible bundle will still be injective on 1-dimensional strata but not on 0-dimensional strata.

[35] As the object R itself may have automorphisms, the functor $\mathsf{Enter}(R)^{\triangleleft} \xrightarrow{\mathsf{mc}_R} \boldsymbol{\mathcal{D}}$ need not be a monomorphism.

the induced morphism

$$\mathsf{lim}\left(\mathcal{K}_0 \longrightarrow \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathcal{Y}} \mathcal{X}\right) \longleftarrow \mathsf{lim}\left(\mathcal{K}_1 \longrightarrow \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathcal{Y}} \mathcal{X}\right)$$

on limits is an equivalence.

Now, our identification of the value of cartesian factorization homology (of category objects) over disk-stratified 1-manifolds is a consequence of the following result.

Proposition A.7. *For any $R \in \mathfrak{D}$, the functor*

$$\mathsf{Enter}(R) \xrightarrow{\mathsf{mc}_R} (\mathbf{\Delta}^{\mathsf{op}})_{R/}$$

is Segal-initial.

In order to prove Proposition A.7, we use the factorization (50) via the following two results.

Lemma A.8. *For any $R \in \mathfrak{D}$, the functor*

$$\mathsf{Enter}(R) \xrightarrow{\mathsf{mc}_R} (\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr}}}$$

is Segal-initial.

Lemma A.9. *For any $R \in \mathfrak{D}$, the functor*

$$(\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr}}} \longrightarrow (\mathbf{\Delta}^{\mathsf{op}})_{R/}$$

is initial.

Proof of Proposition A.7. The claim follows from Lemmas A.8 and A.9, along with the evident facts that initial functors over $\mathbf{\Delta}^{\mathsf{op}}$ are Segal-initial and that Segal-initiality is preserved under composition. □

The proofs of Lemmas A.8 and A.9 will occupy the remainder of this section. We first prove Lemma A.9; the proof of Lemma A.8 will require some preliminaries, upon which embark thereafter.

Proof of Lemma A.9. Fix an arbitrary object

$$(R \to T) \in (\mathbf{\Delta}^{\mathsf{op}})_{R/}~.$$

By Quillen's Theorem A, it suffices to show that the category

$$\left((\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr}}}\right)_{/(R\to T)} := (\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr}}} \underset{(\mathbf{\Delta}^{\mathsf{op}})_{R/}}{\times} \left((\mathbf{\Delta}^{\mathsf{op}})_{R/}\right)_{/(R\to T)} \tag{51}$$

of factorizations

$$\begin{tikzcd} R \arrow[rr] \arrow[dr, dashed, "\mathsf{cls.cr}"'] & & T \\ & T' \arrow[ur, dashed] & \end{tikzcd} \tag{52}$$

has contractible groupoid completion. Observe that the inclusion of the full subcategory

$$(\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr}}/^{\mathsf{act}}T} := \left((\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr}}}\right)_{/^{\mathsf{act}}(R\to T)} \tag{53}$$

of factorizations

$$\begin{tikzcd} R \arrow[rr] \arrow[dr, dashed, "\mathsf{cls.cr}"'] & & T \\ & T' \arrow[ur, dashed, "\mathsf{act}"'] & \end{tikzcd} \tag{54}$$

has a left adjoint, obtained by applying the closed-active factorization system on $\mathbf{\Delta}^{\mathsf{op}}$ to the morphism $T' \to T$ in a factorization (52). Since adjoint functors induces equivalences on groupoid completions, it suffices to show that the category (53) has contractible groupoid completion. For this, we explicitly construct an initial object therein.

Suppose that our chosen morphism $R \to T$ in $\mathfrak{D}$ is classified by the diagram

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d] & F \arrow[d] & T \arrow[l, hook'] \arrow[d] \\ \Delta^{\{0\}} \arrow[r, hook] & \Delta^1 & \Delta^{\{1\}} \arrow[l, hook'] \end{tikzcd}$$

of stratified spaces. Consider the category

$$\Gamma(\mathsf{Enter}(F)) := \Gamma\left(\begin{tikzcd} \mathsf{Enter}(F) \arrow[d] \\ \mathsf{Enter}(\Delta^1) \end{tikzcd}\right) \simeq \lim\left(\begin{tikzcd} & & \mathsf{Enter}(R) \arrow[d] \\ & \mathsf{Ar}(\mathsf{Enter}(F)) \arrow[r, "t"] \arrow[d, "s"'] & \mathsf{Enter}(F) \\ \mathsf{Enter}(T) \arrow[r] & \mathsf{Enter}(F) & \end{tikzcd}\right).$$

Its objects are entering paths in F which start in T and end in R, or equivalently sections of $F \downarrow \Delta^1$; its morphisms are entering paths among such entering paths. Thus, it admits a functor to the poset $\{(11) \to (01) \to (00)\}$ classifying its types of objects: a section is sent to the object (ij) exactly when its value on $\Delta^{\{0\}}$ lies in an i-dimensional stratum of R and its value on $\Delta^{\{1\}}$ lies in a j-dimensional stratum of T.

