

POINCARÉ/KOSZUL DUALITY

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ABSTRACT. We prove a duality for factorization homology which generalizes both usual Poincaré duality for manifolds and Koszul duality for \mathcal{E}_n -algebras. The duality has application to the Hochschild homology of associative algebras and enveloping algebras of Lie algebras. We interpret our result at the level of topological quantum field theory.

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INTRODUCTION

This paper arises from the following question: what is Poincaré duality in factorization homology?

Before describing our solution, we give some background for this question. After Lurie in [Lu2], a factorization homology, or topological chiral homology, theory is a homology-type theory for n -manifolds; these theories are natural with respect to embeddings of manifolds and satisfy a multiplicative generalization of the Eilenberg–Steenrod axioms for ordinary homology—see [AF1]. This relatively new theory has two particularly notable antecedents: the labeled or amalgamated configuration space models of mapping spaces of Salvatore [Sa], Segal [Se4], and Kallel [Ka], after [Bö], [Mc], [May], and [Se1]; the algebraic approaches to conformal field theory of Beilinson & Drinfeld in [BeD], via factorization algebras, and of Segal in [Se3].

In the last few years, significant work has taken place in this subject in addition to the basic investigation of the foundations of factorization homology in [Lu2], [AF1], and [AFT2]. In algebraic geometry, Gaitsgory & Lurie use factorization algebras to prove Weil conjecture’s on Tamagawa numbers for algebraic groups in the function field case—see [GL]. In mathematical physics, Costello has developed in [Co] a renormalization machine for quantizing field theories; by work of Costello & Gwilliam in [CG], this machine outputs a factorization homology theory as a model for the observables in a perturbative quantum field theory. Their work gives an array of interesting examples of factorization homology theories and manifold invariants connected with gauge theory, quantum groups, and knot and 3-manifold invariants.

The question of Poincaré duality finds motivation from all the preceding works. We focus on two veins. First, factorization homology theories are characterized by a monoidal generalization of the Eilenberg–Steenrod axioms for usual homology, so that factorization homology specializes to ordinary homology in the case the target symmetric monoidal category is that of chain complexes with direct sum. As such, it makes sense to ask that different values of factorization homology theories, valued in a general symmetric monoidal category, likewise enjoy a relationship specializing to that of Poincaré duality. From this perspective, the present work fills in the bottom middle in the following table of analogies.

Ordinary/Generalized	Factorization	Physics
topological space M	n -manifold M	spacetime M
abelian group A	n -disk stack X	quantum field theory Z
additivity $\mathbb{I} \rightsquigarrow \oplus$	multiplicativity $\mathbb{I} \rightsquigarrow \otimes$	locality $\mathbb{I} \rightsquigarrow \otimes$
homology $C_*(M, A)$	factorization homology $\int_M X$	observables $Z(M)$
linearity $C_*(M, A) \simeq P_1 C_*(M, A)$	Goodwillie calculus $P_\bullet \int_M X$	perturbative observables $Z_{\text{pert}}(M)$
Poincaré duality $C_*(M, A) \simeq C^*(M, A[n])$	Poincaré/Koszul duality $\int_M X_x^\wedge \simeq \left(\int_M T_x X[-n] \right)^\vee$	

Second, in the perspective espoused in the works [BeD] and [CG], factorization homology theories are algebraic models for physical field theories. In extended topological quantum field theory, as appears in the cobordism hypothesis after Baez & Dolan [BaD] and Lurie [Lu3], there is likewise a fundamental duality: the duality of higher n -categories which appear as the values of the field theories on points. These dual theories take equal values on n -manifolds, and linearly dual values on $(n - 1)$ -manifolds; their values on manifolds of lower dimension is expressible, in a less familiar way, as a higher categorical form of duality.

While both of these perspectives augur for a notion of duality in factorization homology, they likewise both indicate that an essential ingredient is missing. In the first case, usual Poincaré duality is a relationship between usual homology and compactly supported cohomology – this suggests that a notion of compactly supported factorization cohomology is necessary. From the perspective of the

cobordism hypothesis in the 1-dimensional case, the factorization homology theories $\int A$ is closely related to extended topological field theories Z whose value on a point is $Z(*) = \text{Perf}_A$, and the value on the circle is $Z(S^1) \simeq \int_{S^1} A \simeq \text{HH}_*(A)$, the Hochschild chains of A . One would expect the dual field theory Z^\vee to take the value $Z^\vee(S^1) = \text{HH}_*(A)^\vee$, the \mathbb{k} -linear dual of the Hochschild chains of A . However, there is in general no algebra B for which $\text{HH}_*(B)$ is equivalent to $\text{HH}_*(A)^\vee$. That is, the category $Z^\vee(*)$ is not given by perfect modules for some other algebra. Under restrictive conditions, however, this sometimes happens: namely, the algebra $\mathbb{D}A$, Koszul dual to A , is a candidate. One can construct a natural map, which is an instance of our Poincaré/Koszul duality map,

$$\text{HH}_*(\mathbb{D}A) \longrightarrow \text{HH}_*(A)^\vee$$

which is sometimes an equivalence; for instance, this is a formal exercise in the case A is $\text{Sym}(V)$, for V a finite complex in strictly positive homological degrees. In general, we shall see, $\text{HH}_*(A)^\vee$ is a type of completion of $\text{HH}_*(\mathbb{D}A)$.

Our two sources of motivation thus offer two pointers on where to look for Poincaré duality in factorization homology. The first says that the factorization homology with coefficients in A should be equivalent to some other construction, not factorization homology per se, but some cohomological variant. Our TQFT motivation suggests that the choice of coefficients for this factorizable generalization of cohomology should be related to the Koszul dual of A . Assembling these hints, one might conclude that such a Poincaré duality should relate the factorization homology with coefficients in A to a not necessarily perturbative form of factorization homology with coefficients related to the Koszul dual of A , a generalization that one can contemplate after the originating works of [GiK] and [Pr].

There is, however, already a not necessarily perturbative form of factorization homology, defined in [AF1]. Namely, for an n -manifold M and a scheme X whose structure sheaf \mathcal{O}_X is enhanced to form a sheaf of n -disk algebras, then one can define the factorization homology of M with coefficients in X as

$$\int_M X = \Gamma\left(X, \int_M \mathcal{O}_X\right)$$

the global sections over X of the sheaf obtained by computing the factorization homology of the structure sheaf. Intuitively, one can also think of this object as a quantization of functions on maps from M to the underlying space X_0 – the observables in a sigma model formed by quantizing a symplectic structure on the underlying space X_0 . We adopt the point of view, after Costello & Gwilliam, that factorization homology with coefficients in stacks over n -disk algebras offers a suitable generic model for the observables in a sigma model. We regard this as a generalization of the perturbative theory in the following sense: perturbation theory involves the calculation of the theory by formal expansion at a fixed solution; for a sigma model, this involves the formal neighborhood of the mapping space $\text{Map}(M, X)$ at a point $M \rightarrow X$, such as a constant map. If $X = \text{Spf } A$ is itself a formal neighborhood of a point, given by the formal spectrum of an augmented pro-nilpotent algebra $\epsilon : A \rightarrow \mathbb{k}$, then both the mapping space $\text{Map}(M, \text{Spf } A)$ and the factorization homology

$$\int_M A \simeq \Gamma(\text{Spf } A, \int_M \mathcal{O})$$

can be completely understood by perturbation theory about the \mathbb{k} -point $\epsilon \in \text{Spf } A$, which we understand as an instance of Goodwillie calculus. From this point of view, we think of the factorization homology $\int_M X$ as being perturbative if the classical target X_0 is a subspace of the quantization X , and the factorization homology $\int_M X$ can be obtained from $\int_M X_0 \sim \mathcal{O}(\text{Map}(M, X_0))$ by Goodwillie calculus; this should be the case when the quantization X is a formal neighborhood of the classical phase space X_0 , in particular, where \hbar is taken to be a formal parameter.

In this paper, we deal completely with the case in which X_0 is a point and X is a general formal space, which one can think of as perturbation theory around a single constant map. For a general augmented algebra $\epsilon : A \rightarrow \mathbb{k}$, the factorization homology $\int_M A$ cannot be understood by perturbation theory at the augmentation. So one might ask for some more general non-affine

geometric target X associated to A , where the greater global geometry of X similarly reflects the noncoverage properties of $\int_M A$ at ϵ . There is, finally, just such a formal space associated to an n -disk algebra, lifting the functor of Koszul duality.

The original formulations of Koszul duality, after Priddy [Pr] and Moore [Mo], involve only algebra and coalgebra – for a review of Koszul duality in algebraic topology, we suggest [Si]. However, from Quillen’s Lie algebraic model for rational homotopy types [Quil], one can think of a Lie algebra as giving rise to a homotopy type via its Maurer–Cartan space. That is, to a differential graded Lie algebra \mathfrak{g} , one can consider the set of solutions to the Maurer–Cartan equation: elements $x \in \mathfrak{g}_{-1}$ for which $dx + \frac{1}{2}[x, x] = 0$. Applying this procedure to a resolution of the Lie algebra, such as the simplicial object in Lie algebras $\mathfrak{g} \otimes \Omega^*(\Delta^\bullet)$ obtained by tensoring with Sullivan’s polynomial de Rham forms on simplices [Su], gives a simplicial set whose associated homotopy type is the Maurer–Cartan space. More generally, we take the Maurer–Cartan functor of a Lie algebra \mathfrak{g} to be a space-valued functor

$$\text{Artin} \xrightarrow{\text{MC}_{\mathfrak{g}}} \text{Spaces}$$

which assigns to a local Artin \mathbb{k} -algebra R the space of maps of Lie algebras

$$\text{MC}_{\mathfrak{g}}(R) := \text{Map}_{\text{Lie}}(\mathbb{D}R, \mathfrak{g})$$

where $\mathbb{D}R$ is the Lie algebra which is Koszul dual to R ; equivalently $\mathbb{D}R \simeq \mathbb{T}_R[-1]$ is a shift of the tangent complex of R given by the maximal ideal $R \rightarrow \mathbb{k}$ shifted by -1 .

This construction gains importance because of the following principle, which very often holds. If X is an object of interest for which there exists a tangent complex \mathbb{T}_X , then in favorable situations there exists a Lie algebra structure on \mathbb{T}_X , and the deformations of the structure of X is given by solutions to the Maurer–Cartan equation in \mathbb{T}_X . This pattern emerged in the 1950s, particularly in complex geometry (see [KS], [NR], [FN], and [Kur]), where solutions to the Maurer–Cartan equation of the Kodaira–Spencer Lie algebra classify deformations of a complex structure. This construction was then studied more generally in other works; in particular, see [GM1], [GM2], [HS], [Hi1], [Get], and [Lu4].

The above pattern applies to n -disk algebra as well, after [Lu4]. Given an augmented n -disk algebra A , there is a formal moduli functor

$$\text{Artin}_n \xrightarrow{\text{MC}_A} \text{Spaces}$$

which assigns to a local Artin n -disk \mathbb{k} -algebra R the space of maps of augmented n -disk algebras

$$\text{MC}_A(R) := \text{Map}(\mathbb{D}^n R, A)$$

from the Koszul dual of R as an n -disk algebra, to A . This is a lift of usual Koszul duality through moduli, in that the ring of global functions of MC_A is exactly the Koszul dual of A .

We now have the ingredients necessary to state our main theorem; see Theorem 3.2.4.

Theorem 0.0.1 (Poincaré/Koszul duality). *Let M be a compact smooth n -dimensional cobordism with boundary partitioned as $\partial M \cong \partial_{\mathbb{L}} \sqcup \partial_{\mathbb{R}}$. For A an augmented n -disk algebra over a field \mathbb{k} with MC_A the associated formal moduli functor of n -disk algebras, there is a natural equivalence*

$$\left(\int_{M \setminus \partial_{\mathbb{R}}} A \right)^{\vee} \simeq \int_{M \setminus \partial_{\mathbb{L}}} \text{MC}_A$$

between the \mathbb{k} -linear dual of the factorization homology of $M \setminus \partial_{\mathbb{R}}$ with coefficients in A and the factorization homology of $M \setminus \partial_{\mathbb{L}}$ with coefficients in the moduli functor MC_A .

Note that if the boundary of the compact n -manifold M is empty, then the statement above reduces to the simpler expression of an equivalence

$$\left(\int_M A \right)^{\vee} \simeq \int_M \text{MC}_A .$$

Remark 0.0.2. If we write X for the formal moduli functor MC_A , then A itself is equivalent to $\mathrm{T}_x X[-n]$, the tangent space at the distinguished \mathbb{k} -point $x \in X$ shifted by $-n$. The assertion of Theorem 0.0.1 then becomes

$$\int_{M \setminus \partial_L} X \simeq \left(\int_{M \setminus \partial_R} \mathrm{T}_x X[-n] \right)^\vee.$$

This theorem coheres to our dual motivations. First, the result specializes to the dual of usual Poincaré duality by setting A to be an algebra with respect to direct sum; in this case the lefthand side becomes usual homology with coefficients in A , the formal moduli problem is representable, and the righthand side becomes usual cohomology with coefficients in an n -fold shift of A . Second, from the cobordism formulation one can see this result involves a duality for the extended topological field theories defined by the factorization homology with coefficients in A and MC_A . Theorem 0.0.1 is a more general than the duality assured by the cobordism hypothesis, however. Since limits need not commute with tensor products of infinite dimensional vector spaces, the natural map

$$\int_M \mathrm{MC}_A \otimes \int_N \mathrm{MC}_A \longrightarrow \int_{M \amalg N} \mathrm{MC}_A$$

need not be an equivalence. So factorization homology with coefficients in MC_A does not define a symmetric monoidal functor from the bordism category unless an additional requirement is made on A , to ensure the factorization homologies $\int_M A$ and $\int_M \mathrm{MC}_A$ are finite dimensional.

Given connectivity or coconnectivity hypotheses on the algebra A , one can replace the moduli problem MC_A with its algebra of global sections $\mathbb{D}^n A$, the Koszul dual of A . We have the following, combining Theorem 2.1.9, Theorem 2.1.7, and Proposition 3.3.4.

Theorem 0.0.3. *Let M be a compact smooth n -dimensional cobordism with boundary partitioned as $\partial M \cong \partial_L \sqcup \partial_R$. Let A be a finitely presented augmented n -disk algebra such that either:*

- *A has a connected augmentation ideal;*
- *A is an algebra over a field \mathbb{k} and the augmentation ideal of A is $(-n)$ -coconnective.*

There is a natural equivalence

$$\left(\int_{M \setminus \partial_R} A \right)^\vee \simeq \int_{M \setminus \partial_L} \mathbb{D}^n A.$$

This result is interesting even for dimension $n = 1$. The factorization homology of the circle is equivalent to Hochschild homology, so in this case we obtain a linear duality between the Hochschild homology of an associative algebra and either the Hochschild homology of the noncommutative moduli problem MC_A or of the Koszul dual $A \simeq \mathrm{Hom}_A(\mathbb{k}, \mathbb{k})$. See Corollary 4.1.1. This specialization is particularly comprehensible in the case where the algebra $A = \mathrm{U}\mathfrak{g}$ is the enveloping algebra of a Lie algebra over a field of characteristic zero. In this case, we obtain a relation between the enveloping algebra of a Lie algebra \mathfrak{g} and the Lie algebra cohomology of \mathfrak{g} . Then we have the following, which generalizes a result of Feigin & Tsygan in [FT2].

Theorem 0.0.4. *For \mathfrak{g} a Lie algebra over a field of characteristic zero which is finite dimensional and concentrated in either homological degrees less than -1 or in degrees greater than 0, then there is an equivalence*

$$\mathrm{HH}_*(\mathrm{U}\mathfrak{g})^\vee \simeq \mathrm{HH}_*(\mathrm{C}^*\mathfrak{g})$$

between the dual of the Hochschild homology of the enveloping algebra and the Hochschild homology of Lie algebra cochains.

We make two remarks on generalizations, or the lack thereof.

Remark 0.0.5. All of our results are valid, with identical proofs, if smooth n -manifolds are replaced with G -structured smooth n -manifolds, where G is a Lie group with a continuous homomorphism $G \rightarrow \mathrm{Diff}(\mathbb{R}^n)$. However, for visual simplicity we omit this notational clutter from most of the current work.

Remark 0.0.6. We had originally imagined this work in the setting of *topological*, rather than smooth, manifolds, but serious technical obstructions dissuaded us. In topology, one runs into the difficulty that the configuration spaces $\text{Conf}_i(M)$ in a compact topological manifold M might not admit a compactification to a topological manifold with corners. Indeed, without some regularity conditions, which are assured by smoothness, our present methods do not allow us to identify the ∞ -category $\text{Disk}_{n/M}^{\text{top}}$ with the exit-path ∞ -category of the Ran space of M (or even to show that the simplicial space of exit-paths in $\text{Ran}(M)$ indeed satisfy the Segal condition and so form an ∞ -category). This is the key technical point which allows us to analyze the layers of the Goodwillie calculus towers in terms of configuration spaces. In algebra, one runs into difficulties stemming from $\text{Top}(n)$, the topological group of homeomorphisms of \mathbb{R}^n , which is far less understood than the group of diffeomorphisms of \mathbb{R}^n . Specifically, the homology $\mathbf{H}_* \text{Top}(n)$ is not known, at least to us, to be finite rank—unlike $\mathbf{H}_* \mathbf{O}(n)$ —and so the notions of coherent and perfect $\text{Top}(n)$ -modules differ. As a result, Koszul duality of topological n -disk algebras does not enjoy the same good duality properties as that of smooth n -disk algebra. We do not think that a statement at the generality of Theorem 0.0.1 is true for topological n -disk algebras and topological manifolds.

We now overview the contents of this paper, section by section.

In **Section 1**, we review the category \mathcal{ZMfld}_n of zero-pointed n -manifolds and the factorization homology of zero-pointed manifolds from [AF2]. A zero-pointed manifold consists of a pointed topological space M_* , which is a smooth n -manifold M with an extra point $*$ and a conically smooth extension of the topology of M to M_* ; the essential example is a space $M/\partial M$, the quotient of an n -manifold by its boundary. This theory naturally incorporates functoriality for both embeddings and Pontryagin–Thom collapse maps of embeddings, a feature we employ in order to present a unified treatment of duality in homology/cohomology and algebra/coalgebra. More precisely, the zero-pointed theory provides additional functorialities for factorization homology with coefficients in an augmented n -disk algebra algebra which, in particular, endows the factorization homology

$$\int_{(\mathbb{R}^n)^+} A$$

with the structure of an n -disk coalgebra, where $(\mathbb{R}^n)^+$ is the 1-point compactification of \mathbb{R}^n . Consequently, we arrive at a geometric presentation of an n -disk coalgebra structure on the n -fold iterated bar construction of an augmented n -disk algebra. We apply this to construct the Poincaré/Koszul duality map, which goes from factorization homology with coefficients in an n -disk algebra to factorization cohomology with coefficients in the Koszul dual n -disk coalgebra. We lastly recall a version of twisted Poincaré duality, which asserts that our duality map is an equivalence in the case of stable ∞ -category with direct sum.

In **Section 2**, we introduce two (co)filtrations of factorization homology and cohomology. One comes from Goodwillie’s calculus of homotopy functors. A second comes from a cardinality filtration $\mathcal{D}\text{isk}_n^{\leq k}$ of $\mathcal{D}\text{isk}_n$. The cardinality filtration is a common generalization of the Goodwillie–Weiss calculus filtration from topology and the Hodge filtration of Hochschild homology (see Remark 2.1.5). We prove that the Poincaré/Koszul duality map exchanges the Goodwillie and the cardinality cofiltrations. That is, in this instance, we prove that Goodwillie calculus and Goodwillie–Weiss calculus are Koszul dual to one another. As a consequence, we obtain spectral sequences for factorization homology whose E^1 terms are identified as homologies of configuration spaces; for the circle, one of these spectral sequences generalizes the Bökstedt spectral sequence. Finally, in the case of our main theorem, we conclude that the Poincaré/Koszul duality map is an equivalence when the algebra A is connected.

In **Section 3**, we introduce factorization homology with coefficients in a formal moduli problem. We prove that the Poincaré/Koszul duality map is an equivalence in the case of a $(-n)$ -coconnective n -disk algebra over a field. We also prove an instance of Koszul duality proper, that there is equivalence between Artin n -disk algebras and finitely presented $(-n)$ -coconnective n -disk algebras. Using this, we show our main theorem, that the moduli-theoretic Poincaré/Koszul duality map is

an equivalence for any augmented n -disk algebra. We conclude by specializing these results to the case of associative algebras and Lie algebras in **Section 4**.

Remark 0.0.7. In this work, we use Joyal’s *quasi-category* model of ∞ -category theory [Jo]. Boardman & Vogt first introduced these simplicial sets in [BoV], as weak Kan complexes, and their and Joyal’s theory has been developed in great depth by Lurie in [Lu1] and [Lu2], our primary references. See the first chapter of [Lu1] for an introduction. We use this model, rather than model categories or simplicial categories, because of the great technical advantages for constructions involving categories of functors, which are ubiquitous in this work. More specifically, we work inside of the quasi-category associated to this model category of Joyal’s. In particular, each map between quasi-categories is understood to be an iso- and inner-fibration; and (co)limits among quasi-categories are equivalent to homotopy (co)limits with respect to Joyal’s model structure.

We will also make use of topological categories, such as the topological category $\mathcal{M}\text{fld}_n$ of smooth n -manifolds and smooth embeddings. By a functor $\mathcal{M}\text{fld}_n \rightarrow \mathcal{V}$ from a topological category such as $\mathcal{M}\text{fld}_n$ to an ∞ -category \mathcal{V} we will always mean a functor $\mathbf{N}\text{Sing } \mathcal{M}\text{fld}_n \rightarrow \mathcal{V}$ from the simplicial nerve of the simplicial category $\text{Sing } \mathcal{M}\text{fld}_n$ obtained by applying the singular functor Sing to the hom spaces of the topological category.

The reader uncomfortable with this language can substitute the words “topological category” for “ ∞ -category” wherever they occur in this paper to obtain the correct sense of the results, but they should then bear in mind the proviso that technical difficulties may then abound in making the statements literally true. The reader only concerned with algebra in chain complexes, rather than spectra, can likewise substitute “pre-triangulated differential graded category” for “stable ∞ -category” wherever those words appear, with the same proviso.

Terminology. [\otimes -conditions] Throughout this document, we will use the letter \mathcal{V} for a symmetric monoidal ∞ -category, and $\mathbb{1}$ for its symmetric monoidal unit. We will not distinguish in notation between it and its underlying ∞ -category.

- We say \mathcal{V} is \otimes -presentable if its underlying ∞ -category is presentable and its symmetric monoidal structure \otimes distributes over small colimits separately in each variable. We say \mathcal{V} is \otimes -stable-presentable if it is \otimes -presentable and its underlying ∞ -category is stable.
- We say \mathcal{V} is \otimes -cocomplete if its underlying ∞ -category admits small colimits and its symmetric monoidal structure \otimes distributes over small colimits separately in each variable.
- We say \mathcal{V} is \otimes -sifted cocomplete if its underlying ∞ -category admits sifted colimits and its symmetric monoidal structure \otimes distributes over sifted colimits separately in each variable.

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1. REVIEW OF REDUCED FACTORIZATION HOMOLOGY

We recall some notions among zero-pointed manifolds and factorization (co)homology thereof, as established in [AF2].

1.1. Zero-pointed manifolds. We recall the enlargement of the symmetric monoidal topological category $\mathcal{M}\text{fld}_n$ of smooth n -manifolds and open embeddings among them (with compact-open C^∞ topologies), to *zero-pointed* n -manifolds.

Definition 1.1.1 (Zero-pointed manifolds). An object of the symmetric monoidal topological category of zero-pointed manifolds $\mathcal{Z}\text{Mfld}$ is a locally compact Hausdorff based topological space M_*

together with the structure of a smooth manifold on the complement of the base point $M := M_* \setminus *$. The topological space of morphisms $\mathbf{ZEmb}(M_*, M'_*)$ consists of based maps $f: M_* \rightarrow M'_*$ for which the restriction $f|_1: f^{-1}M' \rightarrow M'$ is an open embedding, endowed with the compact-open topology. Composition is given by composing based maps. The symmetric monoidal structure is wedge sum. There is the full sub-symmetric monoidal topological category

$$\mathcal{ZMfld}_n \subset \mathcal{ZMfld}$$

consisting of those zero-pointed manifolds M_* for which M has dimension exactly n . For a smooth n -manifold M , we denote by M_+ the zero-pointed manifold given by M with a disjoint zero-point; M^+ is the zero-pointed manifold defined by the 1-point compactification of M . We denote the full sub-symmetric monoidal categories of \mathcal{ZMfld}_n

$$\mathcal{Disk}_{n,+} \subset \mathcal{ZDisk}_n \supset \mathcal{Disk}_n^+$$

which consist of wedge sums of \mathbb{R}_+^n , of \mathbb{R}_+^n and $(\mathbb{R}^n)^+$, and of $(\mathbb{R}^n)^+$, respectively. Likewise, we consider $\mathcal{Mfld}_{n,+}$ and \mathcal{Mfld}_n^+ as the full sub-symmetric monoidal categories of \mathcal{ZMfld}_n consisting of zero-pointed manifolds of the form M_+ and M^+ , respectively.

Remark 1.1.2. In [AF2] we gave the above definition without reference to smooth structures, and placed a superscript $^{\text{sm}}$ for the throughout the above definition. In this work, because we will always work in a smooth context, we will not use this cluttering notation.

Example 1.1.3 (Cobordisms). Let \overline{M} be a cobordism, i.e., a compact manifold with partitioned boundary $\partial\overline{M} = \partial_L \amalg \partial_R$. The based topological space

$$M_* := * \coprod_{\partial_L} (\overline{M} \setminus \partial_R)$$

is a zero-pointed manifold.

Our results of this paper will make the following requirement of the topology around the zero-point: that it is a conical singularity in the sense of [AFT1].

Definition 1.1.4 (Conically smooth/finite). A *conically smooth* zero-pointed manifold is a zero-pointed manifold M_* together with the structure of a stratified space (that will not appear in its notation) on its 1-point compactification $(M_*)^+$ extending the given smooth structure of M . We say a zero-pointed manifold M_* is *conically finite* if it is the underlying zero-pointed manifold of a conically smooth zero-pointed manifold. The ∞ -category of *conically finite* zero-pointed manifolds is the full sub- ∞ -category

$$\mathcal{ZMfld}^{\text{fin}} \subset \mathcal{ZMfld}$$

consisting of the conically finite ones.

In particular, a conically smooth zero-pointed determines a stratified space structure on the pointed topological space M_* .

Remark 1.1.5. The *unzipping* construction of [AFT1] reveals that a conically smooth zero-pointed manifold is the data of a compact smooth manifold \overline{M} with boundary which is partitioned $\partial\overline{M} = \partial_L \sqcup \partial_R$, as in Example 1.1.3.

Remark 1.1.6. Not every zero-pointed manifold M_* is conically finite. The one-point compactification of an infinite genus surface illustrates this.

Remark 1.1.7. Consider the notation of Definition 1.1.4. A zero-pointed embedding from $* \coprod_{\partial_L} (\overline{M} \setminus \partial_R)$ to $* \coprod_{\partial'_L} (\overline{M}' \setminus \partial'_R)$ is vastly different from an embedding from \overline{M} to \overline{M}' that respect the partitioned boundaries in any sense.

We now catalogue some facts about \mathcal{ZMfld}_n that are proven in [AF2]. Note first the continuous functor $\mathcal{Mfld}_n \rightarrow \mathcal{Mfld}_{n,+}$, given by $M \mapsto M_+$, which is symmetric monoidal.

Theorem 1.1.8 ([AF2]). *Let \mathcal{V} be a symmetric monoidal ∞ -category.*

- *There is a contravariant involution $\neg: \mathcal{Z}\mathcal{M}\text{fld}_n \cong \mathcal{Z}\mathcal{M}\text{fld}_n^{\text{op}}: \neg$ which sends a zero-pointed manifold*

$$M_* \longmapsto (M_*)^+ \setminus * ,$$

the 1-point compactification of M minus the original zero-point.

- *There is a canonical equivalence of ∞ -categories*

$$\text{Fun}^{\otimes}(\mathcal{M}\text{fld}_{n,+}, \mathcal{V}) \xrightarrow{\cong} \text{Fun}^{\otimes, \text{aug}}(\mathcal{M}\text{fld}_n, \mathcal{V})$$

to between symmetric monoidal functors from $\mathcal{M}\text{fld}_{n,+}$ and symmetric monoidal functors from $\mathcal{M}\text{fld}_n$ which are augmented over the constant functor $\mathcal{M}\text{fld}_n \rightarrow \mathcal{V}$ whose value is the unit $\mathbb{1}$ of \mathcal{V} . (That is, an augmented functor F has the data of a projection $F(M) \rightarrow \mathbb{1}$ and a section $\mathbb{1} \rightarrow F(M)$ compatibly for all M .)

For i a finite cardinality, the configuration space $\text{Conf}_i(M)$ of i distinct points in a smooth n -manifold M is the smooth open submanifold of the ni -manifold

$$\text{Conf}_i(M) := \{ \{1, \dots, i\} \hookrightarrow M \} \subset M^{\{1, \dots, i\}}$$

consisting of the injections. Let M_* be a zero-pointed manifold. There is an open embedding $\text{Conf}_i(M) \hookrightarrow (M_*)^{\wedge i}$ of the configuration space into the iterated smash product. We denote

$$\text{Conf}_i(M_*) \quad \text{and} \quad \text{Conf}_i^{\neg}(M_*)$$

for the locally compact Hausdorff spaces given by two different topologies on the underlying set of $\text{Conf}_i(M) \amalg *$: the first is the coarsest topology such for which the evident inclusion $\text{Conf}_i(M) \amalg * \rightarrow (M_*)^{\wedge \{1, \dots, i\}}$ is continuous; the second is the finest topology such for which the collapse map $(M_*)^{\wedge \{1, \dots, i\}} \rightarrow \text{Conf}_i(M) \amalg *$ is continuous.

Proposition 1.1.9 ([AF2]). *For each finite cardinality i and each zero-pointed n -manifold M_* , the based spaces $\text{Conf}_i(M_*)$ and $\text{Conf}_i^{\neg}(M_*)$ exist and are zero-pointed (ni)-manifolds. Further, the given maps*

$$\text{Conf}_i(M_*) \longrightarrow (M_*)^{\wedge i} \longrightarrow \text{Conf}_i^{\neg}(M_*)$$

are morphisms of zero-pointed manifolds. They bear a canonical relation $\text{Conf}_i(M_)^{\neg} \cong \text{Conf}_i^{\neg}(M_*^{\neg})$. If M_* is conically finite, then both $\text{Conf}_i(M_*)$ and $\text{Conf}_i^{\neg}(M_*)$ are conically finite; and for any abelian group A the singular homology $H_q(\text{Conf}_i(M_*); A)$ vanishes for $q > ni + (n-1)(i-\ell)$, where ℓ is the number of components of M_* and i is greater than ℓ .*

Example 1.1.10. In the case $M_* = M_+$, then $\text{Conf}_i(M_*) = \text{Conf}_i(M)_+$, and the coconnectivity statement of Proposition 1.1.9 follows by induction on i through the standard fibration sequence $M \setminus \{x_1, \dots, x_i\} \rightarrow \text{Conf}_{i+1}(M_*) \rightarrow \text{Conf}_i(M_*)$.

Remark 1.1.11 (B -structures). In [AF2] we develop a theory of zero-pointed manifolds equipped with a B -structure, where $B \rightarrow \text{BO}(n)$ is a map of spaces. We will make use of this generalization for the simple case where $B \rightarrow \text{BO}(ni)$ is the classifying space of the block-sum homomorphism $\Sigma_i \wr \text{O}(n) \rightarrow \text{O}(ni)$, and the $\mathbb{B}(\Sigma_i \wr \text{O}(n))$ -structured zero-pointed manifolds are of the form $\text{Conf}_i(M_*)$ and $\text{Conf}_i^{\neg}(M_*)$.

1.2. Reduced factorization homology. Theorem 1.1.8 justifies the following definitions.

Definition 1.2.1 (Reduced factorization (co)homology). Let \mathcal{V} be a symmetric monoidal ∞ -category.

The ∞ -categories of *augmented n -disk algebras* and of *augmented n -disk coalgebras*, respectively, are those of symmetric monoidal functors

$$\text{Alg}_n^{\text{aug}}(\mathcal{V}) := \text{Fun}^{\otimes}(\text{Disk}_{n,+}, \mathcal{V}) \quad \text{and} \quad \text{cAlg}_n^{\text{aug}}(\mathcal{V}) := \text{Fun}^{\otimes}(\text{Disk}_n^+, \mathcal{V}) .$$

Restrictions along the inclusions $\mathcal{D}\text{isk}_{n,+} \hookrightarrow \mathcal{Z}\text{Mfld}_n \hookrightarrow \mathcal{D}\text{isk}_n^+$ have (a priori partially defined) adjoints depicted in the diagram

$$\begin{array}{ccccc}
& & \xrightarrow{f_-} & & \\
\text{Bar: } \text{Alg}_n^{\text{aug}}(\mathcal{V}) & & \text{Fun}^{\otimes}(\mathcal{Z}\text{Mfld}_n, \mathcal{V}) & & \text{cAlg}_n^{\text{aug}}(\mathcal{V}): \text{cBar} \\
& \xleftarrow{|_{\mathcal{D}\text{isk}_{n,+}}} & & \xleftarrow{f^{(-)^{-}}} & \\
& & & & \xleftarrow{|_{\mathcal{D}\text{isk}_n^+}}
\end{array}$$

whose left and right composites are as depicted. Explicitly, for A an augmented n -disk algebra, C an augmented n -disk coalgebra, and M_* a zero-pointed n -manifold, the values of these adjoints are given as

$$\int_{M_*} A := \text{colim} \left((\mathcal{D}\text{isk}_{n,+})_{/M_*} \rightarrow \mathcal{D}\text{isk}_{n,+} \xrightarrow{A} \mathcal{V} \right)$$

and

$$\int^{M_*} C := \lim \left((\mathcal{D}\text{isk}_n^+)^{M_* /} \rightarrow \mathcal{D}\text{isk}_n^+ \xrightarrow{C} \mathcal{V} \right)$$

which we refer to respectively as the *factorization homology* M_* with coefficients in A , and as the *factorization cohomology* of M_* with coefficients in C . Note that factorization cohomology is contravariant, whereas factorization homology is covariant.

Theorem 1.2.2 ([AF2]). *Let \mathcal{V} be a symmetric monoidal ∞ -category which admits sifted colimits. If M_* is conically finite, then $\int_{M_*} A$ exists and $\int_- A$ depicts a covariant functor to \mathcal{V} from conically finite zero-pointed n -manifolds. In addition, $\int_- A$ depicts a symmetric monoidal functor from conically finite zero-pointed n -manifolds provided \mathcal{V} is \otimes -sifted cocomplete. The dual result holds for $\int^- C$.*

Remark 1.2.3. The dual conditions for factorization homology typically do not hold in cases of interest. For instance, when \mathcal{V} is chain complexes, the tensor product does not distribute over cosifted limits limits, although direct sum does.

There is a canonical comparison arrow between factorization homology and factorization cohomology.

Theorem 1.2.4 (Poincaré/Koszul duality map ([AF2])). *Let A be an n -disk algebra in \mathcal{V} , a symmetric monoidal ∞ -category which admits sifted colimits and cosifted limits. Let M_* be a conically finite zero-pointed n -manifold. There is a canonical arrow in \mathcal{V}*

$$(1) \quad \int_{M_*} A \longrightarrow \int^{M_*^-} \text{Bar } A$$

which is functorial in M_* and A .

The present work continues the analysis of this Poincaré/Koszul duality map. In [AF2], we showed that it is an equivalence in several instances. One special case is when \mathcal{V} is the ∞ -category of spaces with Cartesian product; in this case factorization cohomology is a mapping space. Another special case is where \mathcal{V} is chain complexes equipped with direct sum; in this case, factorization homology is usual homology with coefficients in the chain complex A (possibly twisted by the $\text{O}(n)$ action), factorization cohomology is usual generalized cohomology with coefficients in $A[n]$ (where $\text{O}(n)$ acts by the sign representation), and this equivalence this usual Poincaré duality with twisted coefficients. In this work, we study this map when \mathcal{V} is chain complexes with tensor product (or, more generally, a stable symmetric monoidal ∞ -category which is \otimes -cocomplete).

1.3. Exiting disks. The slice ∞ -category $\mathcal{D}isk_{n,+}/M_*$ appears in the defining expression for factorization homology. We review a variant of this ∞ -category, $\mathcal{D}isk_+(M_*)$, of *exiting disks* in M_* , which offers several conceptual and technical advantages. Heuristically, objects of $\mathcal{D}isk_+(M_*)$ are embeddings from finite disjoint unions of disks into M , while morphisms are isotopies of such to embeddings with some of these isotopies witnessing disks slide off to infinity where they are forgotten. Disks are not allowed to slide in *from* infinity, unlike in $\mathcal{D}isk_{n,+}/M_*$. To define this ∞ -category of exiting disks, we require the regularity around the zero-point granted by a conically smooth structure.

For this section, we fix a conically smooth zero-pointed n -manifold M_* . We recall the following notion from §4 of [AFT1].

Definition 1.3.1 ([AFT1]). A *basic*, or *basic singularity type*, is a stratified space of the form

$$\mathbb{R}^i \times C(L)$$

where i is a finite cardinality, L is a compact stratified space, and $C(L) := * \amalg_{L \times \{0\}} L \times [0, 1)$ is its *open cone*, where here the half-open interval is stratified as the two strata $\{0\}$ and $(0, 1)$. The ∞ -category of *basics* is the full ∞ -subcategory consisting of the basics

$$\mathcal{B}sc \subset \mathcal{S}nglr$$

in the ∞ -category of stratified spaces and spaces of open embeddings among them.

In §2 of [AFT2] appears a stratified version of $\mathcal{D}isk_{n/M}$, which we now recall.

Definition 1.3.2 ([AFT2]). For each stratified space X we denote the full ∞ -subcategory

$$\mathcal{D}isk(\mathcal{B}sc)_{/X} \subset \mathcal{S}nglr_{/X}$$

consisting of those open embeddings $U \hookrightarrow X$ for which U is isomorphic to a finite disjoint union of basics.

Definition 1.3.3 ($\mathcal{D}isk_+(M_*)$). The ∞ -category of *exiting disks* of M_* is the full ∞ -subcategory

$$\mathcal{D}isk_+(M_*) \subset \mathcal{D}isk(\mathcal{B}sc)_{/M_*}$$

consisting of those $V \hookrightarrow M_*$ whose image contains $*$. We use the notation

$$\mathcal{D}isk^+(M_*^\square) := \mathcal{D}isk_+(M_*)^{\text{op}}.$$

Explicitly, an object of $\mathcal{D}isk_+(M_*)$ is a conically smooth open embedding $B \sqcup U \hookrightarrow M_*$ where $B \cong C(L)$ is a cone-neighborhood of $* \in M_*$ and U is abstractly diffeomorphic to a finite disjoint union of Euclidean spaces, and a morphism is a isotopy to an embedding among such.

For each zero-pointed manifold M_* , the unique zero-pointed embedding $* \rightarrow M_*$ induces the functor

$$\mathcal{D}isk_{n,+} = \mathcal{D}isk_{n,+}/* \longrightarrow \mathcal{D}isk_{n,+}/M_*.$$

We denote the resulting pushout among ∞ -categories as

$$\begin{array}{ccc} \mathcal{D}isk_{n,+} & \longrightarrow & \mathcal{D}isk_{n,+}/M_* \\ \downarrow & & \downarrow \\ * & \longrightarrow & (\mathcal{D}isk_{n,+}/M_*) / (\mathcal{D}isk_{n,+}). \end{array}$$

The next result makes reduced factorization homology tractable.