Now, purely closed covers are limit diagrams [AFRa] (a fact which we will use repeatedly [?]). It follows that the limit

$$\tilde{F} := \lim\left(\Gamma(\mathsf{Enter}(F)) \xrightarrow{\mathsf{ev}_0} \mathsf{Enter}(R) \xrightarrow{\mathsf{mc}_R} (\mathbf{\Delta}^{\mathsf{op}})_{R/} \longrightarrow \mathbf{\Delta}^{\mathsf{op}} \xhookrightarrow{\langle - \rangle} \mathfrak{D} \hookrightarrow \mathfrak{M}\right)$$

exists. Indeed, the composite functor defining $\tilde{F}$ takes a section of $F \downarrow \Delta^1$ of type (ij) to a copy of $\mathbb{D}^i$, a morphism lying over $(11) \to (01)$ to a closed morphism $\mathbb{D}^1 \to \mathbb{D}^0$, and a morphism lying over $(01) \to (00)$ to the unique equivalence. Thus, $\tilde{F}$ lies in the subcategory $\mathfrak{D} \subset \mathfrak{M}$, and has one edge for each section of $F \downarrow \Delta^1$ of type (11), and its vertices are in bijection with the set of connected components of the groupoid completion of the pullback

$$\Gamma(\mathsf{Enter}(F))_{|((01) \to (00))} . \tag{55}$$

We will show in stages that this object $\tilde{F} \in \mathfrak{D}$ naturally defines an initial object of the category (53), thereby proving the claim.

We begin by showing that the object $\tilde{F} \in \mathfrak{D}$ actually lies in the image of the functor $\langle - \rangle$, i.e. that it is simply a refinement of $\mathbb{D}^1$ or $\mathbb{D}^0$. For this, we consider the blowup $B := \mathsf{Bl}_R(F)$ of F along

R, as well as the link $L := \mathsf{Link}_R(F)$ of R in F. These participate in a commutative diagram

$$\begin{tikzcd}
L \arrow[r, hook] \arrow[d] & B \arrow[d] & T \arrow[l, hook'] \arrow[d, "\wr"] \\
R \arrow[r, hook] \arrow[d] & F \arrow[d] & T \arrow[l, hook'] \arrow[d] \\
\Delta^{\{0\}} \arrow[r, hook] & \Delta^1 & \Delta^{\{1\}} \arrow[l, hook']
\end{tikzcd}$$

of stratified spaces whose vertical morphisms are all proper constructible bundles, through which B determines a refinement morphism $L \xrightarrow{\mathsf{ref}} T$ lying in $\mathfrak{D}$ and hence in $\mathbf{\Delta}^{\mathsf{op}} \subset \mathfrak{D}$. Consider the resulting lax-commutative diagram

$$\begin{tikzcd}
\Gamma(\mathsf{Enter}(B)) \arrow[r, "\mathsf{ev}_0"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \mathsf{Enter}(L) \arrow[dr, "\mathsf{mc}_L"] & & & \\
& & \mathbf{\Delta}^{\mathsf{op}} \arrow[r, hook, "\mathsf{f.f.}"] & \mathfrak{D} \arrow[r, hook, "\mathsf{f.f.}"] & \mathfrak{M} \\
\Gamma(\mathsf{Enter}(F)) \arrow[r, "\mathsf{ev}_0"'] & \mathsf{Enter}(R) \arrow[ur, "\mathsf{mc}_R"'] & & &
\end{tikzcd} \quad . \tag{56}$$

By definition, the limit of the lower composite $\Gamma(\mathsf{Enter}(F)) \to \mathfrak{M}$ is $\tilde{F}$. A priori the limit of the upper composite should define an object $\tilde{B}$, but since B defines a refinement morphism $L \xrightarrow{\mathsf{ref}} T$ then the upper functor ev_0 is initial and this limit is simply L. As we will show momentarily, the left vertical functor in diagram (56) is a localization, hence in particular initial. It follows that the natural transformation of diagram (56) defines a morphism

$$\tilde{F} \longrightarrow L \tag{57}$$

in $\mathfrak{D}$ on limits over $\Gamma(\mathsf{Enter}(B))$. Examining this natural transformation, we see that its components are all either equivalences or the unique creation morphism $\mathbb{D}^0 \xrightarrow{\mathsf{cr}} \mathbb{D}^1$. Since L lies in the image of $\langle - \rangle$, it follows that $\tilde{F}$ indeed lies in the image of $\langle - \rangle$, as claimed.

To argue that the left vertical functor in diagram (56) is a localization, we observe from the construction of the blowup that sections of $\mathsf{Enter}(B)$, considered as morphisms in $\mathsf{Enter}(B)$, are equivalent data to factorizations of their images in $\mathsf{Enter}(F)$: in other words, we have an equivalence

$$\Gamma(\mathsf{Enter}(B)) \xrightarrow{\sim} \mathsf{Fun}([2], \mathsf{Enter}(F))^{|\mathsf{Enter}(T)}_{|\mathsf{Enter}(R)} := \lim \left(\begin{tikzcd}
& & \mathsf{Enter}(T) \arrow[d] \\
& \mathsf{Fun}([2], \mathsf{Enter}(F)) \arrow[r, "\mathsf{ev}_0"'] \arrow[d, "\mathsf{ev}_2"] & \mathsf{Enter}(F) \\
\mathsf{Enter}(R) \arrow[r] & \mathsf{Enter}(F) &
\end{tikzcd} \right)$$

of categories. Moreover, the left vertical functor in diagram (56) factors through the composite

$$\begin{tikzcd}
\Gamma(\mathsf{Enter}(B)) \arrow[rr, "\sim"] \arrow[dd] & & \mathsf{Fun}([2], \mathsf{Enter}(F))^{|\mathsf{Enter}(T)}_{|\mathsf{Enter}(R)} \arrow[dl] \\
& \Gamma(\mathsf{Enter}(F)) \underset{\mathsf{Enter}(R)}{\times} \mathsf{Ar}(\mathsf{Enter}(R)) \arrow[dl] & \\
\Gamma(F) & &
\end{tikzcd}$$

in which the upper diagonal functor is the right adjoint to the evident fully faithful inclusion while the lower diagonal functor is the left adjoint to the evident fully faithful inclusion (so both are given by some manner of composition). In particular, both diagonal functors are localizations, and so the left vertical functor in diagram (56) is indeed a localization, as desired.