Theorem 1.3.4 ([AF2]). (1) *The ∞ -category $\mathcal{D}isk_+(M_*)$ is sifted.*
(2) *There is a final functor*

$$\mathcal{D}isk_+(M_*) \longrightarrow (\mathcal{D}isk_{n,+}/M_*) / (\mathcal{D}isk_{n,+})$$

whose value on $(B \sqcup U \hookrightarrow M_)$ is represented by $(U_+ \hookrightarrow M_*) \in \mathcal{D}isk_{n,+}/M_*$.*

Consider the composite functor

$$(2) \quad \text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \text{Fun}(\text{Disk}_{n,+}/M_*, \mathcal{V}) \xrightarrow{\text{LKan}} \text{Fun}\left(\left(\text{Disk}_{n,+}/M_*\right)/\left(\text{Disk}_{n,+}\right), \mathcal{V}\right) \rightarrow \text{Fun}(\text{Disk}_+(M_*), \mathcal{V}) .$$

The first arrow is restriction along the projection $\text{Disk}_{n,+}/M_* \rightarrow \text{Disk}_{n,+}$; the second arrow is left Kan extension along the quotient functor $\text{Disk}_{n,+}/M_* \rightarrow (\text{Disk}_{n,+}/M_*)/(\text{Disk}_{n,+})$; the third arrow is restriction along that asserted in Theorem 1.3.4.

Notation 1.3.5. Given an augmented n -disk algebra $A: \text{Disk}_{n,+} \rightarrow \mathcal{V}$, we will use the same notation $A: \text{Disk}_+(M_*) \rightarrow \mathcal{V}$ for the value of the functor (2) on A .

We content ourselves with this Notation 1.3.5 because of the immediate corollary of Theorem 1.3.4.

Corollary 1.3.6. *Let \mathcal{V} be a symmetric monoidal ∞ -category whose underlying ∞ -category admits sifted colimits. Let $A: \text{Disk}_{n,+} \rightarrow \mathcal{V}$ be an augmented n -disk algebra, and let $C: \text{Disk}_n^+ \rightarrow \mathcal{V}$ be an augmented n -disk coalgebra. There are canonical identifications in \mathcal{V} :*

$$\int_{M_*} A \simeq \underset{(B \sqcup U \hookrightarrow M_*) \in \mathcal{D}\text{isk}_+(M_*)}{\text{colim}} A(U_+) ,$$

and

$$\int^{M_*} C \simeq \underset{(B \sqcup V \hookrightarrow M_*) \in \mathcal{D}\text{isk}^+(M_*^-)}{\text{lim}} C(V^+) .$$

1.4. Free and trivial algebras. We give two procedures for constructing augmented n -disk algebras. In this subsection we fix a symmetric monoidal ∞ -category \mathcal{V} which is \otimes -presentable. From Corollary 3.2.3.5 of [Lu2], the ∞ -category $\text{Alg}_n^{\text{aug}}(\mathcal{V})$ is presentable.

Definition 1.4.1 ($O(n)$ -modules). Let G be a topological group and let \mathcal{V} be an ∞ -category. The ∞ -category of G -modules is the functor category

$$\text{Mod}_G(\mathcal{V}) := \text{Fun}(BG, \mathcal{V})$$

from the ∞ -groupoid associated to the classifying space of G .

Warning 1.4.2. In the case where the topological group is the orthogonal group, $G = O(n)$, and $\mathcal{V} = \text{Ch}_{\mathbb{k}}$ is chain complexes over a ring \mathbb{k} , there is an equivalence

$$\text{Mod}_{O(n)}(\text{Ch}_{\mathbb{k}}) \simeq \text{Mod}_{\mathcal{C}_*(O(n); \mathbb{k})}(\text{Ch}_{\mathbb{k}})$$

between $O(n)$ -modules (in the sense of Definition 1.4.1) and modules for the differential graded algebra of \mathbb{k} -linear chains on $O(n)$. This should *not* be confused, in the case \mathbb{k} is \mathbb{R} or \mathbb{C} , with the usual category of representations of $O(n)$ as a Lie group. There is a functor from the representation category to the functor category, but it is far from being an equivalence.

Throughout this work we will make use of the basic and essential result from differential topology that the inclusions

$$O(n) \hookrightarrow \text{GL}(n) \hookrightarrow \text{Diff}(\mathbb{R}^n) \hookrightarrow \text{Emb}(\mathbb{R}^n, \mathbb{R}^n)$$

are all homotopy equivalences. Gram–Schmidt orthogonalization defines a deformation retraction onto the inclusion $O(n) \xrightarrow{\simeq} \text{GL}(n)$. Conjugating by scaling and translation, $(f, t) \mapsto (x \mapsto \frac{f(tx) - f(0)}{t} + f(0))$, then defines a deformation retraction onto the inclusion $\text{GL}(n) \xrightarrow{\simeq} \text{Emb}(\mathbb{R}^n, \mathbb{R}^n)$.

Write $\mathcal{V}_{\mathbb{1}}//_{\mathbb{1}}$ for the ∞ -category of objects $E \in \mathcal{V}$ equipped with a retraction onto the symmetric monoidal unit: $\text{id}_{\mathbb{1}}: \mathbb{1} \rightarrow E \rightarrow \mathbb{1}$. Note that if \mathcal{V} is stable, then there is a natural equivalence $\mathcal{V}_{\mathbb{1}}//_{\mathbb{1}} \xrightarrow{\simeq} \mathcal{V}$ sending an object $\mathbb{1} \rightarrow E \rightarrow \mathbb{1}$ to the cokernel of the unit $\text{cKer}(\mathbb{1} \rightarrow E)$. Write $\text{Disk}_{n,+}^{\leq 1} \subset \text{Disk}_{n,+}$ for the full ∞ -subcategory consisting of those zero-pointed Euclidean spaces

with at most one non-base component. By the equivalence $\mathbf{O}(n) \simeq \mathbf{Emb}(\mathbb{R}^n, \mathbb{R}^n)$, this full ∞ -subcategory is initial among pointed ∞ -categories under $\mathbf{BO}(n)$. In other words, there is a canonical equivalence of ∞ -categories

$$\mathbf{Fun}_{\mathbb{1}}(\mathcal{D}\text{isk}_{n,+}^{\leq 1}, \mathcal{V}) \xrightarrow{\simeq} \mathbf{Mod}_{\mathbf{O}(n)}(\mathcal{V}_{\mathbb{1}} // \mathbb{1})$$

where the source is functors whose value on $*$ is a symmetric monoidal unit of \mathcal{V} ; the target is $\mathbf{O}(n)$ -modules in retractive objects over the unit.

Definition 1.4.3 (Free). Restriction along $\mathcal{D}\text{isk}_{n,+}^{\leq 1} \subset \mathcal{D}\text{isk}_{n,+}$ determines the solid arrow, referred to as the *underlying $\mathbf{O}(n)$ -module*:

$$(3) \quad \mathbf{Alg}_n^{\text{aug}}(\mathcal{V}) \begin{array}{c} \xleftarrow{\mathbb{F}^{\text{aug}}} \\ \xrightarrow{\quad \quad \quad} \\ \downarrow \mathcal{D}\text{isk}_{n,+}^{\leq 1} \\ \end{array} \mathbf{Mod}_{\mathbf{O}(n)}(\mathcal{V}_{\mathbb{1}} // \mathbb{1}) .$$

This forgetful functor preserves limits, and so there is a left adjoint, as depicted, referred to as the *augmented free* functor.

If \mathcal{V} is a stable ∞ -category, then the forgetful functor from algebras to $\mathbf{O}(n)$ -modules has an inverse, the *trivial algebra* functor \mathbf{t}^{aug} . Given an $\mathbf{O}(n)$ -module V , the augmented n -disk algebra $\mathbf{t}^{\text{aug}}V$ has V as its underlying $\mathbb{1} \oplus \mathbf{O}(n)$ -module; the restriction to V of the multiplication map $V \otimes V \rightarrow V$ factors as the augmentation followed by the unit: $V \otimes V \rightarrow \mathbb{1} \otimes \mathbb{1} \simeq \mathbb{1} \rightarrow V$. See §7.3 of [Lu2] for a formal construction. This allows the following definition of the adjoint

Definition 1.4.4 (Cotangent space). The *augmented cotangent space* functor is the left adjoint to the *augmented trivial* functor:

$$\mathbf{Alg}_n^{\text{aug}}(\mathcal{V}) \begin{array}{c} \xleftarrow{\mathbf{t}^{\text{aug}}} \\ \xrightarrow{\quad \quad \quad} \\ \downarrow L^{\text{aug}} \\ \end{array} \mathbf{Mod}_{\mathbf{O}(n)}(\mathcal{V}_{\mathbb{1}} // \mathbb{1}) .$$

The cotangent space is an inverse to the free functor.

Lemma 1.4.5. *Let \mathcal{V} be a \otimes -presentable symmetric monoidal ∞ -category. There is a canonical equivalence of endofunctors of $\mathbf{Mod}_{\mathbf{O}(n)}(\mathcal{V}_{\mathbb{1}} // \mathbb{1})$:*

$$L^{\text{aug}} \circ \mathbb{F}^{\text{aug}} \xrightarrow{\simeq} \text{id} .$$

Proof. The composition of the forgetful functor and the trivial algebra functor is equivalent to the identity, therefore the composite of their left adjoints is the identity. \square

1.4.1. **Stable case.** In this section, fix a \otimes -stable-presentable symmetric monoidal ∞ -category \mathcal{V} . Recall that \mathcal{V} is naturally tensored over the ∞ -category of pointed spaces. Using this structure, we define a functor $-\otimes_{\mathbf{O}(n)} - : \mathbf{Mod}_{\mathbf{O}(n)}(\mathbf{Spaces}_*) \times \mathbf{Mod}_{\mathbf{O}(n)}(\mathcal{V}) \rightarrow \mathcal{V}$ by the following composite:

$$\begin{array}{ccc} \mathbf{Mod}_{\mathbf{O}(n)}(\mathbf{Spaces}_*) \times \mathbf{Mod}_{\mathbf{O}(n)}(\mathcal{V}) & \xrightarrow{\quad \quad \quad} & \mathcal{V} \\ \downarrow & & \uparrow \\ \mathbf{Mod}_{\mathbf{O}(n) \times \mathbf{O}(n)}(\mathbf{Spaces}_* \times \mathcal{V}) & \xrightarrow{\quad \quad \quad} \mathbf{Mod}_{\mathbf{O}(n) \times \mathbf{O}(n)}(\mathcal{V}) \xrightarrow{\quad \quad \quad} & \mathbf{Mod}_{\mathbf{O}(n)}(\mathcal{V}) \end{array}$$

where the second step is given by the tensoring operation $\mathbf{Spaces}_* \times \mathcal{V} \rightarrow \mathcal{V}$; the third step is restriction along the diagonal map $\mathbf{O}(n) \rightarrow \mathbf{O}(n) \times \mathbf{O}(n)$; the last step is taking the coinvariants of the action by $\mathbf{O}(n)$. Dually, we define a functor

$$\mathbf{Map}^{\mathbf{O}(n)}(-, -) : \mathbf{Mod}_{\mathbf{O}(n)}(\mathbf{Spaces}_*)^{\text{op}} \times \mathbf{Mod}_{\mathbf{O}(n)}(\mathcal{V}) \rightarrow \mathcal{V}$$

by substituting the cotensor $(\mathbf{Spaces}_*)^{\text{op}} \times \mathcal{V} \rightarrow \mathcal{V}$ for the tensor, and invariants for coinvariants.

In the case that \mathcal{V} is stable there is an equivalence $\text{Ker}^{\text{aug}}: \mathcal{V}_{\mathbb{1} // \mathbb{1}} \simeq \mathcal{V}: \mathbb{1} \oplus (-)$, and thereafter an equivalence

$$(4) \quad \text{Ker}^{\text{aug}}: \text{Mod}_{\text{O}(n)}(\mathcal{V}_{\mathbb{1} // \mathbb{1}}) \simeq \text{Mod}_{\text{O}(n)}(\mathcal{V}): \mathbb{1} \oplus (-) .$$

Notation 1.4.6 (\mathbb{F} and L). In the case that \mathcal{V} is stable, we denote

$$\mathbb{F}: \text{Mod}_{\text{O}(n)}(\mathcal{V}) \underset{(4)}{\simeq} \text{Mod}_{\text{O}(n)}(\mathcal{V}_{\mathbb{1} // \mathbb{1}}) \xrightarrow{\mathbb{F}^{\text{aug}}} \text{Alg}_n^{\text{aug}}(\mathcal{V}) ,$$

$$L: \text{Alg}_n^{\text{aug}}(\mathcal{V}) \underset{\mathfrak{t}^{\text{aug}}}{\overset{L^{\text{aug}}}{\rightleftarrows}} \text{Mod}_{\text{O}(n)}(\mathcal{V}_{\mathbb{1} // \mathbb{1}}) \underset{(4)}{\simeq} \text{Mod}_{\text{O}(n)}(\mathcal{V}): \mathfrak{t} .$$

In this stable case, we make \mathbb{F} explicit and so recognize L . The underlying $\text{O}(n)$ -module of the value of \mathbb{F} on a $\text{O}(n)$ -module V is

$$(5) \quad \mathbb{F}(V) \simeq \bigoplus_{i \geq 0} \left(\text{Conf}_i^{\text{fr}}(\mathbb{R}_+^n) \bigotimes_{\Sigma_i \wr \text{O}(n)} V^{\otimes i} \right) ,$$

because the monoidal structure distributes over colimits. Stability of \mathcal{V} implies stability of $\text{Mod}_{\text{O}(n)}(\mathcal{V})$. In the solid diagram among ∞ -categories

$$\begin{array}{ccc} & \text{Stab}(\text{Alg}_n^{\text{aug}}(\mathcal{V})) & \\ \Sigma^\infty \nearrow & & \dashrightarrow \alpha \\ \text{Alg}_n^{\text{aug}}(\mathcal{V}) & \xrightarrow{L} & \text{Mod}_{\text{O}(n)}(\mathcal{V}) \end{array}$$

there is a canonical filler, from the stabilization, as a colimit preserving functor. See §7.3.4 of [Lu2] or Proposition 2.23 of [Fr2], which state that the functor α is an equivalence, and so the horizontal functor in the above diagram witnesses $\text{Mod}_{\text{O}(n)}(\mathcal{V})$ as the stabilization of $\text{Alg}_n^{\text{aug}}(\mathcal{V})$.

1.5. Linear Poincaré duality. In the case that the symmetric monoidal ∞ -category is of the form \mathcal{S}^\oplus , with underlying ∞ -category \mathcal{S} stable and presentable, and whose symmetric monoidal structure is given by direct sum, factorization homology and factorization cohomology profoundly simplify. Here we state Poincaré duality in this simplified setting.

For \mathcal{X} a small ∞ -category with a zero object, we denote by $\text{PShv}_*(\mathcal{X})$ the ∞ -category of those (space-valued) presheaves on \mathcal{X} whose value on the zero object is $*$.

Definition 1.5.1 (Frame bundle). The *frame bundle* functor is the composition

$$\text{Fr}: \mathcal{Z}\text{Mfld} \rightarrow \text{PShv}_*(\mathcal{Z}\text{Mfld}) \rightarrow \text{PShv}(\text{BO}(n))^{*/} \simeq \text{Mod}_{\text{O}(n)^{\text{op}}}(\text{Spaces}_*)$$

of the Yoneda embedding, followed by restriction along the full subcategory $\text{Disk}_{n,+}^{\leq 1} \subset \mathcal{Z}\text{Mfld}$ – this subcategory is initial among ∞ -categories under $\text{BO}(n)$ with a zero object. Explicitly, Fr_{M_*} can be identified as the pointed space $\text{ZEmb}(\mathbb{R}_+^n, M_*)$ with $\text{O}(n)$ -action given by precomposition by homeomorphisms of \mathbb{R}^n .

Notation 1.5.2 (Framed configurations). For M_* a zero-pointed n -manifold, we denote the $\Sigma_i \wr \text{O}(n)$ -module in pointed spaces

$$\text{Conf}_i^{\text{fr}}(M_*) := \text{Fr}_{\text{Conf}_i(M_*)} .$$

The following is a formulation of Poincaré or Atiyah duality for a suitable class of zero-pointed manifolds.

Theorem 1.5.3 (Linear Poincaré duality [AF2]). *Let \mathcal{S} be a stable and presentable ∞ -category, and let E and F be $\text{O}(n)$ -modules in \mathcal{S} . Let M_* be a conically finite zero-pointed n -manifold. A morphism of $\text{O}(n)$ -modules $\alpha: (\mathbb{R}^n)^+ \otimes E \rightarrow F$ canonically determines a morphism in \mathcal{S}*

$$\alpha_{M_*}: \text{Fr}_{M_*} \bigotimes_{\text{O}(n)} E \longrightarrow \text{Map}^{\text{O}(n)}(\text{Fr}_{M_*}, F) .$$

Furthermore, if α is an equivalence then so is α_{M_*} .

Remark 1.5.4 (With B -structures). We follow up on Remark 1.1.11. There is a version of Theorem 1.5.3 that is also true in the context of B -manifolds – it is stated and proved there. We will make use of this version as it applies to $\mathbf{B}(\Sigma_i \wr \mathcal{O}(n))$ -structured zero-pointed manifolds of the form $\text{Conf}_i(M_*)$.

2. FILTRATIONS

In this section we establish the cardinality (co)filtration of factorization (co)homology, as well as the Goodwillie cofiltration of factorization homology. We use these to give partial results for Poincaré/Koszul duality for when the monoidal structure of \mathcal{V} does not distribute over totalizations, as with the case of chain complexes with tensor product or spectra with smash product. Throughout this section, if not otherwise specified, the following parameters are assumed to be fixed.

- A conically smooth zero-pointed n -manifold M_* (see Definition 1.1.4).
- A \otimes -stable-presentable symmetric monoidal ∞ -category \mathcal{V} .
- An augmented n -disk algebra $A: \mathcal{D}\text{isk}_{n,+} \rightarrow \mathcal{V}$.
- An augmented n -disk coalgebra $C: \mathcal{D}\text{isk}_n^+ \rightarrow \mathcal{V}$.

Example 2.0.5. Here are some standard examples of such entities.

- Let \overline{M} be a smooth cobordism from $\partial_{\mathbf{L}} =: \partial_{\mathbf{L}}$ to $\partial_{\mathbf{R}} =: \partial_{\mathbf{R}}$. In this case, $M_*^\square = * \coprod_{\partial_{\mathbf{R}}} (\overline{M} \setminus \partial_{\mathbf{L}})$, and

$$\text{Conf}_i(M_*) = * \prod_B \{f: \{1, \dots, i\} \hookrightarrow \overline{M} \setminus \partial_{\mathbf{R}}\}$$

where $B = \{f \mid \emptyset \neq f^{-1}\partial_{\overline{M}}\}$, and

$$\text{Conf}_i^\square(M_*^\square) = * \prod_{B'} \{f: \{1, \dots, i\} \rightarrow \overline{M} \setminus \partial_{\mathbf{L}}\}$$

where $B' = \{f \mid \emptyset \neq f^{-1}\partial_{\overline{M}} \text{ or } |f^{-1}x| > 1 \text{ for some } x \in \overline{M}\}$.

- Write $\text{Ch}_{\mathbb{k}}$ for the ∞ -category of chain complexes over a commutative ring \mathbb{k} . Then $\text{Ch}_{\mathbb{k}}^\oplus$ and $\text{Ch}_{\mathbb{k}}^\otimes$, equipped with direct sum and tensor product, are examples of such symmetric monoidal ∞ -categories. In general, any such \mathcal{V} is symmetric monoidally tensored and cotensored over finite spaces.
- For \mathcal{S} a stable presentable ∞ -category, let $E \in \mathcal{S}$ be an object. The assignment $A: U_* \mapsto E^{U_*^\square}$ depicts an augmented n -disk algebra in \mathcal{S}^\oplus – its underlying object is (non-canonically) identified as $E[-n] \simeq \Omega^n E$. Likewise, the assignment $C: U_* \mapsto U_* \otimes E$ depicts an augmented n -disk coalgebra in \mathcal{S}^\oplus – its underlying object is (non-canonically) identified as $E[n] \simeq \Sigma^n E$. Constructing n -(co)algebras in more general \mathcal{V} is substantially more interesting, and also more involved, depending on the specifics of \mathcal{V} .