We now proceed to construct a canonical factorization

$$\begin{tikzcd}
R \arrow[rr] \arrow[dr, dashed, "\mathsf{cls.cr}"'] & & T \\
& \tilde{F} \arrow[ur, dashed, "\mathsf{act}"'] &
\end{tikzcd} \tag{58}$$

in $\mathfrak{D}$, thereby obtaining an object of the category (53). Firstly, the morphism $R \to \tilde{F}$ arises from the functoriality of limits as it applies to the morphism

$$\begin{tikzcd}
\Gamma(\mathsf{Enter}(F)) \arrow[rr, "\mathsf{ev}_0"] \arrow[dr] & & \mathsf{Enter}(R) \arrow[dl, "\mathsf{mc}_R"] \\
& \mathfrak{M} &
\end{tikzcd}$$

in $\mathsf{Cat}_{/\mathfrak{M}}$. Examining the definitions, we see that it is indeed closed-creation: it corresponds contravariantly to the proper constructible bundle taking each stratum of $\tilde{F}$ to its "stratum of origin" in R. Secondly, we take our morphism $\tilde{F} \to T$ to be the composite

$$\tilde{F} \xrightarrow{(57)} L \xrightarrow{B} T ,$$

in which both morphisms are active by construction. It remains to show that the composite

$$R \longrightarrow \tilde{F} \longrightarrow T$$

recovers the morphism $R \xrightarrow{F} T$. For this, we explicitly construct the composition data of the commutative triangle (58) by pulling back the proper constructible bundle $B \downarrow \Delta^1$ along the morphism $\Delta^2 \xrightarrow{\sigma_0} \Delta^1$ of stratified spaces and then taking the pushout of the fiber $L \times \Delta^{\{0<1\}}$ along the evident proper constructible bundle

$$\begin{tikzcd}
L \times \Delta^{\{0<1\}} \arrow[rr] \arrow[dr] & & \mathsf{Cylr}(R \longleftarrow \tilde{F}) \arrow[dl] \\
& \Delta^{\{0<1\}} &
\end{tikzcd}$$

among proper constructible bundles over $\Delta^{\{0<1\}}$.

Finally, we show that the object (58) $\in$ (53) is initial. Given an arbitrary object (54) $\in$ (53), we first construct a morphism (58) $\to$ (54) and then show that it is unique. For this, observe first that the composite functor

$$(53) \longrightarrow \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{(-)^{(0)}} \mathsf{Fin}^{\mathsf{op}} \tag{59}$$

consists of faithful functors – the first being faithful because $\mathfrak{D}$ is a 1-category. Thus, we proceed as follows: we construct a function $T'^{(0)} \to \tilde{F}^{(0)}$ on sets of vertices, we show that it admits a lift to (53), and finally we show that any morphism (58) $\to$ (54) has the same image in $\mathsf{Fin}^{\mathsf{op}}$.

So, choose any vertex t' of T'. We choose sections of the proper constructible bundles classifying the morphisms $R \to T'$ and $T' \to T$ that restrict to the vertex t': the first is unique by proper-constructibility, and the second exists by the assumed activeness. Since the simplicial space $\mathcal{E}\mathsf{nter}$ (the opposite of $\mathcal{E}\mathsf{xit}$) is a Segal space [AFRb, Theorem 6.4.2], these sections compose uniquely to a section of the morphism $R \xrightarrow{F} T$, which defines a vertex of $\tilde{F}$ by its construction. That this resulting vertex of $\tilde{F}$ is independent of the choice of active extension follows immediately from our description of the vertices of $\tilde{F}$. So we obtain a well-defined function $T'^{(0)} \to \tilde{F}^{(0)}$, as desired.

To show that the corresponding morphism in $\mathsf{Fin}^{\mathsf{op}}$ lifts to a morphism in (53), we first show that it lifts to a morphism in $\mathbf{\Delta}^{\mathsf{op}}$. For this, recall that the vertices of $\tilde{F}$ are simply equivalence classes of certain sections of $\mathsf{Exit}(F)$, which inherit an ordering from the functor $\Gamma(\mathsf{Exit}(F)) \xrightarrow{\mathsf{ev}_1} \mathsf{Exit}(T)$ and the evident ordering on the strata of T. So it follows from the construction of the function $T'^{(0)} \to \tilde{F}^{(0)}$ that it preserves linear orders, and hence lifts uniquely to a morphism $T'^{\circ} \to \tilde{F}^{\circ}$ in $\mathbf{\Delta}$.

Now, by construction, we have a commutative diagram

$$\begin{tikzcd}
& \tilde{F}^{(0)} \arrow[dr] \arrow[dd] & \\
R^{(0)} \arrow[ur] \arrow[rr] \arrow[dr] & & T^{(0)} \\
& T'^{(0)} \arrow[ur] &
\end{tikzcd} , \tag{60}$$

which we consider as a functor $[3] \to \mathsf{Fin}^{\mathsf{op}}$. Since the functor $\mathbf{\Delta}^{\mathsf{op}} \xrightarrow{(-)^{(0)}} \mathsf{Fin}^{\mathsf{op}}$ is conservative, there exists a unique lift of the restriction $\{1 < 2 < 3\} \to \mathsf{Fin}^{\mathsf{op}}$ to $\mathbf{\Delta}^{\mathsf{op}} \subset \mathfrak{D}$. Since $\mathfrak{D}$ is an ordinary category, to obtain a lift $[3] \to \mathfrak{D}$ it suffices to obtain a lift $\{0 < 1 < 2\} \cong [2] \to \mathfrak{D}$ extending the given lift $\partial[2] \to \mathfrak{D}$. For this, we check over each term in mc_R. Over a vertex of R, there is nothing to check since the object $\mathbb{D}^0 \in \mathfrak{D}$ is initial. Over a closed-creation morphism $R \xrightarrow{\mathsf{cls.cr}} \mathbb{D}^1$, we have that $\tilde{F}$ is a disjoint union of copies of $\mathbb{D}^0$ and $\mathbb{D}^1$, while T' is a disjoint union of objects in the image of $\langle - \rangle$, and we have already constructed a morphism from the former to the latter in $\mathfrak{D}$. The commutativity of this new triangle starting at $\mathbb{D}^1$ follows from the commutativity of its image in $\mathsf{Fin}^{\mathsf{op}}$, which in turn follows from the commutativity of the triangle $\{0 < 1 < 2\} \to \mathsf{Fin}^{\mathsf{op}}$.

We now conclude the proof by showing that this morphism (58) $\to$ (54) in the category (53) is unique. So suppose we are given an arbitrary such morphism, i.e. a commutative diagram

$$\begin{tikzcd}
 & \tilde{F} \arrow[dr, "\mathsf{act}"] \arrow[dd] & \\
R \arrow[ur, "\mathsf{cls.cr}"] \arrow[rr] \arrow[dr, "\mathsf{cls.cr}"'] & & T \\
 & T' \arrow[ur, "\mathsf{act}"'] &
\end{tikzcd} \tag{61}$$

in $\mathfrak{D}$. We will see that the postcomposition

$$[3] \xrightarrow{(61)} \mathfrak{D} \xrightarrow{(-)^{(0)}} \mathsf{Fin}^{\mathsf{op}}$$

agrees with the diagram (60), which by the conservativity of the latter functor completes the proof. In fact, because we are only considering a double slice category (i.e. a subcategory of $\mathsf{Fun}([3], \mathfrak{D})$ whose values on $0 \in [3]$ and $3 \in [3]$ are fixed), it suffices to show this after precomposition with the functor $\{1 < 2\} \hookrightarrow [3]$: in other words, it suffices to show that the induced morphisms

$$T'^{(0)} \longrightarrow \tilde{F}^{(0)}$$

agree.

We will show this by considering the proper constructible bundle over Δ^3 which classifies the commutative diagram (61). For any face $\Delta^{\{i_0 < \cdots < i_k\}} \subset \Delta^3$, let us write $\Gamma'_{i_0,\ldots,i_k}$ for the subset of those of its sections over $\Delta^{\{i_0 < \cdots < i_k\}}$ which select 0-dimensional strata over all vertices that they contain except possibly $\Delta^{\{3\}}$. Then, our argument consists in chasing the commutative diagram

$$\begin{tikzcd}
 & \Gamma'_{023} \arrow[r, "\sim"] \arrow[dl] & \Gamma'_{23} \arrow[r, two heads] & \Gamma'_{2} \\
\Gamma'_{03} \arrow[rrr, dashed, bend right] & \Gamma'_{0123} \arrow[r, "\sim"] \arrow[u, "\wr"'] \arrow[d] & \Gamma'_{123} \arrow[r, two heads] \arrow[u, "\wr"'] \arrow[d] & \Gamma'_{12} \arrow[u, "\wr"] \arrow[d] \\
 & \Gamma'_{013} \arrow[r, "\sim"'] \arrow[ul, two heads, bend left] & \Gamma'_{13} \arrow[r, two heads] & \Gamma'_{1}
\end{tikzcd} \tag{62}$$

in Fin, which we now proceed to explain. First of all, all solid arrows are given by restriction of sections, which implies that the solid part of the diagram commutes. Those marked as equivalences are so due to the unique existence of a leftwards extension of a section selecting a 0-dimensional stratum, while those rightwards arrows marked as surjections are so due to the existence of an extension by the definition of active morphisms in $\mathfrak{D}$ and the fact that $\mathcal{E}\mathsf{nter}$ is a Segal space. The existence of the dashed morphism is special to the situation: it is due to the definition of $\tilde{F}$, as are the surjectivity of the restriction $\Gamma'_{013} \to \Gamma'_{03}$ and the commutativity of the triangle in which these two morphisms participate. Now, by definition, the morphism (58) $\xrightarrow{(61)}$ (54) induces the function

$$T'^{(0)} \simeq \Gamma'_2 \xleftarrow{\sim} \Gamma'_{12} \longrightarrow \Gamma'_1 \simeq \tilde{F}^{(0)}$$

displayed along the right column of diagram (62), while the morphism we have constructed operates (which only makes reference to the face $\Delta^{\{0<2<3\}} \subset \Delta^3$, note) by composing any dotted section in the diagram

$$\begin{tikzcd}
T'^{(0)} \simeq \Gamma'_2 \arrow[r, dotted, bend left] & \Gamma'_{23} \arrow[l, two heads] & \Gamma'_{023} \arrow[l, "\sim"'] \arrow[r] & \Gamma'_{03} \arrow[r, dashed] & \Gamma'_1 \simeq \tilde{F}'
\end{tikzcd}$$

with the remaining sequence of composable functions extracted from diagram (62), which we argued previously to be independent of the choice of section. The equality of the two functions thus follows from the commutativity of diagram (62). $\square$

We now work towards the proof of Lemma A.8.