2.1. Main results. Here we display the main results in this section and prove them based upon results developed in latter subsections.

2.1.1. Cardinality (co)filtration. We observe a natural filtration of factorization homology, and identify the filtration quotients; we do likewise for factorization cohomology. Write $[M_*]$ for the set of connected components of M_* , which we regard as a based set. For each finite cardinality i , we will denote the subcategories of based finite sets

$$(\text{Fin}_*^{\leq i})_{/[M_*]} \subset (\text{Fin}_*^{\text{surj}})_{/[M_*]} \subset (\text{Fin}_*)_{/[M_*]}$$

where an object of the middle is a *surjective* based map $I_+ \rightarrow [M_*]$, and a morphism between two such is a *surjective* map over $[M_*]$; and where the left is the full subcategory consisting of those $I_+ \rightarrow [M_*]$ for which the cardinality $|I| \leq i$ is bounded. Taking connected components gives a functor

$$[-]: \mathcal{D}\text{isk}_+(M_*) \rightarrow (\text{Fin}_*)_{/[M_*]}$$

to based finite sets over the based set of connected components of M_* .

Definition 2.1.1 ($\mathcal{D}\text{isk}_+^{\leq i}(M_*)$). We define the ∞ -category

$$\mathcal{D}\text{isk}_+^{\text{surj}}(M_*) := \mathcal{D}\text{isk}_+(M_*)_{|(\text{Fin}_*^{\text{surj}})_{/[M_*]}} \subset \mathcal{D}\text{isk}_+(M_*)$$

and, for each finite cardinality i , the full ∞ -subcategory

$$\mathcal{D}\text{isk}_+^{\leq i}(M_*) := \mathcal{D}\text{isk}_+^{\text{surj}}(M_*)_{|(\text{Fin}_*^{\leq i})_{/[M_*]}} \subset \mathcal{D}\text{isk}_+^{\text{surj}}(M_*).$$

We denote the opposites:

$$\mathcal{D}\text{isk}_+^{+, \text{surj}}(M_*^\neg) := (\mathcal{D}\text{isk}_+^{\text{surj}}(M_*))^\text{op} \quad \text{and} \quad \mathcal{D}\text{isk}_+^{+, \leq i}(M_*^\neg) := (\mathcal{D}\text{isk}_+^{\leq i}(M_*))^\text{op}.$$

Definition 2.1.2. Let i be a finite cardinality. We define the object of \mathcal{V}

$$\tau^{\leq i} \int_{M_*} A := \text{colim}_{(B \sqcup U \hookrightarrow M_*) \in \mathcal{D}\text{isk}_+^{\leq i}(M_*)} A(U_+) = \text{colim} \left(\mathcal{D}\text{isk}_+^{\leq i}(M_*) \rightarrow \mathcal{D}\text{isk}_+(M_*) \xrightarrow{A} \mathcal{V} \right).$$

We define the object of \mathcal{V}

$$\tau^{\leq i} \int^{M_*^\neg} C := \lim_{(B \sqcup V \hookrightarrow M_*) \in \mathcal{D}\text{isk}_+^{+, \leq i}(M_*)} C(V^+) = \lim \left(\mathcal{D}\text{isk}_+^{+, \leq i}(M_*^\neg) \rightarrow \mathcal{D}\text{isk}_n^+(M_*^\neg) \xrightarrow{C} \mathcal{V} \right).$$

We likewise define such objects for the comparison “ $\leq i$ ” replaced by other comparisons among finite cardinalities, such as “ $\geq i$ ” and “ $= i$ ”.

The $\mathbb{Z}_{\geq 0}$ -indexed sequence of fully faithful functors

$$\dots \longrightarrow \mathcal{D}\text{isk}_+^{\leq i}(M_*) \longrightarrow \mathcal{D}\text{isk}_+^{\leq i+1}(M_*) \longrightarrow \dots \longrightarrow \mathcal{D}\text{isk}_+^{\text{surj}}(M_*)$$

witnesses $\mathcal{D}\text{isk}_+^{\text{surj}}(M_*)$ as a sequential colimit. There results a canonical sequence in \mathcal{V}

$$(6) \quad \dots \longrightarrow \tau^{\leq i-1} \int_{M_*} A \longrightarrow \tau^{\leq i} \int_{M_*} A \longrightarrow \dots \longrightarrow \int_{M_*} A.$$

Dually, there is a canonical sequence in \mathcal{V}

$$(7) \quad \int^{M_*^\neg} C \longrightarrow \dots \longrightarrow \tau^{\leq i} \int^{M_*^\neg} C \longrightarrow \tau^{\leq i-1} \int^{M_*^\neg} C \longrightarrow \dots.$$

There are likewise sequences with $\tau^{\leq i}$ replaced by $\tau^{\geq i}$.

Lemma 2.1.3 (Cardinality convergence). *The morphism in \mathcal{V} from the colimit of the cardinality sequence*

$$\tau^{\leq \infty} \int_{M_*} A \xrightarrow{\cong} \int_{M_*} A$$

is an equivalence. Likewise, the morphism in \mathcal{V} to the limit

$$\int^{M_*^\neg} C \xrightarrow{\cong} \tau^{\leq \infty} \int^{M_*^\neg} C$$

is an equivalence.

Proof. Directly apply Corollary 1.3.6 and Lemma 2.3.2 which states that $\mathcal{D}\text{isk}_+^{\text{surj}}(M_*) \rightarrow \mathcal{D}\text{isk}_+(M_*)$ is final. □

The following main result of this subsection identifies the layers of the filtration (6) and the cofiltration (7) in terms of configuration spaces. We give a proof of this result at the end of §2.3.2.

Theorem 2.1.4 (Cardinality cokernels and kernels). *Each arrow in the cardinality filtration of factorization homology in (6) belongs to a canonical cofiber sequence*

$$(8) \quad \tau^{\leq i-1} \int_{M_*} A \longrightarrow \tau^{\leq i} \int_{M_*} A \longrightarrow \text{Conf}_i^{\neg, \text{fr}}(M_*) \bigotimes_{\Sigma_i \wr \mathcal{O}(n)} A^{\otimes i} .$$

Likewise, each arrow in the cardinality cofiltration of factorization cohomology in (7) belongs to a canonical fiber sequence

$$(9) \quad \text{Map}^{\Sigma_i \wr \mathcal{O}(n)}(\text{Conf}_i^{\neg, \text{fr}}(M_*), C^{\otimes i}) \longrightarrow \tau^{\leq i} \int^{M_*} C \longrightarrow \tau^{\leq i-1} \int^{M_*} C .$$

Remark 2.1.5. In the $n = 1$ case where the manifold M is the circle and A is an associative algebra, the cardinality filtration specializes to the Hodge filtration of Hochschild homology studied by Burghelea–Vigu-Poirrier [BuV], Feigin–Tsygan [FT1], Gerstenhaber–Schack [GS], and Loday [Lo].

2.1.2. Goodwillie cofiltration. We are about to apply Goodwillie’s calculus to functors of the form $\text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \mathcal{V}$. This calculus was developed by Goodwillie in [Go] for application to functors from pointed spaces and has since been generalized; see [Lu2], [Kuh1], and [Kuh2]. We begin by recalling this formalism.

Let i be a finite cardinality. Let \mathcal{X} and \mathcal{Y} be presentable ∞ -categories, each with a zero object. The ∞ -category of *polynomial functors of degree i* is the full ∞ -subcategory of reduced functors

$$\text{Poly}_i(\mathcal{X}, \mathcal{Y}) \subset \text{Fun}_0(\mathcal{X}, \mathcal{Y})$$

consisting of those that send strongly coCartesian $(i + 1)$ -cubes to Cartesian $(i + 1)$ -cubes. This inclusion admits a left adjoint

$$P_i : \text{Fun}_0(\mathcal{X}, \mathcal{Y}) \longrightarrow \text{Poly}_i(\mathcal{X}, \mathcal{Y})$$

implementing a localization. There is the full ∞ -subcategory of *homogeneous functors of degree i*

$$\text{Homog}_i(\mathcal{X}, \mathcal{Y}) \subset \text{Poly}_i(\mathcal{X}, \mathcal{Y})$$

consisting of those polynomial functors H of degree i for which $P_j H \simeq 0$ is the zero functor provided $j < i$. Consequently, each reduced functor $F : \mathcal{X} \rightarrow \mathcal{Y}$ canonically determines a cofiltration of reduced functors

$$F \rightarrow P_\infty F \rightarrow \cdots \rightarrow P_i F \rightarrow P_{i-1} F \rightarrow \cdots$$

with each composite arrow $F \rightarrow P_i F$ the unit for the above adjunction, and with each kernel $\text{Ker}(P_i F \rightarrow P_{i-1} F)$ homogeneous of degree i . Here we denote $P_\infty F := \varprojlim_i P_i F$ for the inverse limit of the cofiltration.

For a fixed zero-pointed n -manifold M_* , we apply this discussion to the functor $\int_{M_*} : \text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \mathcal{V}_{\mathbb{1}} //_{\mathbb{1}} \simeq \mathcal{V}$ – here we have used the identification of retractive objects over the unit of \mathcal{V} with \mathcal{V} , as discussed in §1.4. In particular, there is a canonical arrow between functors

$$(10) \quad \int_{M_*} \longrightarrow P_\infty \int_{M_*} .$$

The next result identifies the layers of the Goodwillie cofiltration of factorization homology in terms of configuration spaces and the cotangent space. We give a proof of this result at the end of §2.6.

Theorem 2.1.6. *There is a canonical fiber sequence among functors $\text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \mathcal{V}$:*

$$\text{Conf}_i^{\text{fr}}(M_*) \bigotimes_{\Sigma_i \wr \mathcal{O}(n)} L(-)^{\otimes i} \longrightarrow P_i \int_{M_*} \longrightarrow P_{i-1} \int_{M_*}$$

for every i a finite cardinality.

Unlike the cardinality (co)filtration of factorization (co)homology of the previous subsection, the Goodwillie cofiltration of factorization homology does not always converge. The next result provides an understood class of parameters for which the Goodwillie cofiltration converges.

We will reference the notion of a *t-structure* on \mathcal{V} , which is just a usual t-structure on the homotopy category of \mathcal{V} (see Definition 1.2.1.4 of [Lu2]). That is, it consists of fully faithful inclusions

$$\mathcal{V}_{>0} \hookrightarrow \mathcal{V} \hookleftarrow \mathcal{V}_{\leq 0} .$$

These inclusions admit a right adjoint $\pi_{>0} : \mathcal{V} \rightarrow \mathcal{V}_{>0}$ and a left adjoint $\pi_{\leq 0} : \mathcal{V} \rightarrow \mathcal{V}_{\leq 0}$. For any object V , the units and counits of these adjunctions define a natural cofiber sequence

$$\pi_{>0}V \rightarrow V \rightarrow \pi_{\leq 0}V$$

We say that a t-structure on \mathcal{V} is *compatible with the symmetric monoidal structure* of \mathcal{V} if the restriction of the functor $\mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$ to $\mathcal{V}_{\geq 0} \times \mathcal{V}_{\geq 0}$ factors through $\mathcal{V}_{\geq 0}$, and if the unit $\mathbb{1} \in \mathcal{V}_{\geq 0}$ is connective. A t-structure is *cocomplete* if the natural map $V \rightarrow \varprojlim \pi_{\leq i}V$ is an equivalence for every object $V \in \mathcal{V}$. Equivalently, cocompleteness means that the inverse limit $\varprojlim \pi_{> i}V$ is the zero object for every V .

We give a proof of this next result at the end of §2.5.

Theorem 2.1.7. *Suppose there exists a cocomplete t-structure on \mathcal{V} that is compatible with the symmetric monoidal structure. The value of the canonical arrow (10) evaluated on A*

$$\int_{M_*} A \longrightarrow P_\infty \int_{M_*} A$$

is an equivalence in \mathcal{V} provided either of the following criteria is satisfied:

- the augmentation ideal $\text{Ker}(A \rightarrow \mathbb{1})$ is connected (with respect to the given t-structure);
- the topological space M_* is connected and compact and the augmentation ideal $\text{Ker}(A \rightarrow \mathbb{1})$ is connective (with respect to the given t-structure).

Remark 2.1.8. A similar result is treated by Matsuoka in [Mat].

2.1.3. Comparing cofiltrations. We compare the cardinality and Goodwillie cofiltrations. Recall the Poincaré duality arrow (1) of Theorem 1.2.4.

We give a proof of the next result at the end of §2.7.

Theorem 2.1.9. *The Poincaré/Koszul duality arrow $\int_{(-)} \rightarrow \int^{(-)\neg} \text{Bar}$ extends to an equivalence of cofiltrations of functors $\text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \text{Fun}(\mathcal{Z}\text{Mfld}_n^{\text{fin}}, \mathcal{V})$:*

$$(11) \quad P_\bullet \int_{(-)} \xrightarrow{\cong} \tau^{\leq \bullet} \int^{(-)\neg} \text{Bar} .$$

Specifically, for each smoothable compact n -manifold \overline{M} with partitioned boundary $\partial \overline{M} = \partial_{\mathbb{L}} \sqcup \partial_{\mathbb{R}}$, each augmented n -disk algebra A in \mathcal{V} , and each finite cardinality i , there is an equivalence in \mathcal{V}

$$P_i \int_{\overline{M} \setminus \partial_{\mathbb{R}}} A \xrightarrow{\cong} \tau^{\leq i} \int^{\overline{M} \setminus \partial_{\mathbb{L}}} \text{Bar}(A) .$$

Corollary 2.1.10. *The Poincaré/Koszul duality arrow $\int_{M_*} A \rightarrow \int^{M_*\neg} \text{Bar} A$ of (1) canonically factors through an equivalence in \mathcal{V}*

$$P_\infty \int_{M_*} A \xrightarrow{\cong} \int^{M_*\neg} \text{Bar} A$$

from the limit of the Goodwillie cofiltration. This factorization is functorial in M_* and A .

2.2. Non-unital algebras. In an ambient stable situation, there is a convenient equivalence between augmented algebras and non-unital algebras.

Recall the connected component functor $\mathcal{D}isk_{n,+} \xrightarrow{\pi_0} \mathbf{Fin}_*$, which is symmetric monoidal. We denote $\mathbf{Fin}_*^{\text{surj}} \subset \mathbf{Fin}_*$ for the subcategory of based finite sets and surjections among them; this is a symmetric monoidal subcategory.

Definition 2.2.1. $\mathcal{D}isk_{n,+}^{\text{surj}}$ is the pullback ∞ -category

$$\mathcal{D}isk_{n,+}^{\text{surj}} := (\mathcal{D}isk_{n,+})|_{\mathbf{Fin}_*^{\text{surj}}} .$$

For \mathcal{V} a symmetric monoidal ∞ -category, the ∞ -category of *non-unital* n -disk algebras in \mathcal{V} is

$$\mathbf{Alg}_n^{\text{nu}}(\mathcal{V}) := \mathbf{Fun}_0^{\otimes}(\mathcal{D}isk_{n,+}^{\text{surj}}, \mathcal{V})$$

the ∞ -category of non-unital symmetric monoidal functors that respect terminal objects.

Proposition 2.2.2 (Proposition 5.4.4.10 of [Lu2]). *There is an equivalence of ∞ -categories*

$$\mathbf{Ker}^{\text{aug}}: \mathbf{Alg}_n^{\text{aug}}(\mathcal{V}) \simeq \mathbf{Alg}_n^{\text{nu}}(\mathcal{V}): \mathbb{1} \oplus (-)$$

– the values of the left functor are depicted as $\mathbf{Ker}^{\text{aug}} A: \mathbb{V}\mathbb{R}_+^n \mapsto \mathbf{Ker}(A(\mathbb{V}\mathbb{R}_+^n) \rightarrow A(*) \simeq \mathbb{1})$. This equivalence is natural among such \mathcal{V} .

2.3. Support for Theorem 2.1.4 (cardinality layers).

2.3.1. Configuration space quotients. Here we pull some results from [AFT1], [AFT2], and [AF2] to identify the cofiber of $\mathcal{D}isk_+^{\leq i}(M_*) \rightarrow \mathcal{D}isk_+^{\leq i}(M_*)$. For this subsection we fix a finite cardinality i .

In §3 of [AFT1] we construct, for each stratified space X , a stratified space $\mathbf{Ran}_{\leq i}(X)$ whose points are finite subsets $S \subset X$ for which the map to connected components $S \rightarrow [X]$ is surjective; and we show that $\mathbf{Ran}_{\leq i}(-)$ is continuously functorial among conically smooth embeddings among stratified spaces which are surjective on connected components. In §2 of [AFT2] we explain that the resulting functor

$$(12) \quad \mathbf{Ran}_{\leq i}: \mathcal{D}isk^{\text{surj}, \leq i}(\mathcal{B}\text{sc})/X \xrightarrow{\simeq} \mathcal{B}\text{sc}/\mathbf{Ran}_{\leq i}(X)$$

is an equivalence.

Corollary 2.3.1. *The functor (12) restricts as an equivalence of ∞ -categories:*

$$\mathbf{Ran}_{\leq i}: \mathcal{D}isk_+^{\text{surj}, \leq i}(M_*) \xrightarrow{\simeq} \mathcal{D}isk_+^{\text{surj}, \leq 1}(\mathbf{Ran}_{\leq i}(M_*)) .$$

Proof. Let $B \subset M_*$ be a basic neighborhood of $*$. In [AFT1] it is shown that any conically smooth open embedding from a basic $U \hookrightarrow M_*$ whose image contains $*$ is canonically based-isotopic to one that factors through a based isomorphism $U \cong B \hookrightarrow M_*$. We conclude that the projection from the slice

$$(13) \quad (\mathcal{D}isk(\mathcal{B}\text{sc})/M_*)^{B/} \xrightarrow{\simeq} \mathcal{D}isk_+(M_*)$$

is an equivalence of ∞ -categories. Likewise, we conclude that the projection from the slice

$$(\mathcal{D}isk_+^{\leq 1}(\mathbf{Ran}_{\leq i}(M_*)))^{\mathbf{Ran}_{\leq i}(B)/} \xrightarrow{\simeq} \mathcal{D}isk_+^{\leq 1}(\mathbf{Ran}_{\leq i}(M_*))$$

is an equivalence of ∞ -categories. The result follows from the equivalence (12). □

This identification affords the following essential consequence.

Lemma 2.3.2. *The functor*

$$\mathcal{D}isk_+^{\text{surj}}(M_*) \longrightarrow \mathcal{D}isk_+(M_*)$$

is final.

Proof. Both $\mathcal{D}\text{isk}_+^{\text{surj}}(M_*) \rightarrow \mathcal{D}\text{isk}^{\text{surj}}(\mathcal{B}\text{sc})_{/M_*}$ and $\mathcal{D}\text{isk}_+(M_*) \rightarrow \mathcal{D}\text{isk}(\mathcal{B}\text{sc})_{/M_*}$ are final. By the partial two-of-three property of finality, the assertion follows from showing that the functor

$$\mathcal{D}\text{isk}^{\text{surj}}(\mathcal{B}\text{sc})_{/M_*} \longrightarrow \mathcal{D}\text{isk}(\mathcal{B}\text{sc})_{/M_*}$$

is final. Writing $M_* \cong \bigvee_{i \in [M_*]} M_{i,*}$ as a wedge over its components, this functor is expressible as a product

$$\mathcal{D}\text{isk}^{\text{surj}}(\mathcal{B}\text{sc})_{/M_*} \cong \prod_{i \in [M_*]} \mathcal{D}\text{isk}^{\text{surj}}(\mathcal{B}\text{sc})_{/M_{i,*}} \longrightarrow \prod_{i \in [M_*]} \mathcal{D}\text{isk}(\mathcal{B}\text{sc})_{/M_{i,*}} \cong \mathcal{D}\text{isk}(\mathcal{B}\text{sc})_{/M_*} .$$

Since a product of final functors is final, we can reduce to the case of a factor, i.e., the case in which M_* is irreducible. Applying Quillen's Theorem A, it suffices to show the weak contractibility of the classifying space of $\mathcal{D}\text{isk}^{\text{surj}}(\mathcal{B}\text{sc})_{/M_*}^{V/}$ for each $V \in \mathcal{D}\text{isk}(\mathcal{B}\text{sc})_{/M_*}$ for M_* connected.