Observation A.10. The inclusion $\mathfrak{M}^{\mathsf{cls.cr}} \hookrightarrow \mathfrak{M}$ preserves all pushouts that exist in the source; this is direct from the closed-active factorization system on $\mathfrak{M}$. Moreover, recalling Remark 1.9, we see that those pushouts in $\mathfrak{M}^{\mathsf{cls.cr}}$ that exist can be described quite explicitly; in particular, the pushout of a closed-creation map along a closed map exists, and is simply computed by pullback of proper constructible bundles.

Observation A.11. Consider a fixed object

$$(R \xrightarrow{\varphi} \langle n\rangle) \in (\mathbf{\Delta}^{\mathsf{op}})_{R/} := \mathbf{\Delta}^{\mathsf{op}} \underset{\mathfrak{D}}{\times} \mathfrak{D}_{R/} ~.$$

For another object $(R \xrightarrow{\psi} \langle m\rangle) \in (\mathbf{\Delta}^{\mathsf{op}})_{R/}$, by Observation A.10, the pushout of the cospan

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\psi"'] & \langle n\rangle \\ \langle m\rangle & \end{tikzcd}$$

in $\mathfrak{D}$ is guaranteed to exist when φ is closed-creation and ψ is closed, in which case it will lie in the image of the embedding

$$\mathbf{\Delta}^{\mathsf{op}} \xhookrightarrow{\langle - \rangle} \mathfrak{D} ~.$$

It follows that when φ is closed-creation, cobase change of the maximal cover of R defines a purely closed cover

$$\mathsf{Enter}(R) \longrightarrow (\mathbf{\Delta}^{\mathsf{op}})_{\langle n\rangle/} \simeq (\mathbf{\Delta}^{\mathsf{op}})_{[n]^\circ/}$$

of $\langle n\rangle \in \mathfrak{D}$.

Remark A.12. In other words, in Observation A.11 we're disallowing the possibility that an internal vertex of an edge of R is refined away, as in the map $[2]^\circ \xrightarrow{\delta_1} [1]^\circ$ in $\mathbf{\Delta}^{\mathsf{op}} \subset \mathfrak{D}$.

Proof of Lemma A.8. For any object

$$(R \to \langle n\rangle) \in (\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr}}} ~,$$

by Observation A.11 cobase change gives a diagram

$$\mathsf{Enter}(R) \longrightarrow (\mathbf{\Delta}^{\mathsf{op}})_{[n]^\circ/} ~,$$

which is adjoint to a purely closed cover

$$\mathsf{Enter}(R)^{\triangleleft} \longrightarrow \mathbf{\Delta}^{\mathsf{op}}$$

(where the cone point is sent to $[n]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$). This construction assembles into a functor

$$(\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr}}} \longrightarrow \mathsf{Fun}\left(\mathsf{Enter}(R)^{\triangleleft}, \mathbf{\Delta}^{\mathsf{op}}\right) ~,$$

which factors through the full subcategory

$$\mathsf{Fun}^{\mathsf{cls.lim}}\left(\mathsf{Enter}(R)^{\triangleleft}, \mathbf{\Delta}^{\mathsf{op}}\right) \subset \mathsf{Fun}\left(\mathsf{Enter}(R)^{\triangleleft}, \mathbf{\Delta}^{\mathsf{op}}\right)$$

on those objects which factor through the subcategory $\mathbf{\Delta}^{\mathsf{op,cls}} \subset \mathbf{\Delta}^{\mathsf{op}}$ and define limit diagrams (therein or equivalently in $\mathbf{\Delta}^{\mathsf{op}}$).

Now suppose we are given any category object $\mathcal{Y} \in \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{X})$ in a complete ∞-category $\mathcal{X}$. Then, we obtain a factorization

$$\begin{tikzcd}
\mathsf{Fun}^{\mathsf{cls.lim}}\left(\mathsf{Enter}(R)^{\triangleleft}, \mathbf{\Delta}^{\mathsf{op}}\right) \arrow[r, hook, "\mathsf{f.f.}"] \arrow[drr, dashed, "\mathcal{Y}\circ-"'] & \mathsf{Fun}\left(\mathsf{Enter}(R)^{\triangleleft}, \mathbf{\Delta}^{\mathsf{op}}\right) \arrow[r, "\mathcal{Y}\circ-"] & \mathsf{Fun}\left(\mathsf{Enter}(R)^{\triangleleft}, \mathcal{X}\right) \\
 & & \mathsf{Fun}^{\mathsf{lim}}\left(\mathsf{Enter}(R)^{\triangleleft}, \mathcal{X}\right) \arrow[u, hook, "\mathsf{f.f.}"']
\end{tikzcd}$$

through the full subcategory of limit diagrams. Consider the resulting composite

$$\left(\mathbf{\Delta}^{\mathsf{op}}\right)_{R/^{\mathsf{cls.cr}}} \longrightarrow \mathsf{Fun}^{\mathsf{cls.lim}}\left(\mathsf{Enter}(R)^{\triangleleft}, \mathbf{\Delta}^{\mathsf{op}}\right) \xrightarrow{\mathcal{Y}\circ-} \mathsf{Fun}^{\mathsf{lim}}\left(\mathsf{Enter}(R)^{\triangleleft}, \mathcal{X}\right) \ . \tag{63}$$

Consider the value of $\mathsf{lim}(63)$ on the cone point of $\mathsf{Enter}(R)^{\triangleleft}$. Since limits are computed pointwise, this is simply

$$\mathsf{lim}\left(\left(\mathbf{\Delta}^{\mathsf{op}}\right)_{R/^{\mathsf{cls.cr}}} \hookrightarrow \left(\mathbf{\Delta}^{\mathsf{op}}\right)_{R/} \longrightarrow \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathcal{Y}} \mathcal{X}\right) \ .$$

On the other hand, by the Fubini theorem for limits, we can also identify the value of $\mathsf{lim}(63)$ on the cone point of $\mathsf{Enter}(R)^{\triangleleft}$ as

$$\mathsf{lim}\left(\mathsf{Enter}(R) \xrightarrow{\mathsf{mc}_R} \left(\mathbf{\Delta}^{\mathsf{op}}\right)_{R/} \longrightarrow \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathcal{Y}} \mathcal{X}\right) \ ,$$

which proves the claim. □

We end this section by proving a variant of Proposition A.7, which is necessary for the proof of Lemma 3.8.

Observation A.13. The commutative triangle (50) admits a refinement to a commutative diagram

$$\begin{tikzcd}
\mathsf{Enter}(R) \arrow[d, dashed, "\alpha"', "\mathsf{mc}_R"] \arrow[dr, "\mathsf{mc}_R", "\gamma"'] & \\
\mathbf{\Delta}^{\mathsf{op}} \underset{\mathfrak{D}}{\times} \mathfrak{D}_{R/} \underset{\mathsf{Ar}(\mathfrak{D})}{\times} \mathsf{Ar}\left(\mathfrak{D}^{\mathsf{cls.cr}}_{|\mathsf{Fin}_*}\right) =: \left(\mathbf{\Delta}^{\mathsf{op}}\right)_{R/^{\mathsf{cls.cr},*}} \arrow[r, hook] \arrow[d, hook, "\beta"'] & \left(\mathbf{\Delta}^{\mathsf{op}}\right)_{R/^{\mathsf{cls.cr}}} \arrow[d, hook, "\delta"] \\
\mathbf{\Delta}^{\mathsf{op}} \underset{\mathfrak{D}}{\times} \mathfrak{D}_{R/} \underset{\mathsf{Ar}(\mathfrak{D})}{\times} \mathsf{Ar}\left(\mathfrak{D}_{|\mathsf{Fin}_*}\right) =: \left(\mathbf{\Delta}^{\mathsf{op}}\right)_{R/^{*}} \arrow[r, hook] & \left(\mathbf{\Delta}^{\mathsf{op}}\right)_{R/}
\end{tikzcd} \tag{64}$$

(in which it sits as the uppermost composite).

We proved in Proposition A.7 that the composite functor $\delta \circ \gamma$ of diagram (64) is Segal-initial, by proving in Lemma A.8 that the functor γ is Segal-initial and in Lemma A.9 that the functor δ is initial. In fact, a nearly identical argument gives the following result.

Proposition A.14. *The composite*

$$\mathsf{Enter}(R) \xrightarrow{\alpha} (\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr},*}} \xrightarrow{\beta} (\mathbf{\Delta}^{\mathsf{op}})_{R/^*}$$

of diagram (64) *is Segal-initial.*

Proof. First of all, the proof of Lemma A.8 carries over without change to show that the functor α is Segal-initial. We will show that the proof of Lemma A.9 likewise implies that the functor β is initial. The claim will then follow (just as in the proof of Proposition A.7) from the fact that initial functors over $\mathbf{\Delta}^{\mathsf{op}}$ are Segal-initial and that Segal-initiality is preserved under composition.

We now turn to the modification of the proof of Lemma A.9. Throughout, we freely use the notation and terminology introduced there.

To begin, choose any object $(R \to T) \in (\mathbf{\Delta}^{\mathsf{op}})_{R/^*}$. Then, the same reduction takes us from considering the category

$$(\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr},*}} \underset{(\mathbf{\Delta}^{\mathsf{op}})_{R/^*}}{\times} \left((\mathbf{\Delta}^{\mathsf{op}})_{R/^*} \right)_{/(R \to T)} \tag{65}$$

of factorizations

$$\begin{tikzcd} R \arrow[rr, "*"] \arrow[dr, dashed, "\mathsf{cls.cr},*"'] & & T \\ & T' \arrow[ur, dashed] & \end{tikzcd}$$

to the category

$$(\mathbf{\Delta}^{\mathsf{op}})_{R/^{\mathsf{cls.cr},*}} \underset{(\mathbf{\Delta}^{\mathsf{op}})_{R/^*}}{\times} \left((\mathbf{\Delta}^{\mathsf{op}})_{R/^*} \right)_{/(R \to T)} \tag{66}$$

of factorizations

$$\begin{tikzcd} R \arrow[rr, "*"] \arrow[dr, dashed, "\mathsf{cls.cr},*"'] & & T \\ & T' \arrow[ur, dashed, "\mathsf{act}"'] & \end{tikzcd} \tag{67}$$