We show this in two cases: V empty or nonempty. If V is empty, we have an identification of this classifying space as

$$\mathbb{B}\left(\mathcal{D}\text{isk}^{\text{surj}}(\mathcal{B}\text{sc})_{/M_*}\right) \simeq \text{Ran}(M_*) ,$$

the Ran space of M_* . This follows from Corollary 2.3.1 by taking sequential colimits over i ; the formation of both sides preserves sequential colimits. The result now follows from the weakly contractible of the Ran space – to see this, we refer to a now standard argument of [BeD]: since the Ran space carries a natural H-space structure by taking unions of subsets, and the composition of the diagonal and the H-space multiplication is the identity, therefore its homotopy groups are zero. If V is nonempty, then the map $V \hookrightarrow M_*$ is surjective on components hence defines an object of $\mathcal{D}\text{isk}^{\text{surj}}(\mathcal{B}\text{sc})_{/M_*}$. Consequently, $(\mathcal{D}\text{isk}^{\text{surj}}(\mathcal{B}\text{sc})_{/M_*})^{V/}$ has an initial object in this case, hence this ∞ -category has a weakly contractible classifying space. \square

The following is an essential result, a description of the layers in the cardinality filtration of the ∞ -category $\mathcal{D}\text{isk}_+(M_*)$.

Lemma 2.3.3. *There is a canonical cofiber sequence among ∞ -categories*

$$\mathcal{D}\text{isk}_+^{<i}(M_*) \longrightarrow \mathcal{D}\text{isk}_+^{\leq i}(M_*) \longrightarrow \mathcal{D}\text{isk}_+^{\leq 1}(\text{Conf}_i^{\neg}(M_*)_{\Sigma_i})$$

for any i a finite cardinality.

Proof. We explain the diagram among ∞ -categories

$$\begin{array}{ccccc} \mathcal{D}\text{isk}_+^{<i}(M_*) & \longrightarrow & \mathcal{D}\text{isk}_+^{\leq i}(M_*) & & \\ \downarrow (\text{Lem } \approx 2.3.1) & & \downarrow (\text{Lem } \approx 2.3.1) & & \\ \mathcal{D}\text{isk}_+^{\leq 1}(\text{Ran}_{<i}(M_*)) & \longrightarrow & \mathcal{D}\text{isk}_+^{\leq 1}(\text{Ran}_{\leq i}(M_*)) & \longrightarrow & \mathcal{D}\text{isk}_+^{\leq 1}(\text{Conf}_i^{\neg}(M_*)_{\Sigma_i}) . \end{array}$$

The top horizontal functor is the one whose cofiber is being examined. The vertical equivalences are directly from Lemma 2.3.1, as indicated. A main result of [AFT1] gives that, for

$$X \rightarrow Y \rightarrow Z$$

a cofiber sequence of stratified spaces comprised of constructible maps, then there is a resulting cofiber sequence of ∞ -categories

$$\mathcal{B}\text{sc}_{/X} \longrightarrow \mathcal{B}\text{sc}_{/Y} \longrightarrow \mathcal{B}\text{sc}_{/Z} .$$

Applying this to

$$\text{Ran}_{<i}(M_*) \longrightarrow \text{Ran}_{\leq i}(M_*) \longrightarrow \text{Conf}_i^{\neg}(M_*) ,$$

followed up by the same logic as in the proof of Lemma 2.3.1, gives that the bottom sequence in the diagram is a cofiber sequence. \square

2.3.2. Reduced extensions. We explain a couple general maneuvers concerning left Kan extensions in the presence of zero objects. For $f : \mathcal{K} \rightarrow \mathcal{K}'$ functor among small ∞ -categories, and \mathcal{V} a presentable ∞ -category, there is an adjunction among functor ∞ -categories

$$f_! : \mathcal{V}^{\mathcal{K}} \rightleftarrows \mathcal{V}^{\mathcal{K}'} : f^*$$

with the right adjoint f^* given by precomposing with f , and with the left adjoint $f_!$ given by left Kan extension. In the following, let \mathcal{V} be a presentable ∞ -category with a zero object. For $* \xrightarrow{x} \mathcal{K}$ a pointed ∞ -category (i.e., an ∞ -category with a distinguished object—which need not be zero), then we define the full ∞ -subcategory

$$\mathrm{Fun}_0(\mathcal{K}, \mathcal{V}) \subset \mathcal{V}^{\mathcal{K}}$$

as the fiber over the zero object in the sequence $\mathrm{Fun}_0(\mathcal{K}, \mathcal{V}) \rightarrow \mathcal{V}^{\mathcal{K}} \xrightarrow{x^*} \mathcal{V}$.

Lemma 2.3.4. *Under the hypotheses above, the inclusion $\mathrm{Fun}_0(\mathcal{K}, \mathcal{V}) \rightarrow \mathcal{V}^{\mathcal{K}}$ admits a left adjoint $(-)^{\mathrm{red}} : \mathcal{V}^{\mathcal{K}} \rightarrow \mathrm{Fun}_0(\mathcal{K}, \mathcal{V})$. Further, this left adjoint fits into a cofiber sequence in $\mathcal{V}^{\mathcal{K}}$*

$$x_! x^* \longrightarrow \mathrm{id}_{\mathcal{V}^{\mathcal{K}}} \longrightarrow (-)^{\mathrm{red}} .$$

Proof. Because x^* preserves colimits, the universal morphism $\mathrm{cKer}(x^* x_! x^* \rightarrow x^*) \xrightarrow{\simeq} x^* \mathrm{cKer}(x_! x^* \rightarrow \mathrm{id})$ is an equivalence in \mathcal{V} . Because the inclusion $* \xrightarrow{x} \mathcal{K}$ is fully faithful, the morphism $x^* x_! x^* \xrightarrow{\simeq} x^*$ is an equivalence in \mathcal{V} . So the values of the cofiber $\mathrm{cKer}(x_! x^* \rightarrow \mathrm{id})$ lie in $\mathrm{Fun}_0(\mathcal{K}, \mathcal{V})$. Because $0 \in \mathcal{V}$ is zero-pointed, then the value $x_!(0) \in \mathcal{V}^{\mathcal{K}}$ is the zero functor. The endofunctor $x_! x^*$ restricts to the zero functor on $\mathrm{Fun}_0(\mathcal{K}, \mathcal{V})$. Consequently, $(-)^{\mathrm{red}}$ restricts to the identity functor on $\mathrm{Fun}_0(\mathcal{K}, \mathcal{V})$. Finally, the arrow $\mathrm{id} \rightarrow (-)^{\mathrm{red}}$ witnesses $(-)^{\mathrm{red}}$ as a left adjoint as claimed. \square

Lemma 2.3.5. *Let $i : \mathcal{K}_0 \rightarrow \mathcal{K}$ be a fully faithful functor among ∞ -categories, and consider the functor $j : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{K}_0 := * \coprod_{\mathcal{K}_0} \mathcal{K}$ to the cone, regarded as a pointed ∞ -category. There is a cofiber sequence in the functor ∞ -category $\mathcal{V}^{\mathcal{K}}$*

$$i_! i^* \longrightarrow \mathrm{id}_{\mathcal{V}^{\mathcal{K}}} \longrightarrow j^* j_!^{\mathrm{red}}$$

for any \mathcal{V} a stable presentable ∞ -category.

Proof. There is a canonical fiber sequence of ∞ -categories

$$\mathrm{Fun}_0(\mathcal{K}/\mathcal{K}_0, \mathcal{V}) \xrightarrow{j^*} \mathcal{V}^{\mathcal{K}} \xrightarrow{i^*} \mathcal{V}^{\mathcal{K}_0} .$$

The inclusion of the constant zero functor $\{0\} \subset \mathcal{V}^{\mathcal{K}_0}$ is fully faithful, therefore the functor j^* is fully faithful since fully faithfulness is preserved by the formation of pullbacks. We therefore identify $\mathrm{Fun}_0(\mathcal{K}/\mathcal{K}_0, \mathcal{V})$ with the corresponding full ∞ -subcategory of $\mathcal{V}^{\mathcal{K}}$ in the following.

Because i^* preserves colimits, the universal morphism $\mathrm{cKer}(i^* i_! i^* \rightarrow i^*) \xrightarrow{\simeq} i^* \mathrm{cKer}(i_! i^* \rightarrow \mathrm{id})$ is an equivalence in $\mathcal{V}^{\mathcal{K}_0}$. Because $i : \mathcal{K}_0 \rightarrow \mathcal{K}$ is fully faithful, the morphism $i^* i_! i^* \xrightarrow{\simeq} i^*$ is an equivalence in $\mathcal{V}^{\mathcal{K}_0}$. It follows that the values of $\mathrm{cKer}(i_! i^* \rightarrow \mathrm{id})$ lie in $\mathrm{Fun}_0(\mathcal{K}/\mathcal{K}_0, \mathcal{V})$. Because $0 \in \mathcal{V}$ is a zero object, the value $i_!(0)$ is the zero functor. It then follows that $i_! i^*$ restricts to the zero functor on $\mathrm{Fun}_0(\mathcal{K}/\mathcal{K}_0, \mathcal{V})$; therefore $\mathrm{cKer}(i_! i^* \rightarrow \mathrm{id})$ restricts to the identity functor on $\mathrm{Fun}_0(\mathcal{K}/\mathcal{K}_0, \mathcal{V})$. In summary, the endofunctor $\mathrm{cKer}(i_! i^* \rightarrow \mathrm{id}) : \mathcal{V}^{\mathcal{K}} \rightarrow \mathcal{V}^{\mathcal{K}}$ factors through $\mathrm{Fun}_0(\mathcal{K}/\mathcal{K}_0, \mathcal{V}) \xrightarrow{j^*} \mathcal{V}^{\mathcal{K}}$. Since the factorizing functor $j_!^{\mathrm{red}} : \mathcal{V}^{\mathcal{K}} \rightarrow \mathrm{Fun}_0(\mathcal{K}/\mathcal{K}_0, \mathcal{V})$ is the composite of left adjoints, $j_!$ and $(-)^{\mathrm{red}}$, therefore $j_!^{\mathrm{red}}$ is again left adjoint to j^* . \square

Proof of Theorem 2.1.4. We explain the cofiber sequence (8). The argument concerning the fiber sequence (9) is dual.

From Lemma 2.3.5, with that notation, there is a cofiber sequence

$$\tau^{\leq i-1} \int_{M_*} A \longrightarrow \tau^{\leq i} \int_{M_*} A \longrightarrow \operatorname{colim} \left((\mathcal{D}\operatorname{isk}_+^{\leq i}(M_*)) / (\mathcal{D}\operatorname{isk}_+^{\leq i-1}(M_*)) \xrightarrow{j^* j_i^{\operatorname{red}} A} \mathcal{V} \right)$$

Through Lemma 2.3.3 the colimit expression is canonically identified as the colimit

$$\operatorname{colim} \left((\mathcal{D}\operatorname{isk}_+^{\leq 1}(\operatorname{Conf}_i^-(M_*)) \xrightarrow{A^{\otimes i}} \mathcal{V} \right).$$

Unwinding definitions, this colimit is $\operatorname{Conf}_i^{-, \operatorname{fr}}(M_*) \otimes_{\Sigma_i \wr \mathcal{O}(n)} A^{\otimes i}$ – the tensor over based modules in spaces. □

2.4. Free calculation. Here we give the calculation of the factorization homology of a free algebra. This calculation is a fundamental input to a number of our arguments. To make this calculation, we assume throughout this subsection that the symmetric monoidal structure of \mathcal{V} distributes over colimits, and we will use the notation \oplus for the coproduct on \mathcal{V} . For the next result, note that each $\mathcal{O}(n)$ -module V in \mathcal{V} determines a $\Sigma_i \wr \mathcal{O}(n)$ -module $V^{\otimes i}$ in \mathcal{V} .

Theorem 2.4.1. *Let \mathcal{V} be a \otimes -cocomplete symmetric monoidal ∞ -category. Let $V \in \operatorname{Mod}_{\mathcal{O}(n)}(\mathcal{V}_{\mathbf{1}} // \mathbf{1})$ be a $\mathcal{O}(n)$ -module in retractive objects over the unit of \mathcal{V} . There is a canonical identification of the factorization homology of the free augmented algebra on V in terms of configuration spaces of M :*

$$\int_{M_*} \mathbb{F}^{\operatorname{aug}} V \simeq \bigoplus_{i \geq 0} \left(\operatorname{Conf}_i^{\operatorname{fr}}(M_*) \otimes_{\Sigma_i \wr \mathcal{O}(n)} V^{\otimes i} \right).$$

Proof. The term on the righthand side depicts a functor $\mathcal{Z}\operatorname{Mfld}_n \rightarrow \mathcal{V}$, naturally in V . This functor canonically extends as a symmetric monoidal functor, because of the distribution assumption on the symmetric monoidal structure of \mathcal{V} . It is manifest that the restriction to $\mathcal{D}\operatorname{isk}_{n,+}$ of the righthand side satisfies the universal property of the free functor $\mathbb{F}^{\operatorname{aug}}$. This proves the theorem for the case of $M_* = \bigvee_J \mathbb{R}_+^n$ a finite disjoint union of Euclidean spaces.

We explain the following sequence of equivalences in \mathcal{V} :

$$\begin{aligned} \int_{M_*} \mathbb{F}^{\operatorname{aug}} V &\stackrel{(1)}{\simeq} \operatorname{colim} \left(\mathcal{D}\operatorname{isk}_+(M_*) \xrightarrow{\mathbb{F}^{\operatorname{aug}} V} \mathcal{V} \right) \\ &\stackrel{(2)}{\simeq} \bigoplus_{i \geq 0} \operatorname{colim}_{U_+ \in \mathcal{D}\operatorname{isk}_+(M_*)} \left(\operatorname{Conf}_i^{\operatorname{fr}}(U_+) \otimes_{\Sigma_i \wr \mathcal{O}(n)} V^{\otimes i} \right) \\ &\stackrel{(3)}{\simeq} \bigoplus_{i \geq 0} \operatorname{colim}_{U_+ \in \mathcal{D}\operatorname{isk}_+(M_*)} \left(\operatorname{colim} \left(\mathcal{D}\operatorname{isk}_+^{\leq i}(U_+) \xrightarrow{V^{\otimes i}} \mathcal{V} \right) \right) \\ &\stackrel{(4)}{\simeq} \bigoplus_{i \geq 0} \operatorname{colim} \left(\mathcal{X}^i \xrightarrow{V^{\otimes i}} \mathcal{V} \right) \\ &\stackrel{(5)}{\simeq} \bigoplus_{i \geq 0} \operatorname{colim} \left(\mathcal{D}\operatorname{isk}_+^{\leq i}(M_*) \xrightarrow{V^{\otimes i}} \mathcal{V} \right) \\ &\stackrel{(6)}{\simeq} \bigoplus_{i \geq 0} \operatorname{colim} \left((\mathcal{D}\operatorname{isk}_+^{\leq 1}(\operatorname{Conf}_i^-(M_*)) \xrightarrow{V^{\otimes i}} \mathcal{V} \right) \end{aligned}$$

The equivalence (1) is the finality of the functor $\mathcal{D}\operatorname{isk}_+(M_*) \rightarrow (\mathcal{D}\operatorname{isk}_{n,+}/M_*) / \mathcal{D}\operatorname{isk}_{n,+}$ (Theorem 1.3.4). The equivalence (2) follows from the first paragraph, combined with the distribution of \bigoplus over sifted colimits (using Theorem 1.3.4). The equivalences (3) and (6) are the identifications $\mathcal{D}\operatorname{isk}_+^{\leq i}(-) \simeq \mathcal{D}\operatorname{isk}_+^{\leq 1}(\operatorname{Conf}_i^-(-))$ of Lemma 2.3.3.

Consider the ∞ -category of arrows $\operatorname{Fun}([1], \mathcal{D}\operatorname{isk}_+(M_*))$. Evaluation at 0 gives a functor

$$\operatorname{ev}_0: \operatorname{Fun}([1], \mathcal{D}\operatorname{isk}_+(M_*)) \longrightarrow \mathcal{D}\operatorname{isk}_+(M_*).$$

We denote the pullback ∞ -category

$$\mathcal{X}^i := \mathcal{D}\text{isk}_+^{\leq i}(M_*) \times_{\mathcal{D}\text{isk}_+(M_*)} \text{Fun}([1], \mathcal{D}\text{isk}_+(M_*)) .$$

There is thus a functor $\mathcal{X}^i \xrightarrow{\text{ev}_0} \mathcal{D}\text{isk}_+^{\leq i}(M_*)$, which is a Cartesian fibration. For each object $e: \bigsqcup_i \mathbb{R}^n \rightarrow M$ in $\mathcal{D}\text{isk}_+^{\leq i}(M_*)$, the object $(e = e)$ is initial in the fiber ∞ -category $\text{ev}_0^{-1}e$. It follows that the functor $\mathcal{X}^i \xrightarrow{\text{ev}_0} \mathcal{D}\text{isk}_+^{\leq i}(M_*)$ is final. This explains the equivalence (5). The equivalence (4) is formal, because nested colimits agree with colimits. \square

Corollary 2.4.2. *Let \mathcal{V} be a \otimes -stable-presentable symmetric monoidal ∞ -category, and let V be a $\text{O}(n)$ -module in \mathcal{V} . For each finite cardinality i , the diagram in \mathcal{V}*

$$\tau^{>i} \int_{M_*} \mathbb{F}^{\text{aug}} V \longrightarrow \int_{M_*} \mathbb{F}^{\text{aug}} V \longrightarrow P_i \int_{M_*} \mathbb{F}^{\text{aug}} V$$

can be canonically identified with the diagram

$$\bigoplus_{j>i} \left(\text{Conf}_j^{\text{fr}}(M_*) \otimes_{\Sigma_j \wr \text{O}(n)} V^{\otimes j} \right) \longrightarrow \bigoplus_{j \geq 0} \left(\text{Conf}_j^{\text{fr}}(M_*) \otimes_{\Sigma_j \wr \text{O}(n)} V^{\otimes j} \right) \longrightarrow \bigoplus_{j \leq i} \left(\text{Conf}_j^{\text{fr}}(M_*) \otimes_{\Sigma_j \wr \text{O}(n)} V^{\otimes j} \right)$$

given by inclusion and projection of summands.

Proof. The identification of the left term can be seen by inspecting the definition of $\tau^{>i} \int_{M_*}$ and tracing the proof of Theorem 2.4.1. The identification of the right term follows by induction on i , for which both the base case and the inductive step are supported by Corollary 2.6.2, using the equivalence $L\mathbb{F}V \xrightarrow{\cong} V$ of Lemma 1.4.5. That the arrows are as claimed is manifest. \square

A basic feature of Koszul duality in general is that it sends free algebras to trivial algebras. We are about to reference the notion of a trivial augmented n -coalgebra, the definition of which is exactly dual to that of trivial algebras (see Definition 1.4.4).

Lemma 2.4.3. *Let \mathcal{V} be a \otimes -presentable symmetric monoidal ∞ -category. Let V be a $\text{O}(n)$ -module in $\mathcal{V}_{\mathbb{1} // \mathbb{1}}$ and consider the diagonal $\text{O}(n)$ -module $(\mathbb{R}^n)^+ \otimes V$ in $\mathcal{V}_{\mathbb{1} // \mathbb{1}}$. There is a canonical equivalence of augmented n -disk coalgebras in \mathcal{V}*

$$\text{Bar}(\mathbb{F}^{\text{aug}} V) \simeq \mathfrak{t}_{\text{cAlg}}^{\text{aug}}((\mathbb{R}^n)^+ \otimes V)$$

from the bar construction of the free augmented algebra to the trivial augmented coalgebra.

Proof. Let J be a finite set and consider the action

$$\text{ZEmb}((\mathbb{R}^n)^+, \bigvee_J (\mathbb{R}^n)^+) \otimes \int_{(\mathbb{R}^n)^+} \mathbb{F}^{\text{aug}} V \longrightarrow \int_{\bigvee_J (\mathbb{R}^n)^+} \mathbb{F}^{\text{aug}} V \simeq \left(\int_{(\mathbb{R}^n)^+} \mathbb{F}^{\text{aug}} V \right)^{\otimes J} .$$

Through Theorem 2.4.1, using the pigeonhole principle, for each $i_1, \dots, i_j > 1$ the restriction of this morphism to $\text{Conf}_1((\mathbb{R}^n)^+) \otimes V$ followed by the projection to $\bigotimes_{j \in J} \text{Conf}_{i_j}((\mathbb{R}^n)^+) \otimes_{\Sigma_{i_j}} V^{\otimes i_j}$ is canonically equivalent to the zero morphism.

For each $\epsilon > 0$ consider the subspace $\text{Conf}_i^\epsilon((\mathbb{R}^n)^+) \subset \text{Conf}_i((\mathbb{R}^n)^+)$ consisting of those based maps $f: \{1, \dots, i\}_+ \rightarrow (\mathbb{R}^n)^+$, whose restriction $f|_1: f^{-1}\mathbb{R}^n \rightarrow \mathbb{R}^n$ is injective and $x = y \in \mathbb{R}^n$ whenever $\|f_1(x) - f_1(y)\| < \epsilon$ – the inclusion of this subspace is a based weak homotopy equivalence. Flowing the vector field $x \mapsto x$ on \mathbb{R}^n for infinite time witnesses a deformation retraction of $\text{Conf}_i^\epsilon((\mathbb{R}^n)^+)$ onto $*$ provided $i > 1$. We conclude that $\text{Conf}_i((\mathbb{R}^n)^+)$ is weakly contractible for $i > 1$. Through Theorem 2.4.1 we arrive at a canonical identification

$$\mathbb{1} \oplus \text{Conf}_1((\mathbb{R}^n)^+) \otimes V \xrightarrow{\cong} \int_{(\mathbb{R}^n)^+} \mathbb{F}^{\text{aug}} V .$$

Combined with the above paragraph, we conclude that $\text{Bar}(\mathbb{F}^{\text{aug}}V)$ is canonically identified as the trivial augmented n -disk coalgebra on the $O(n)$ -module $\text{Conf}_1((\mathbb{R}^n)^+) \otimes V = ((\mathbb{R}^n)^+ \otimes V)$. \square

2.5. Support for Theorem 2.1.7 (convergence).

Lemma 2.5.1. *Let \mathcal{V} be a \otimes -sifted cocomplete symmetric monoidal ∞ -category. The functors $\text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \mathcal{V}$*

$$\tau^{>i} \int_{M_*} \quad \text{and} \quad \int_{M_*} \quad \text{and} \quad P_i \int_{M_*}$$

preserve sifted colimits for any i a finite cardinality.

Proof. We first prove the statement for \int_{M_*} . Let $A: \mathcal{J} \rightarrow \text{Alg}_n^{\text{aug}}(\mathcal{V})$ be a diagram of augmented n -disk algebras in \mathcal{V} , indexed by a sifted ∞ -category. The canonical arrow $\text{colim}_{j \in \mathcal{J}} \int_{M_*} A_j \rightarrow \int_{M_*} \text{colim}_{j \in \mathcal{J}} A_j$ in \mathcal{V} is a composition

$$\text{colim}_{j \in \mathcal{J}} \text{colim}_{U_+ \in \mathcal{D}\text{isk}_+(M_*)} A_j(U_+) \simeq \text{colim}_{U_+ \in \mathcal{D}\text{isk}_+(M_*)} \text{colim}_{j \in \mathcal{J}} A_j(U_+) \longrightarrow \text{colim}_{U_+ \in \mathcal{D}\text{isk}_+(M_*)} (\text{colim}_{j \in \mathcal{J}} A_j)(U_+)$$

where the outer objects are in terms of the defining expression for factorization homology, the left equivalence is through commuting colimits, and the right arrow is a colimit of canonical arrows. Because of the hypotheses on \mathcal{V} , each arrow $\text{colim}_{j \in \mathcal{J}} A_j(U_+) \rightarrow (\text{colim}_{j \in \mathcal{J}} A_j)(U_+)$ is an equivalence if and only if it is for U connected. This is the case provided the forgetful functor $\text{ev}_{\mathbb{R}_+^n}: \text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \mathcal{V}$ preserves sifted colimits. This assertion is Proposition 3.2.3.1 of [Lu2].

Because P_i is a left adjoint, it commutes with sifted colimits. The statement is thus true for $P_i \int_{M_*}$ after the first paragraph.

The functor $\tau^{>i} \int_{M_*}: \text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \text{Fun}(\mathcal{D}\text{isk}_+^{>i}(M_*), \mathcal{V}) \xrightarrow{\text{colim}} \mathcal{V}$ is a composition of two functors, the latter of which exists on the image of the first and it commutes with those sifted colimits on which it is defined. Colimits in the middle ∞ -category are given objectwise, and so it is enough to show that the restriction $\text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \text{Fun}(\mathcal{D}\text{isk}_+^{=i}(M_*), \mathcal{V})$ preserves sifted colimits for each finite cardinality i . For $i = 1$, this follows from the first paragraph as the case $M_* = \mathbb{R}_+^n$. For general i this follows because the functor $\otimes: \mathcal{V}^{\times i} \rightarrow \mathcal{V}$ preserves sifted colimits, by assumption. \square

Lemma 2.5.2 (Free resolutions). *Let \mathcal{V} be a \otimes -sifted cocomplete symmetric monoidal ∞ -category. Every augmented n -disk algebra in \mathcal{V} is a sifted colimit of free augmented n -disk algebras. That is, there are no proper full ∞ -subcategories of $\text{Alg}_n^{\text{aug}}(\mathcal{V})$ that contain the essential image of the functor $\mathbb{F}^{\text{aug}}: \text{Mod}_{O(n)}(\mathcal{V}_{\mathbb{1}} // \mathbb{1}) \rightarrow \text{Alg}_n^{\text{aug}}(\mathcal{V})$ and that is closed under the formation of sifted colimits.*

Proof. We apply Lurie's ∞ -categorical Barr–Beck theorem. The forgetful functor $\text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \mathcal{V}_{\mathbb{1}} // \mathbb{1}$ is conservative and preserves sifted colimits (by Proposition 3.2.3.1 of [Lu2]). The result then follows from Proposition 4.7.4.14 of [Lu2]. \square

Lemma 2.5.3. *Let A be an augmented n -disk algebra in \mathcal{V} . The canonical sequence of arrows among functors $\text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \mathcal{V}$*

$$\tau^{>i} \int_{M_*} \longrightarrow \int_{M_*} \longrightarrow P_i \int_{M_*}$$

is a cofiber sequence for any i a finite cardinality.

Proof. Corollary 2.4.2 immediately gives the result for the case that A is free. Every augmented n -disk algebra is a sifted colimit of free augmented n -disk algebras. Lemma 2.5.1 states that the left two functors commute with sifted colimits. Because P_i is a left adjoint, the functor $P_i \int_{M_*}$ commutes with sifted colimits. \square

Proof of Theorem 2.1.7. Choose such a t-structure as in the statement of the theorem. It suffices to show that the kernel of the canonical arrow is the zero object. There is a pair of equivalences in \mathcal{V}

$$\mathrm{Ker}\left(\int_{M_*} A \rightarrow P_\infty \int_{M_*} A\right) \underset{(1)}{\simeq} \varprojlim_i \mathrm{Ker}\left(\int_{M_*} A \rightarrow P_i \int_{M_*} A\right) \underset{(2)}{\simeq} \varprojlim_i \tau^{>i} \int_{M_*} A.$$

The equivalence labeled (1) follows because formation of kernels commutes with sequential limits. Equivalence (2) follows from Lemma 2.5.3, which identifies the term of cardinality cofiltration $\tau^{>i} \int_{M_*} A$ as the kernel of $\int_{M_*} A \rightarrow P_i \int_{M_*} A$. We claim that, under either of two criteria, the object $\tau^{>i} \int_{M_*} A$ is i -connected with respect to the t-structure. This claim then implies $\varprojlim_i \tau^{>i} \int_{M_*} A$ is infinitely connected under the named criteria; this completes the proof because \mathcal{V} being cocomplete with respect to the given t-structure implies that only the zero object is infinitely connected.

We now prove the above mentioned claim. Since connectivity is preserved under colimits, it suffices to resolve the algebra A by free n -disk algebras. We can thus reduce to showing that $\tau^{>i} \int_{M_*} \mathbb{F}^{\mathrm{aug}} V$ is i -connected, the case in which $A \simeq \mathbb{F}^{\mathrm{aug}} V$ is the free augmented n -disk algebra on a $\mathcal{O}(n)$ -module V . By Corollary 2.4.2, we have a calculation of this truncation as

$$\tau^{>i} \int_{M_*} \mathbb{F}^{\mathrm{aug}} V \simeq \bigoplus_{\ell > i} (\mathrm{Conf}_\ell^{\mathrm{fr}}(M_*) \otimes_{\Sigma_\ell \mathcal{O}(n)} V^{\otimes \ell}).$$

Since the inclusion $\mathcal{V}_{\geq i} \rightarrow \mathcal{V}$ is a left adjoint, the essential image is closed under colimits – see §1.2.1 of [Lu2]. So it is enough to show that $\mathrm{Conf}_i^{\mathrm{fr}}(M_*) \otimes_{\Sigma_i \mathcal{O}(n)} V^{\otimes i}$ is i -connected under either of the named

criteria, for all i . Apply this fact to the functor $\mathrm{Disk}_+^{\leq 1}(\mathrm{Conf}_i(M_*)) \xrightarrow{j^* j_i^{\mathrm{red}} V^{\otimes i}} \mathcal{V}$ from the proof of Theorem 2.1.4 (using the notation of §2.3) supposing one of the criteria is satisfied: under the first criterion, $V^{\otimes i}$ is i -connected and $\mathrm{Conf}_i(M_*)$ is connected; under the second criterion, $\mathrm{Conf}_i(M_*)$ is i -connected and $V^{\otimes i}$ is connected. The claim follows since colimits preserve connectivity. \square

2.6. Support for Theorem 2.1.6 (Goodwillie layers). For the next result we denote the diagonal functor as $\mathrm{diag}_i: \mathrm{Mod}_{\mathcal{O}(n)}(\mathcal{V}) \rightarrow \mathrm{Mod}_{\mathcal{O}(n)}(\mathcal{V})^{\times i}$, where \mathcal{V} is a \otimes -cocomplete stable symmetric monoidal ∞ -category.

Lemma 2.6.1. *There is an equivalence of ∞ -categories*

$$\mathrm{Poly}_1(\mathrm{Mod}_{\mathcal{O}(n)}(\mathcal{V})_{\Sigma_i}^{\times i}, \mathcal{V}) \xrightarrow{\simeq} \mathrm{Homog}_i(\mathrm{Alg}_n^{\mathrm{aug}}(\mathcal{V}), \mathcal{V})$$

for each i a finite cardinality. It assigns to D the functor $D(\mathrm{diag}_i \circ L(-))$.

Proof. Theorem 6.1.4.7 of [Lu2] gives the following identification: there is a canonical fully faithful functor

$$\mathrm{Homog}_i(\mathrm{Alg}_n^{\mathrm{aug}}(\mathcal{V}), \mathcal{V}) \hookrightarrow \mathrm{Fun}(\mathrm{Stab}(\mathrm{Alg}_n^{\mathrm{aug}}(\mathcal{V}))_{\Sigma_i}^{\times i}, \mathcal{V})$$

whose essential image consists of the Σ_i -invariant functors that preserve colimits in each variable. For \mathcal{O} an operad, there is the general canonical equivalence of ∞ -categories $\mathrm{Stab}(\mathrm{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{V})) \simeq \mathrm{Mod}_{\mathcal{O}(1)}(\mathcal{V})$ – see §7.3.4 of [Lu2]. Through the general equivalence above, the named expression depicts an inverse to this fully faithful functor on its essential image. \square

Corollary 2.6.2. *Let M_* be a zero-pointed n -manifold, and let i be a finite cardinality. The functor*

$$\mathrm{Conf}_i^{\mathrm{fr}}(M_*) \otimes_{\Sigma_i \mathcal{O}(n)} L(-)^{\otimes i}: \mathrm{Alg}_n^{\mathrm{aug}}(\mathcal{V}) \longrightarrow \mathcal{V}$$

is homogeneous of degree i .

Proof. The coend $\text{Conf}_i^{\text{fr}}(M_*) \otimes_{\Sigma_i \wr \mathcal{O}(n)} -: \text{Mod}_{\mathcal{O}(n)}(\mathcal{V})^{\times i} \rightarrow \mathcal{V}$ preserves colimits and is Σ_i -invariant, so we can apply Lemma 2.6.1. \square

Proof of Theorem 2.1.6. Through Lemma 2.6.1, the i -homogeneous layer of the Goodwillie cofiltration of the functor \int_{M_*} is a symmetric functor of i -variables of $\mathcal{O}(n)$ -modules. To identify this multi-variable functor we evaluate on free augmented n -disk algebras. Through Corollary 2.4.2, there is a canonical identification

$$\text{Ker}\left(P_i \int_{M_*} \mathbb{F}V \rightarrow P_{i-1} \int_{M_*} \mathbb{F}V\right) \simeq \text{Conf}_i^{\text{fr}}(M_*) \otimes_{\Sigma_i \wr \mathcal{O}(n)} V^{\otimes i}$$

functorially in the $\mathcal{O}(n)$ -module V . This verifies the theorem in this free case because of the canonical equivalence $L\mathbb{F}V \simeq V$ of Lemma 1.4.5. \square

2.7. Support for Theorem 2.1.9 (comparing cofiltrations). In this subsection we fix a \otimes -stable-presentable symmetric monoidal ∞ -category \mathcal{V} .

We use the following result, which generalizes Lemma 2.4.3 away from the case of free algebras, at the level of $\mathcal{O}(n)$ -modules.

Theorem 2.7.1 (Corollary 2.29 of [Fr2]). *There is a canonical equivalence between functors $\text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \text{Mod}_{\mathcal{O}(n)}(\mathcal{V})$*

$$(\mathbb{R}^n)^+ \otimes L(-) \simeq \text{cKer}^{\text{aug}}(\text{Bar}(-)) .$$

Corollary 2.7.2. *For each conically finite zero-pointed (ni) -manifold P_* , equipped with a $\Sigma_i \wr \mathcal{O}(n)$ -structure, there is a canonical equivalence of functors $\text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \mathcal{V}$*

$$\text{Fr}_{P_*} \otimes_{\Sigma_i \wr \mathcal{O}(n)} L(-)^{\otimes i} \xrightarrow{\simeq} \text{Map}^{\Sigma_i \wr \mathcal{O}(n)}(\text{Fr}_{P_*}, \text{Bar}(-)^{\otimes i}) .$$

Proof. Theorem 2.7.1 gives a canonical equivalence among $\Sigma_i \wr \mathcal{O}(n)$ -modules in \mathcal{V} :

$$(\mathbb{R}^{ni})^+ \otimes L(-)^{\otimes i} \simeq ((\mathbb{R}^n)^+ \otimes L(-))^{\otimes i} \simeq (\text{cKer}^{\text{aug}}(\text{Bar}(-)))^{\otimes i} .$$

The result is a direct corollary of Theorem 1.5.3, according to the $\mathbf{B}(\Sigma_i \wr \mathcal{O}(n))$ -structured version of Remark 1.5.4. \square

Corollary 2.7.3. *For each finite cardinality i , the functor $\tau^{\leq i} \int_{M_*} \text{Bar}: \text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \mathcal{V}$ is polynomial of degree i .*

Proof. Through Corollary 2.7.2, Corollary 2.6.2 implies $\text{Map}^{\Sigma_i \wr \mathcal{O}(n)}(\text{Fr}_{P_*}, \text{Bar}(-)^{\otimes i})$ is homogeneous of degree i . The result then follows by induction using the fibration sequence of Theorem 2.1.4. \square

Proof of Theorem 2.1.9. Let M_* be a zero-pointed n -manifold. Let i be a finite cardinality. Corollary 2.7.3 asserts that the functor $\tau^{\leq i} \int_{M_*} \text{Bar}: \text{Alg}_n^{\text{aug}}(\mathcal{V}) \rightarrow \mathcal{V}$ is polynomial of degree i . The morphism of cofiltrations $P_\bullet \int_{M_*} \rightarrow \tau^{\leq \bullet} \int_{M_*} \text{Bar}$ follows through the universal property of the Goodwillie cofiltration. There results a morphism of i -homogeneous layers:

$$\text{Ker}\left(P_i \int_{M_*} \rightarrow P_{i-1} \int_{M_*}\right) \longrightarrow \text{Ker}\left(\tau^{\leq i} \int_{M_*} \text{Bar} \rightarrow \tau^{\leq i-1} \int_{M_*} \text{Bar}\right) .$$

Through Theorem 2.1.4 and Theorem 2.1.6, this morphism is canonically equivalent to the morphism of functors

$$(14) \quad \text{Conf}_i^{\text{fr}}(M_*) \otimes_{\Sigma_i \wr \mathcal{O}(n)} L(-)^{\otimes i} \longrightarrow \text{Map}^{\Sigma_i \wr \mathcal{O}(n)}(\text{Conf}_i^{\text{fr}}(M_*), \text{Bar}(-)^{\otimes i}) .$$

This is the arrow of Corollary 2.7.2 applied to the $\mathbf{B}(\Sigma_i \wr \mathbf{O}(n))$ -structured zero-pointed manifold $\mathbf{Conf}_i(M_*)$ of Proposition 1.1.9, and so the arrow is an equivalence. \square

3. FACTORIZATION HOMOLOGY OF FORMAL MODULI PROBLEMS

In this section, we will be concerned with a generalization of factorization homology which allows for a more general coefficient system, a moduli functor of n -disk algebras (as in the works on derived algebraic geometry [Lu4], [TV], and [Fr1]).

Definition 3.0.4 (Linear dual). Let \mathcal{V} be a symmetric monoidal ∞ -category with \mathcal{V} presentable. There is a functor $\mathcal{V}^{\text{op}} \rightarrow \mathbf{PShv}(\mathcal{V})$ given by $c \mapsto \mathcal{V}(- \otimes c, \mathbb{1})$, morphisms to the symmetric monoidal unit. Should \mathcal{V} be \otimes -presentable, this functor canonically factors through the Yoneda embedding as a functor

$$(-)^\vee : \mathcal{V}^{\text{op}} \rightarrow \mathcal{V}$$

which we refer to as *linear dual*.

We choose to simplify and familiarize matters by restricting our generality.

Convention (Focus on $\mathbf{Ch}_{\mathbb{k}}^{\otimes}$). Henceforward, we fix a field \mathbb{k} and, unless otherwise stated, work over the background symmetric monoidal ∞ -category $\mathbf{Ch}_{\mathbb{k}}^{\otimes}$ of chain complexes over \mathbb{k} – its equivalences are quasi-isomorphisms. With tensor product it becomes a \otimes -stable-presentable symmetric monoidal ∞ -category, and it is endowed with a standard t-structure. Our choice to work over a field is for the basic but fundamental property that the duality functor exchanges i -connected and $(-i)$ -coconnected objects.

Notation 3.0.5. Let i be an integer. We use the familiar notation $\mathbf{Ch}_{\mathbb{k}}^{\geq i} \subset \mathbf{Ch}_{\mathbb{k}} \supset \mathbf{Ch}_{\mathbb{k}}^{\leq i}$ for the full ∞ -subcategories consisting of those chain complexes whose non-zero homology has the indicated degree bounds.

We simplify the notation

$$\mathbf{Alg}_n^{\text{nu}} := \mathbf{Alg}_n^{\text{nu}}(\mathbf{Ch}_{\mathbb{k}}) \underset{\text{Prop 2.2.2}}{\simeq} \mathbf{Alg}_n^{\text{aug}}(\mathbf{Ch}_{\mathbb{k}}) =: \mathbf{Alg}_n^{\text{aug}} .$$

We denote the full ∞ -subcategories $\mathbf{Alg}_n^{\text{nu}, \geq i} \subset \mathbf{Alg}_n^{\text{nu}} \supset \mathbf{Alg}_n^{\text{nu}, \leq i}$ consisting of those non-unital n -disk algebras whose underlying chain complex lies in $\mathbf{Ch}_{\mathbb{k}}^{\geq i}$ and $\mathbf{Ch}_{\mathbb{k}}^{\leq i}$, respectively. We likewise denote the full ∞ -subcategories $\mathbf{Alg}_n^{\text{aug}, \geq i} \subset \mathbf{Alg}_n^{\text{aug}} \supset \mathbf{Alg}_n^{\text{aug}, \leq i}$ consisting of those augmented n -disk algebras A whose associated non-unital algebra lies in $\mathbf{Alg}_n^{\text{nu}, \geq i}$ and $\mathbf{Alg}_n^{\text{nu}, \leq i}$, respectively.