(where in both cases $T, T' \in \mathbf{\Delta}^{\mathsf{op}} \subset \mathfrak{D}$, with T fixed and T' varying). Indeed, the reflective localization

$$\begin{tikzcd} (51) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (53) \arrow[l, hook', shift left] \end{tikzcd}$$

restricts to a reflective localization

$$\begin{tikzcd} (65) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (66) \arrow[l, hook', shift left] \end{tikzcd}$$

on non-full subcategories, because the closed-active factorization system on $\mathfrak{D}$ restricts to one on the subcategory $\mathfrak{D}_{|\mathsf{Fin}_*} \subset \mathfrak{D}$. So it suffices to show that in the case at hand, the initial object of (53) lies in the subcategory (66) $\subset$ (53) and that moreover its unique morphism in (53) to any other object of (66) actually lies in (66).

Now, we claim that both maps $R \to \tilde{F} \to T$ lie in the subcategory $\mathfrak{D}_{|\mathsf{Fin}_*} \subset \mathfrak{D}$. For the map $R \to \tilde{F}$, recall that the 1-dimensional strata of $\tilde{F}$ are in bijection with the sections of $F \downarrow \Delta^1$ of type (11), which evidently inject into the set of 1-dimensional strata of R. And the second map $\tilde{F} \to T$ lies in the subcategory $\mathbf{\Delta}^{\mathsf{op}} \subset \mathfrak{D}$, so it automatically lies in the subcategory $\mathfrak{D}_{|\mathsf{Fin}_*} \subset \mathfrak{D}$. So indeed, in the present situation the initial object (58) $\in$ (53) lies in the subcategory (66) $\subset$ (53).

And then, suppose that we have an arbitrary object (67) $\in$ (66) $\subset$ (53), and consider the unique map (58) $\to$ (67) in (53). To see that this morphism lies in the subcategory (66) $\subset$ (53), we must check that its component $\tilde{F} \to T'$ lies in the subcategory $\mathfrak{D}_{|\mathsf{Fin}_*} \subset \mathfrak{D}$. But this is a morphism in $\mathbf{\Delta}^{\mathsf{op}} \subset \mathfrak{D}$, and so it automatically lies in the subcategory $\mathfrak{D}_{|\mathsf{Fin}_*} \subset \mathfrak{D}$. $\square$

Appendix B. Factorization systems

Proposition B.1. *Let $\mathcal{B}$ be an ∞-category equipped with a factorization system $[\mathcal{B}_0;\mathcal{B}_1]$. There exists a left adjoint*

$$\begin{tikzcd} \mathsf{Cat}^{\mathcal{B}_0}_{\mathsf{cocart}/\mathcal{B}} \arrow[r, dashed, shift left] \arrow[r, phantom, "\perp"] & \mathsf{coCart}_{\mathcal{B}} \arrow[l, hook', two heads, shift left] \end{tikzcd}$$

to the surjective monomorphism, which takes an object $(\mathcal{E}\to\mathcal{B})$ to the horizontal composite in the diagram

$$\begin{tikzcd} \mathcal{E}\underset{\mathcal{B}}{\times}\mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B}) \arrow[r] \arrow[d] & \mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B}) \arrow[r, "\mathsf{ev}_t"] \arrow[d, "\mathsf{ev}_s"] & \mathcal{B} \\ \mathcal{E} \arrow[r] & \mathcal{B} \end{tikzcd} . \tag{68}$$

The proof of Proposition B.1 requires the following easy result.

Lemma B.2. *Let $\mathcal{B}$ be an ∞-category equipped with a factorization system $[\mathcal{B}_0;\mathcal{B}_1]$. Then the restricted evaluation functor*

$$\mathsf{ev}_t : \mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B}) \longrightarrow \mathcal{B} \tag{69}$$

is a cocartesian fibration. Moreover, given an object $(\tilde{b}\to b)\in\mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B})$ and a morphism $b\to b'$ in $\mathcal{B}$, a cocartesian lift is given by the commutative square

$$\begin{tikzcd} \tilde{b} \arrow[r, dashed, "\mathcal{B}_0"] \arrow[d, "\mathcal{B}_1"'] & \tilde{b}' \arrow[d, dashed, "\mathcal{B}_1"] \\ b \arrow[r] & b' \end{tikzcd} \tag{70}$$

involving the unique indicated factorization of the composite $\tilde{b}\to b\to b'$ (thought of as a morphism in $\mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B})$ by reading horizontally).

Proof. First of all, by the evaluation functor

$$\mathsf{Ar}(\mathcal{B}) \xrightarrow{\mathsf{ev}_t} \mathcal{B}$$

is a cocartesian fibration, with cocartesian morphisms those which become equivalences under the functor ev_s. Then, observe that the fully faithful inclusion

$$\mathsf{Ar}(\mathcal{B}) \hookleftarrow \mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B})$$

admits a left adjoint (given by taking a morphism φ with canonical factorization $[\varphi_1;\varphi_2]$ to the morphism φ_2), and moreover that any morphism which this left adjoint takes to an equivalence is also taken to an equivalence by $\mathsf{Ar}(\mathcal{B}) \xrightarrow{\mathsf{ev}_t} \mathcal{B}$. Thus, the claim follows from [Lura, Lemma 2.2.4.11 and Remark 2.2.4.12]. ☐