3.1. Formal moduli. We begin with a few essential notions from derived algebraic geometry of n -disk algebras.

Definition 3.1.1 ($\mathbf{Perf}_{\mathbb{k}}$). We denote the full ∞ -subcategory $\mathbf{Perf}_{\mathbb{k}} \subset \mathbf{Ch}_{\mathbb{k}}$ consisting of those chain complexes V over \mathbb{k} for which the \mathbb{k} -module $\bigoplus_{q \in \mathbb{Z}} \mathbf{H}_q V$ is finite rank over \mathbb{k} . We denote the intersection

$\mathbf{Perf}_{\mathbb{k}}^{\geq i} := \mathbf{Perf}_{\mathbb{k}} \cap \mathbf{Ch}_{\mathbb{k}}^{\geq i}$ consisting of those perfect complexes whose homology vanishes below dimension i .

Definition 3.1.2 (\mathbf{Triv}_n and \mathbf{Artin}_n). The full ∞ -subcategory $\mathbf{Triv}_n \subset \mathbf{Alg}_n^{\text{nu}, \geq 0}$ is the essential image of $\mathbf{Mod}_{\mathbf{O}(n)}(\mathbf{Perf}_{\mathbb{k}}^{\geq 0})$ under the functor that assigns a complex the associated trivial algebra – it consists of trivial connective non-unital n -disk algebras whose underlying $\mathbf{O}(n)$ -module is a perfect chain complex. The ∞ -category \mathbf{Artin}_n of non-unital Artin n -disk algebras is the smallest full ∞ -subcategory of $\mathbf{Alg}_n^{\text{nu}, \geq 0}$ that contains \mathbf{Triv}_n and is closed under small extensions. That is:

If B is in \mathbf{Artin}_n , V is in \mathbf{Triv}_n , and the following diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathbb{k} & \longrightarrow & V \end{array}$$

forms a pullback square in $\mathbf{Alg}_n^{\text{nu}, \geq 0}$, then A is in \mathbf{Artin}_n .

Remark 3.1.3. One might also justifiably call these *local* non-unital Artin algebras, but for economy we omit this extra adjective. In [Lu4], Lurie uses the terminology *small* for an equivalent definition in the case of n -disk algebras with trivializations of the tangent bundles, i.e., \mathcal{E}_n -algebras.

Definition 3.1.4 (Moduli functor). The ∞ -category of formal n -disk moduli functors

$$\mathbf{Moduli}_n := \mathbf{Fun}(\mathbf{Artin}_n, \mathbf{Spaces})$$

is the ∞ -category of copresheaves on non-unital Artin n -disk algebras.

Remark 3.1.5. To obtain a workable geometric theory of formal moduli functors such as that of [Lu4], [TV], or [Hi2], one should restrict to those presheaves that satisfy some gluing condition, such as preserving limits of small extensions (after [Sc]). We will not use these conditions, so for simplicity of presentation we omit them.

Example 3.1.6 (Formal spectrum). There is the composite functor

$$\mathbf{Spf}: (\mathbf{Alg}_n^{\geq 0, \text{aug}})^{\text{op}} \longrightarrow \mathbf{PShv}(\mathbf{Alg}_n^{\text{aug}, \geq 0}) \longrightarrow \mathbf{Moduli}_n$$

of Yoneda followed by restriction. Its values are given by $\mathbf{Spf}(A): R \mapsto \mathbf{Alg}_n^{\text{nu}}(A, R)$, which we refer to as the *formal spectrum of A* .

We choose to conceptually simplify our duality formalism by using linear duals and never consider coalgebras.

Definition 3.1.7. The Koszul duality functor is the composite

$$\mathbb{D}^n: (\mathbf{Alg}_n^{\text{aug}})^{\text{op}} \xrightarrow{\mathbf{Bar}^{\text{op}}} (\mathbf{cAlg}_n^{\text{aug}})^{\text{op}} \xrightarrow{(-)^\vee} \mathbf{Alg}_n^{\text{aug}}$$

which is \mathbf{Bar} followed by linear dual.

Definition 3.1.8. For an Artin n -disk algebra R and an augmented n -disk algebra A over a field \mathbb{k} , the Maurer–Cartan space

$$\mathbf{MC}_A(R) := \mathbf{Alg}_n^{\text{aug}}(\mathbb{D}^n R, A)$$

is the space of maps from the Koszul dual of R to A . The Maurer–Cartan functor $\mathbf{MC}: \mathbf{Alg}_n^{\text{aug}} \rightarrow \mathbf{Moduli}_n$ is the adjoint of the pairing

$$\mathbf{Artin}_n \times \mathbf{Alg}_n^{\text{aug}} \longrightarrow \mathbf{Alg}_n^{\text{aug}, \text{op}} \times \mathbf{Alg}_n^{\text{aug}} \longrightarrow \mathbf{Spaces}$$

where the first functor is $\mathbb{D}^n \times \text{id}$ and the second functor is the mapping space; \mathbf{MC} sends A to the functor \mathbf{MC}_A .

Remark 3.1.9. Our definition of the moduli functor \mathbf{MC}_A has the same form as that given by Lurie in [Lu4], there denoted ΨA . Our construction of the functor \mathbb{D}^n , on the other hand, is somewhat different: we use the geometry of zero-pointed manifolds, whereas Lurie uses twisted arrow categories. To verify that these two constructions agree requires a relationship between twisted arrow categories and zero-pointed manifolds. We defer this problem to a separate work.

The formal spectrum functor \mathbf{Spf} has a right adjoint.

Definition 3.1.10 (Algebra of functions). We denote the (*augmented*) n -disk algebra of functions functor $\mathcal{O}: \mathbf{Moduli}_n \rightarrow (\mathbf{Alg}_n^{\text{aug}})^{\text{op}}$ that is given as

$$\mathcal{O}(X) := \lim_{(\mathbf{Spf}(R) \rightarrow X) \in ((\mathbf{Artin}_n^{\text{op}})_{/X})^{\text{op}}} R = \lim \left(((\mathbf{Artin}_n^{\text{op}})_{/X})^{\text{op}} \rightarrow \mathbf{Artin}_n \rightarrow \mathbf{Alg}_n^{\text{aug}} \right).$$

In other words, the functor \mathcal{O} is the right Kan extension of the inclusion $(\mathbf{Artin}_n)^{\text{op}} \hookrightarrow (\mathbf{Alg}_n^{\text{aug}})^{\text{op}}$ along the functor \mathbf{Spf} :

$$\begin{array}{ccc} (\mathbf{Artin}_n)^{\text{op}} & \xrightarrow{\quad} & (\mathbf{Alg}_n^{\text{aug}})^{\text{op}} \\ \mathbf{Spf} \downarrow & \dashrightarrow \mathcal{O} & \nearrow \\ \mathbf{Moduli}_n & & \end{array} .$$

The Maurer–Cartan functor is a lift of the duality functor \mathbb{D}^n . Namely, there is a canonical equivalence

$$\mathbb{D}^n A \simeq \mathcal{O}(\mathrm{MC}_A)$$

between the Koszul dual of A and the augmented n -disk algebra of functions on the Maurer–Cartan functor of A . This can be seen as a special case of Theorem 3.2.4 for $\overline{M} = \mathbb{D}^n$, the closed n -disk.

3.2. Factorization homology with formal moduli coefficients. We have a notion of factorization homology with coefficients in a formal n -disk moduli functor.

Definition 3.2.1 (Factorization homology with formal moduli). We extend factorization homology to formal moduli functors

$$\begin{array}{ccc} \mathrm{Artin}_n & \xrightarrow{f} & \mathrm{Fun}(\mathcal{Z}\mathrm{Mfld}_n^{\mathrm{fin}}, \mathrm{Ch}) \\ \mathrm{Spf} \downarrow & \dashrightarrow f & \\ \mathrm{Moduli}_n^{\mathrm{op}} & & \end{array}$$

as the right Kan extension of \int along Spf , denoted with the same symbol. Explicitly, the factorization homology of a zero-pointed n -manifold M_* with coefficients in a formal n -disk moduli functor X is

$$\int_{M_*} X := \lim_{(\mathrm{Spf}(R) \rightarrow X) \in ((\mathrm{Artin}_n^{\mathrm{op}})_{/X})^{\mathrm{op}}} \int_{M_*} R = \lim \left(((\mathrm{Artin}_n^{\mathrm{op}})_{/X})^{\mathrm{op}} \rightarrow \mathrm{Artin}_n \xrightarrow{f_{M_*}} \mathrm{Ch}_{\mathbb{k}} \right).$$

Remark 3.2.2. One can think of $\int_{M_*} X$ as $\Gamma(X, \int_{M_*} \mathcal{O})$, the global sections of the sheaf on X given by calculating the factorization homology of M_* with coefficients in the structure sheaf of X . Importantly, the canonical arrow $\int_{M_*} X \rightarrow \int_{M_*} \mathcal{O}(X)$ is typically not an equivalence unless X is affine; the arrow can be regarded as a type of completion.

Remark 3.2.3. Unless a formal moduli functor X is affine, factorization homology $\int_{M_*} X$ will typically fail to satisfy \otimes -excision and the Goodwillie tower for factorization homology will typically fail to converge.

We now state our main theorem.

Theorem 3.2.4 (Poincaré/Koszul duality for formal moduli). *For \mathbb{k} a field, there is a canonical equivalence of functors $\mathrm{Alg}_n^{\mathrm{aug}} \rightarrow \mathrm{Fun}(\mathcal{Z}\mathrm{Mfld}_n^{\mathrm{fin}}, \mathrm{Ch}_{\mathbb{k}})$,*

$$\left(\int_{(-)} \right)^{\vee} \simeq \int_{(-)^{\vee}} \mathrm{MC}.$$

In particular, for each augmented n -disk algebra A in chain complexes over \mathbb{k} , and each n -dimensional cobordism \overline{M} with boundary $\partial \overline{M} = \partial_{\mathrm{L}} \amalg \partial_{\mathrm{R}}$, there is a canonical equivalence of chain complexes over \mathbb{k}

$$\left(\int_{\overline{M} \setminus \partial_{\mathrm{L}}} A \right)^{\vee} \simeq \int_{\overline{M} \setminus \partial_{\mathrm{R}}} \mathrm{MC}_A$$

between the linear dual of the factorization homology with coefficients in A , and the factorization homology with coefficients in the Maurer–Cartan moduli functor of A .

Remark 3.2.5. Let A be an augmented n -disk algebra in chain complexes over \mathbb{k} , and let M be a closed $(n - d)$ -manifold. Taking products with d -dimensional Euclidean spaces defines augmented d -disk algebras $\int_{M_+ \wedge -} A: \mathrm{Disk}_{d,+} \rightarrow \mathrm{Ch}_{\mathbb{k}}^{\otimes}$ and $\int_{M_+ \wedge -} \mathrm{MC}_A: \mathrm{Disk}_{d,+} \rightarrow \mathrm{Ch}_{\mathbb{k}}^{\otimes}$. By our factorization homology construction of our Koszul dual functor \mathbb{D}^d on augmented d -algebras in chain complexes over \mathbb{k} , there is an equivalence

$$\mathbb{D}^d \left(\int_{M_+ \wedge \mathbb{R}_+^d} A \right) \simeq \left(\int_{(\mathbb{R}^d)_+} \int_{M_+ \wedge \mathbb{R}_+^d} A \right)^{\vee}.$$

Our result then specializes to a canonical equivalence of augmented d -algebras

$$\mathbb{D}^d \left(\int_{M_+ \wedge \mathbb{R}_+^d} A \right) \simeq \int_{M_+ \wedge \mathbb{R}_+^d} \mathrm{MC}_A .$$

We now prove our main theorem, making use of the three results which will be developed in the coming subsections.

Proof of Theorem 3.2.4. Let A be an augmented n -disk algebra in chain complexes over \mathbb{k} , and let M_* be a zero-pointed n -manifold. We explain the diagram of canonical equivalences in $\mathrm{Ch}_{\mathbb{k}}$, each natural in all of their arguments:

$$\begin{array}{ccc} \left(\int_{M_*} A \right)^\vee & \xleftarrow{\quad \simeq \quad} & \int_{M_*^\neg} \mathrm{MC}_A \\ \downarrow \text{Prop 3.5.5} \simeq & & \parallel \text{Def 3.2.1} \\ \left(\operatorname{colim}_{F \in (\mathrm{FPres}_n^{\leq -n})/A} \int_{M_*} F \right)^\vee & & \lim_{R \in ((\mathrm{Artin}_n^{\mathrm{op}})_{/\mathrm{MC}_A})^{\mathrm{op}}} \int_{M_*^\neg} R \\ \downarrow \simeq & & \downarrow \simeq \text{Thm 3.4.3} \\ \lim_{F \in ((\mathrm{FPres}_n^{\leq -n})/A)^{\mathrm{op}}} \left(\int_{M_*} F \right)^\vee & \xleftarrow[\text{Thm 3.3.5}]{\simeq} & \lim_{F \in ((\mathrm{FPres}_n^{\leq -n})/A)^{\mathrm{op}}} \int_{M_*^\neg} \mathbb{D}^n F . \end{array}$$

By Proposition 3.5.5, we can calculate factorization homology with coefficients in A as a colimit over finitely presented $(-n)$ -coconnective n -disk algebras: $\operatorname{colim}_{F \in (\mathrm{FPres}_n^{\leq -n})/A} \int_{M_*} F \xrightarrow{\simeq} \int_{M_*} A$. By Theorem 3.3.5, for a finitely presented $(-n)$ -coconnective n -disk algebra F , there is natural equivalence $\left(\int_{M_*} F \right)^\vee \xleftarrow{\simeq} \int_{M_*^\neg} \mathbb{D}^n F$. By Theorem 3.4.3, Koszul duality restricts to an equivalence $\mathbb{D}^n : \mathrm{FPres}_n^{\leq -n} \simeq (\mathrm{Artin}_n)^{\mathrm{op}} : \mathbb{D}^n$ between finitely presented and Artin n -disk algebras. By definition $\mathrm{Map}(\mathrm{Spf}(R), \mathrm{MC}_A) \simeq \mathrm{Map}(\mathbb{D}^n R, A)$ for R Artin, and so this last equivalence gives an equivalence of slice categories $(\mathrm{FPres}_n^{\leq -n})/A \simeq (\mathrm{Artin}_n^{\mathrm{op}})_{/\mathrm{MC}_A}$. \square

3.3. Finitely presented coconnective algebras. We first prove the main result for the special case in which A is a finitely presented $(-n)$ -coconnective augmented n -disk algebra. The definition of finite presentation is exactly dual to our definition of Artinian.

Recall from Definition 1.4.3 the free algebra functor $\mathbb{F} : \mathrm{Mod}_{\mathrm{O}(n)}(\mathrm{Ch}_{\mathbb{k}}) \rightarrow \mathrm{Alg}_n^{\mathrm{aug}}$.

Notation 3.3.1. For $i > 0$, we will ongoingly make use of the following full ∞ -subcategories

$$\mathrm{Alg}_n^{\mathrm{aug}} \supset \mathrm{Alg}_n^{\leq -i} \supset \mathrm{Free}_n^{\mathrm{all}, \leq -i} \supset \mathrm{Free}_n^{\leq -i} \supset \mathrm{Free}_n^{\mathrm{perf}, \leq -i}$$

respectively consisting of: augmented n -disk algebras whose augmentation ideal is $(-i)$ -coconnective as a \mathbb{k} -module; free augmented n -disk algebras on $\mathrm{O}(n)$ -modules whose underlying \mathbb{k} -module is $(-i)$ -coconnective; free augmented n -disk algebras on $\mathrm{O}(n)$ -modules whose underlying \mathbb{k} -module is $(-i)$ -coconnective and finite; and free augmented n -disk algebras on $\mathrm{O}(n)$ -modules which are $(-i)$ -coconnective truncations of perfect $\mathrm{O}(n)$ -modules.

While the outer inclusions in Notation 3.3.1 are manifest, the inner inclusion is a particular feature of n -disk algebras, which we shall see in the proof of Theorem 3.3.5. We make use of another class of augmented n -disk algebras which is dual to Artin algebras.

Definition 3.3.2 ($\mathrm{FPres}_n^{\leq -n}$). We denote the intermediate full ∞ -subcategory

$$\mathrm{Free}_n^{\mathrm{perf}, \leq -n} \subset \mathrm{FPres}_n^{\leq -n} \subset \mathrm{Alg}_n^{\mathrm{aug}, \leq -n} ,$$

which is the smallest among all such that satisfy the following property:

Let A and B be $(-n)$ -coconnective n -disk algebras with

$$\begin{array}{ccc} \mathbb{F}V & \longrightarrow & \mathbb{k} \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

a pushout square in $\mathbf{Alg}_n^{\text{aug}}$ in which V is a $O(n)$ -module in $\text{Perf}_{\mathbb{k}}^{\leq -n}$; then if A is in $\mathbf{FPres}_n^{\leq -n}$, then B is in $\mathbf{FPres}_n^{\leq -n}$.

This next result verifies that cofibers of augmented algebras can inherit coconnectivity, which assures the existence of many $(-N)$ -coconnective algebras that are not free.

Lemma 3.3.3. *Let $N \geq n$. Consider a pushout square*

$$\begin{array}{ccc} F & \longrightarrow & \mathbb{k} \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

among augmented n -disk algebras in which the augmentation ideals of F and of A are $(-N)$ -coconnective. Should the morphism $F \rightarrow A$ induce an injection $H_{-N}(F; \mathbb{k}) \hookrightarrow H_{-N}(A; \mathbb{k})$ on degree- $(-N)$ homology, then the augmentation ideal of B is $(-N)$ -coconnective.

Proof. This is a spectral sequence argument for the skeletal filtration of functorial free resolutions.