Proof of Proposition B.1. First of all, the horizontal composite in (68) is indeed a cocartesian fibration by Lemma B.2, since cocartesian fibrations are stable under pullback and composition. So this construction defines a functor

$$\mathsf{Cat}^{\mathcal{B}_0}_{\mathsf{cocart}/\mathcal{B}} \longrightarrow \mathsf{Cat}_{\mathsf{cocart}/\mathcal{B}} .$$

Next, to see that this functor actually factors through the subcategory

$$\mathsf{coCart}_{\mathcal{B}} \subset \mathsf{Cat}_{\mathsf{cocart}/\mathcal{B}} ,$$

we must show that given a morphism

$$\mathcal{E}_1 \longrightarrow \mathcal{E}_2 \tag{71}$$

in $\mathsf{Cat}^{\mathcal{B}_0}_{\mathsf{cocart}/\mathcal{B}}$, the horizontal functor F in the resulting commutative diagram

$$\begin{tikzcd} \mathcal{E}_1 \underset{\mathcal{B}}{\times} \mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B}) \arrow[rr, "F"] \arrow[dr] & & \mathcal{E}_2 \underset{\mathcal{B}}{\times} \mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B}) \arrow[dl] \\ & \mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B}) \arrow[d, "\mathsf{ev}_t"'] & \\ & \mathcal{B} & \end{tikzcd} \tag{72}$$

preserves cocartesian lifts of morphisms in $\mathcal{B}$. So, fix a morphism $b \xrightarrow{\varphi} b'$ in $\mathcal{B}$. Suppose we are given a lift of its source $b \in \mathcal{B}$ in

$$\mathcal{E}_1 \underset{\mathcal{B}}{\times} \mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B}) \ :$$

this consists of an object $(\tilde{b} \to b) \in \mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B})$ as well as an object $e \in \mathcal{E}_1$ lying over $\tilde{b} \in \mathcal{B}$. Then, by Lemma B.2 and parts (2) and (3) of [Lur09, Proposition 2.4.1.3], we can obtain a cocartesian lift of the morphism $b \to b'$ by first extracting the unique factorization

$$\begin{tikzcd} \tilde{b} \arrow[r, dashed, "\mathcal{B}_0"] \arrow[d, "\mathcal{B}_1"'] & \tilde{b}' \arrow[d, dashed, "\mathcal{B}_1"] \\ b \arrow[r] & b' \end{tikzcd}$$

and then extracting a cocartesian lift of the morphism $\tilde{b} \to \tilde{b}'$ in $\mathcal{B}$ at $e \in \mathcal{E}_1$. So clearly the functor F preserves cocartesian lifts over morphisms in the subcategory $\mathcal{B}_0 \subset \mathcal{B}$. On the other hand, if the morphism φ instead lies in the subcategory $\mathcal{B}_1 \subset \mathcal{B}$, then the map $\tilde{b} \to \tilde{b}'$ will be an equivalence, and hence the cocartesian lift in $\mathcal{E}_1$ will be an equivalence as well. So the functor F preserves cocartesian lifts over morphisms in the subcategory $\mathcal{B}_1 \subset \mathcal{B}$ as well. Since cocartesian lifts in a cocartesian fibration compose, the existence of the factorization system $[\mathcal{B}_0; \mathcal{B}_1]$ on $\mathcal{B}$ implies that the functor F preserves all cocartesian lifts. So our construction does indeed define a functor

$$\mathsf{Cat}^{\mathcal{B}_0}_{\mathsf{cocart}/\mathcal{B}} \longrightarrow \mathsf{coCart}_{\mathcal{B}} \ .$$

Thus, it remains to show that this functor is indeed a left adjoint. For this, we first observe that the identity section of the functor ev_s in the diagram (68) induces a section

$$\begin{tikzcd} \mathcal{E} \underset{\mathcal{B}}{\times} \mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B}) \arrow[d] \\ \mathcal{E} \arrow[u, dashed, bend left] \end{tikzcd} \ ,$$

which lies in $\mathsf{Cat}^{\mathcal{B}_0}_{\mathsf{cocart}/\mathcal{B}}$ by our previous considerations. We claim that if we consider $(\mathcal{E} \to \mathcal{B}) \in \mathsf{Cat}^{\mathcal{B}_0}_{\mathsf{cocart}/\mathcal{B}}$ then this section will be the unit map, while if we consider $(\mathcal{E} \to \mathcal{B}) \in \mathsf{coCart}_{\mathcal{B}}$ then the downwards functor will be the counit map. Indeed, if we are given any $(\mathcal{E} \to \mathcal{B}) \in \mathsf{Cat}^{\mathcal{B}_0}_{\mathsf{cocart}/\mathcal{B}}$ and any $(\mathcal{D} \to \mathcal{B}) \in \mathsf{coCart}_{\mathcal{B}}$, it is straightforward to verify that these maps induce inverse equivalences

$$\mathsf{hom}_{\mathsf{Cat}^{\mathcal{B}_0}_{\mathsf{cocart}/\mathcal{B}}}(\mathcal{E}, \mathcal{D}) \simeq \mathsf{hom}_{\mathsf{coCart}_{\mathcal{B}}}\left(\mathcal{E} \underset{\mathcal{B}}{\times} \mathsf{Ar}^{\mathcal{B}_1}(\mathcal{B}), \mathcal{D}\right)$$

of spaces. ∎

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