For $q \geq 0$, we use the simplified notation $\overline{\mathbb{F}}^q$ for the q -fold composition of the endofunctor on $\mathbf{Mod}_{O(n)}$ which is the composition of mutual adjoints

$$\overline{\mathbb{F}}: \mathbf{Alg}_n^{\text{aug}} \xrightarrow{\text{Ker}(- \rightarrow \mathbb{k})} \mathbf{Mod}_{O(n)} \xrightarrow{\mathbb{F}} \mathbf{Alg}_n^{\text{aug}} .$$

With this notation, recognize the given morphism $F \rightarrow A$ as the geometric realization of the morphism $\overline{\mathbb{F}}^\bullet(\mathbb{F}(F)) \rightarrow \overline{\mathbb{F}}^\bullet(\mathbb{F}(A))$ between their functorial free resolutions. Denote the cokernel $A/F := \text{cKer}(F \rightarrow A)$ between underlying \mathbb{k} -modules. Because the functor $\overline{\mathbb{F}}$ is a left adjoint, the diagram

$$\begin{array}{ccc} \overline{\mathbb{F}}^\bullet(\mathbb{F}(F)) & \longrightarrow & \mathbb{k} \\ \downarrow & & \downarrow \\ \overline{\mathbb{F}}^\bullet(\mathbb{F}(A)) & \longrightarrow & \overline{\mathbb{F}}^\bullet(\mathbb{F}(A/F)) \end{array}$$

is a pushout among simplicial objects in augmented n -disk algebras. Because geometric realizations commute with colimits, there is a canonical identification

$$|\overline{\mathbb{F}}^\bullet(\mathbb{F}(A/F))| \simeq B$$

between augmented n -disk algebras. Because each of the forgetful functors $\mathbf{Alg}_n^{\text{aug}} \xrightarrow{\text{Ker}(- \rightarrow \mathbb{k})} \mathbf{Mod}_{O(n)} \rightarrow \mathbf{Mod}_{\mathbb{k}}$ preserves sifted colimits, we obtain an identification of the underlying \mathbb{k} -modules

$$(15) \quad |\text{Ker}(\overline{\mathbb{F}}^\bullet(\mathbb{F}(A/F)) \rightarrow \mathbb{k})| \simeq \text{Ker}(B \rightarrow \mathbb{k}) .$$

As this proof proceeds, we use the condensed notation for this simplicial \mathbb{k} -module

$$\mathbb{I}^\bullet(\mathbb{F}(A/F)) := \text{Ker}(\overline{\mathbb{F}}^\bullet(\mathbb{F}(A/F)) \rightarrow \mathbb{k}) .$$

Now, the standard Reedy structure on $\mathbf{\Delta}^{\text{op}}$ determines, for each simplicial \mathbb{k} -module X_\bullet , the sequential diagram of \mathbb{k} -modules

$$\text{Sk}_0(X_\bullet) \longrightarrow \cdots \longrightarrow \text{Sk}_q(X_\bullet) \longrightarrow \cdots$$

whose colimit is identified as the geometric realization $|X_\bullet|$, as well as, for each $q \geq 0$, the latching object

$$L_q(X_\bullet) := \operatorname{colim}_{[q] \xrightarrow[\neq]{\text{surj}} [p]} X_p .$$

Inspecting the category indexing this colimit, we recognize this q th latching object as a pushout among finite coproducts

$$(16) \quad L_{q-1}(X_\bullet) \coprod_{\coprod_{q-1} L_{q-1}(X_\bullet)} \left(\coprod_{q-1} X_{q-1} \right) \simeq L_q(X_\bullet)$$

involving the $(q-1)$ st latching object. For each $q \geq 0$, these objects fit into a pushout diagram among \mathbb{k} -modules:

$$(17) \quad \begin{array}{ccc} L_q(X_\bullet) \oplus_{\partial\Delta^q \otimes L_q(X_\bullet)} \partial\Delta^q \otimes X_q & \longrightarrow & \operatorname{Sk}_{q-1}(X_\bullet) \\ \downarrow & & \downarrow \\ X_q & \longrightarrow & \operatorname{Sk}_q(X_\bullet) . \end{array}$$

In particular, through the identification (15) we recognize the augmentation ideal of B as a sequential colimit

$$\operatorname{colim} \left(\operatorname{Sk}_0(\mathbb{I}^\bullet(\mathbb{F}(A/F))) \rightarrow \cdots \rightarrow \operatorname{Sk}_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) \rightarrow \cdots \right) \simeq \operatorname{Ker}(B \rightarrow \mathbb{k})$$

of skeleta. Because, for each integer k , the \mathbb{k} -module $\mathbb{k}[k]$ is compact as a \mathbb{k} -module, there is an identification of degree- k homology groups

$$(18) \quad \operatorname{colim}_{q \geq 0} \operatorname{H}_k(\operatorname{Sk}_q(\mathbb{I}^\bullet(\mathbb{F}(A/F)))) \simeq \operatorname{H}_k(\operatorname{Ker}(B \rightarrow \mathbb{k})) .$$

Let $\mathbb{k}[k] \xrightarrow{\sigma} \operatorname{Ker}(B \rightarrow \mathbb{k})$ represent a non-zero homology class. The isomorphism (18) grants the existence of a minimal q for which σ factors as

$$\sigma : \mathbb{k}[k] \xrightarrow{\tilde{\sigma}} \operatorname{Sk}_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) \longrightarrow \operatorname{Ker}(B \rightarrow \mathbb{k}) .$$

The minimality of q , together with the non-triviality of σ , ensures that the composite morphism between \mathbb{k} -modules to the cokernel,

$$(19) \quad \mathbb{k}[k] \xrightarrow{\tilde{\sigma}} \operatorname{Sk}_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) \longrightarrow \operatorname{Sk}_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) / \operatorname{Sk}_{q-1}(\mathbb{I}^\bullet(\mathbb{F}(A/F))) ,$$

does not factor through 0. In this way, we see that the augmentation ideal of B is r -coconnective provided this cokernel is r -coconnective for each $q \geq 0$. We are thus reduced to establishing that each such cokernel is $(-N)$ -coconnective.

Through the pushout square (17) applied to this simplicial \mathbb{k} -module $\mathbb{I}^\bullet(\mathbb{F}(A/F))$, we identify the cokernel in (19) as the pushout of cokernels:

$$(20) \quad \begin{array}{ccc} \mathbb{I}^q(\mathbb{F}(A/F)) / \partial\Delta^q \otimes L_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) & \longrightarrow & \mathbb{I}^q(\mathbb{F}(A/F)) / \partial\Delta^q \otimes \mathbb{I}^q(\mathbb{F}(A/F)) \\ \downarrow & & \downarrow \\ \mathbb{I}^q(\mathbb{F}(A/F)) / L_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) & \longrightarrow & \operatorname{Sk}_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) / \operatorname{Sk}_{q-1}(\mathbb{I}^\bullet(\mathbb{F}(A/F))) . \end{array}$$

In a moment we will prove the following two facts about the canonical morphism

$$(21) \quad L_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) \longrightarrow \mathbb{I}^q(\mathbb{F}(A/F))$$

between \mathbb{k} -modules for each $q \geq 0$.

- (1) For each $q \geq 0$, the morphism (21) is a section of a retraction.
- (2) For each $q \geq 0$, the cokernel of the morphism (21) is $(-N - q)$ -coconnective.

We finish the proof of this result premised on these two facts. Using the fact (1), the pushout square (20) simplifies to the pushout square

$$(22) \quad \begin{array}{ccc} \Sigma^q(L_q(\mathbb{I}^\bullet(\mathbb{F}(A/F)))) \oplus (\mathbb{I}^q(\mathbb{F}(A/F))/L_q(\mathbb{I}^\bullet(\mathbb{F}(A/F)))) & \longrightarrow & \Sigma^q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) \\ \downarrow & & \downarrow \\ \mathbb{I}^q(\mathbb{F}(A/F))/L_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) & \longrightarrow & \mathrm{Sk}_q(\mathbb{I}^\bullet(\mathbb{F}(A/F)))/\mathrm{Sk}_{q-1}(\mathbb{I}^\bullet(\mathbb{F}(A/F))). \end{array}$$

Again using the fact (1), this pushout square determines an identification between \mathbb{k} -modules

$$\Sigma^q\left(\mathbb{I}^q(\mathbb{F}(A/F))/L_q(\mathbb{I}^\bullet(\mathbb{F}(A/F)))\right) \simeq \mathrm{Sk}_q(\mathbb{I}^\bullet(\mathbb{F}(A/F)))/\mathrm{Sk}_{q-1}(\mathbb{I}^\bullet(\mathbb{F}(A/F))).$$

Using fact (2), we conclude that the cokernel in (19) is $(-N)$ -coconnective, thereby completing the argument provided facts (1) and (2).

We now prove the fact (1). Because the ∞ -category $\mathbf{Mod}_{\mathbb{k}}$ is stable, each morphism between \mathbb{k} -modules $X \rightarrow Y$ admits a unique retraction $Y \rightarrow X$ if any at all. As so, there is an ∞ -subcategory of $\mathbf{Mod}_{\mathbb{k}}$ consisting of sections of retractions, and it is equivalent to the opposite of the ∞ -subcategory of $\mathbf{Mod}_{\mathbb{k}}$ consisting of retractions. Now, there is the standard expression for the augmentation ideal of the free augmented n -disk algebra on a \mathbb{k} -module V :

$$(23) \quad \mathrm{Ker}(\mathbb{F}(V) \rightarrow \mathbb{k}) \simeq \bigoplus_{i>0} \mathrm{Conf}_i(\mathbb{R}^n) \otimes_{\Sigma_i} V^{\otimes i}.$$

With respect to this expression, the canonical morphism $V \rightarrow \mathrm{Ker}(\mathbb{F}(V) \rightarrow \mathbb{k})$ is the inclusion of the 1st cofactor; in particular, it is a section of a retraction. It follows that, for $[q] \xrightarrow{\sigma} [p]$ a surjective morphism in $\mathbf{\Delta}$, the induced morphism $\sigma^*: \mathbb{I}^p(\mathbb{F}(A/F)) \rightarrow \mathbb{I}^q(\mathbb{F}(A/F))$ between \mathbb{k} -modules is a section of a retraction that we denote as $\sigma_*: \mathbb{I}^q(\mathbb{F}(A/F)) \rightarrow \mathbb{I}^p(\mathbb{F}(A/F))$. In other words, the restricted functor $\mathbb{I}^\bullet(\mathbb{F}(A/F)): (\mathbf{\Delta}^{\mathrm{surj}})^{\mathrm{op}} \rightarrow \mathbf{Mod}_{\mathbb{k}}$ factors through this ∞ -subcategory of sections, thereby determining a functor $\mathbb{I}^\bullet(\mathbb{F}(A/F)): \mathbf{\Delta}^{\mathrm{surj}} \rightarrow \mathbf{Mod}_{\mathbb{k}}$ that factors through the ∞ -subcategory of retractions. In particular, for each $q \geq 0$, we obtain a morphism between \mathbb{k} -modules

$$\mathbb{I}^q(\mathbb{F}(A/F)) \longrightarrow \lim_{[q] \xrightarrow{\sigma} [p]} \mathbb{I}^p(\mathbb{F}(A/F)) =: L^q(\mathbb{I}^\bullet(\mathbb{F}(A/F)))$$

whose projection to the $([q] \xrightarrow{\sigma} [p])$ -factor is the retraction $\sigma_*: \mathbb{I}^q(\mathbb{F}(A/F)) \rightarrow \mathbb{I}^p(\mathbb{F}(A/F))$. Composing with the morphism (21) gives a morphism

$$L_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) \longrightarrow L^q(\mathbb{I}^\bullet(\mathbb{F}(A/F))).$$

As with the expression (16), we recognize this morphism as the canonical one:

$$L_{q-1}(\mathbb{I}^\bullet(\mathbb{F}(A/F))) \amalg_{\amalg_{L_{q-1}}(\mathbb{I}^\bullet(\mathbb{F}(A/F)))} \amalg_{\amalg_{\mathbb{I}^{q-1}}(\mathbb{F}(A/F))} \longrightarrow L^{q-1}(\mathbb{I}^\bullet(\mathbb{F}(A/F))) \amalg_{\amalg_{L^{q-1}}(\mathbb{I}^\bullet(\mathbb{F}(A/F)))} \amalg_{\amalg_{\mathbb{I}^{q-1}}(\mathbb{F}(A/F))}.$$

By induction on q , this morphism is an equivalence because the ∞ -category $\mathbf{Mod}_{\mathbb{k}}$ is stable. This concludes the proof of fact (1).

We now prove fact (2). Let $q \geq 0$. Fact (1) gives that the morphism (21) is injective on homology groups. Consequently, it is sufficient to prove that (21) has the following two properties.

- (a) The codomain of (21) is $(-N)$ -coconnective
- (b) The morphism (21) is surjective on degree- k homology groups for each $-N - q < k \leq -N$.

We prove (a) after some preliminary observations about free algebras. Let V be an $\mathcal{O}(n)$ -module whose underlying \mathbb{k} -module is $(-N)$ -coconnective. Inspecting the expression (23) for the augmentation ideal of a free augmented n -disk algebra on V reveals that the i th cofactor in the expression is $(-Ni + (n-1)(i-1))$ -coconnective as a \mathbb{k} -module. Using that $N \geq n$, this has some immediate implications.

- The augmentation ideal of $\mathbb{F}(V)$ is $(-N)$ -coconnective as a \mathbb{k} -module.
- The cokernel of the morphism $V \rightarrow \text{Ker}(\mathbb{F}(V) \rightarrow \mathbb{k})$ between underlying \mathbb{k} -modules is $(-N-1)$ -coconnective.

Let $V \rightarrow W$ a morphism between $\text{O}(n)$ -modules that is $(-N)$ -coconnective on underlying \mathbb{k} -modules. Suppose this morphism is injective on degree- $(-N)$ homology. Then the morphism between augmentation ideals $\text{Ker}(\mathbb{F}(V) \rightarrow \mathbb{k}) \rightarrow \text{Ker}(\mathbb{F}(W) \rightarrow \mathbb{k})$ is also injective on $(-N)$ -degree homology. Furthermore, the underlying \mathbb{k} -module W/V of the cokernel is in fact $(-N)$ -coconnective, from which we conclude that the augmentation ideal of $\mathbb{F}(W/V)$ is $(-N)$ -coconnective as a \mathbb{k} -module. Using the assumptions on the given morphism $F \rightarrow A$, we conclude that the \mathbb{k} -module $\mathbb{I}^q(\mathbb{F}(A/F))$ is $(-N)$ -coconnective for each $q \geq 0$. This proves (a).

We now prove (b). Using the definition of the q th latching object as a colimit, it is enough to show that the composite morphism between \mathbb{k} -modules,

$$(24) \quad \bigoplus_{[q] \xrightarrow{\text{surj}} [q-1]} \mathbb{I}^{q-1}(\mathbb{F}(A/F)) \longrightarrow L_q(\mathbb{I}^\bullet(\mathbb{F}(A/F))) \longrightarrow \mathbb{I}^q(\mathbb{F}(A/F)),$$

is surjective on degree- k homology groups for $-N-q \leq k \leq -N$. We prove this by induction on $q \geq 0$. The case $q = 0$ is the assertion that the cokernel of the unit morphism $\mathbb{k} \rightarrow \mathbb{F}(A/F)$ is $(-N)$ -coconnective. This is a restatement of the first bullet-point above. The case $q = 1$ is the assertion that the cokernel of the unit morphism $\mathbb{F}(A/F) \rightarrow \overline{\mathbb{F}}(\mathbb{F}(A/F))$ is $(-N-1)$ -coconnective. This is an application of the second bullet-point above. We now establish the inductive step. Using fact (1), we rewrite the composite morphism (24) as a coproduct preserving morphism

$$\overline{\mathbb{F}}^{q-1}(\mathbb{F}(A/F)) \oplus \bigoplus_{\bigoplus}^{q-1} \overline{\mathbb{F}}^{q-1}(\mathbb{F}(A/F)) \longrightarrow \overline{\mathbb{F}}^{q-1}(\overline{\mathbb{F}}(\mathbb{F}(A/F))) \simeq \overline{\mathbb{F}}^{q-1}(\mathbb{F}(A/F)) \oplus \overline{\mathbb{F}}^{q-1}\left(\frac{\overline{\mathbb{F}}(\mathbb{F}(A/F))}{\mathbb{F}(A/V)}\right)$$

that is the identity on the first cofactor. Using the above bullet-points, the second cofactor in the domain of this morphism is $(-N-1)$ -coconnective. This reduces the problem to showing the morphism

$$\bigoplus_{\bigoplus}^{q-1} \overline{\mathbb{F}}^{q-1}(\mathbb{F}(A/F)) \longrightarrow \overline{\mathbb{F}}^{q-1}\left(\frac{\overline{\mathbb{F}}(\mathbb{F}(A/F))}{\mathbb{F}(A/V)}\right)$$

is surjective on degree- k homology groups for $-N-q \leq k$. This morphism factors as

$$\bigoplus_{\bigoplus}^{q-1} \overline{\mathbb{F}}^{q-1}(\mathbb{F}(A/F)) \longrightarrow \bigoplus_{\bigoplus}^{q-1} \overline{\mathbb{F}}^{q-2}\left(\frac{\overline{\mathbb{F}}(\mathbb{F}(A/F))}{\mathbb{F}(A/V)}\right) \longrightarrow \overline{\mathbb{F}}^{q-1}\left(\frac{\overline{\mathbb{F}}(\mathbb{F}(A/F))}{\mathbb{F}(A/V)}\right).$$

As so, it is sufficient to argue that this latter morphism has the desired surjectivity. Because $\frac{\overline{\mathbb{F}}(\mathbb{F}(A/F))}{\mathbb{F}(A/V)}$ is $(-N-1)$ -coconnective, this is the case by induction applied to $q-1$. This completes this proof. \square

Before addressing this affine case, we require the following easy, but important, comparison of factorization homology and factorization cohomology in the stable setting.

Proposition 3.3.4. *Let \mathcal{V} be a \otimes -stable-presentable symmetric monoidal ∞ -category. Let A be an augmented n -disk algebra in \mathcal{V} for which the natural map*

$$(A^\vee)^{\otimes i} \xrightarrow{\simeq} (A^{\otimes i})^\vee$$

is an equivalence for all $i \geq 0$. The composition $\text{Disk}_n^+ \cong \text{Disk}_{n,+}^{\text{op}} \xrightarrow{A^\vee} \mathcal{V}^{\text{op}} \xrightarrow{(-)^\vee} \mathcal{V}$ canonically extends to an augmented n -disk coalgebra $A^\vee: \text{Disk}_n^+ \rightarrow \mathcal{V}$. Furthermore, for any conically finite zero-pointed n -manifold M_ the canonical arrow*

$$\int^{M_*} A^\vee \xrightarrow{\simeq} \left(\int_{M_*} A \right)^\vee$$

is an equivalence.

Proof. The first statement is clear, by hypothesis. Furthermore, the hypothesis on A supports the leftward arrow in the string of canonical equivalences in \mathcal{V} :

$$\left(\int_{M_*} A\right)^\vee = \left(\underset{U_+ \in \mathcal{D}\text{isk}_+(M_*)}{\text{colim}} A(U_+)\right)^\vee \xrightarrow{\simeq} \underset{U_+ \in (\mathcal{D}\text{isk}_+(M_*)^{\text{op}})}{\text{lim}} A(U_+)^\vee \xleftarrow{\simeq} \underset{U^+ \in ((\mathcal{D}\text{isk}_+^+)_{M_*})}{\text{lim}} A^\vee(U^+) = f^{M_*} A^\vee.$$

□

We now turn to the affine case in the proof of our main theorem.

Theorem 3.3.5. *Let \mathbb{k} be a field. There is an equivalence between functors*

$$\left(\int_{(-)}\right)^\vee \simeq \int_{(-)^\neg} \mathbb{D}^n : \text{FPres}_n^{\leq -n} \rightarrow \text{Fun}(\mathcal{Z}\text{Mfld}_n^{\text{fin}}, \text{Ch}_\mathbb{k}).$$

Specifically, for each finitely presented $(-n)$ -coconnective augmented n -disk algebra A in chain complexes over \mathbb{k} , and each n -dimensional cobordism \bar{M} with boundary $\partial\bar{M} = \partial_L \amalg \partial_R$, there is a canonical equivalence of chain complexes over \mathbb{k}

$$\left(\int_{\bar{M} \setminus \partial_L} A\right)^\vee \simeq \int_{\bar{M} \setminus \partial_R} \mathbb{D}^n A.$$

Proof. Fix a conically finite zero-pointed n -manifold M_* , and a finitely presented $(-n)$ -coconnective augmented n -disk algebra A . We will explain the diagram in $\text{Ch}_\mathbb{k}$

$$\left(\int_{M_*} A\right)^\vee \xleftarrow{(1)} \left(P_\infty \int_{M_*} A\right)^\vee \xleftarrow{(2)} \left(\int_{M_*^\neg} \text{Bar}(A)\right)^\vee \xrightarrow{(3)} \left(\int_{M_*^\neg} \mathbb{D}^n A\right)^{\vee\vee} \xleftarrow{(4)} \int_{M_*^\neg} \mathbb{D}^n A$$

and that each arrow in it is an equivalence. The arrow (1) is the linear dual of the canonical map to the limit of the Goodwillie cofiltration. The arrow (2) is the linear dual of that from Theorem 2.1.9, which factors the Poincaré/Koszul duality morphism and is an equivalence by that result. The arrow (3) is the linear dual of that of Proposition 3.3.4. The arrow (4) is the standard one. All of these maps are natural in M_*

We verify that (3) is an equivalence. Recall that since A is finitely presented, therefore LA is perfect. The equivalence of chain complexes $\mathbb{D}^n A \simeq \mathbb{k} \oplus ((\mathbb{R}^n)^+ \otimes LA)^\vee$ given by Theorem 2.7.1 implies $\mathbb{D}^n A$ too is perfect. We can thus apply Proposition 3.3.4 to conclude that (3) is an equivalence.

We next show (1) is an equivalence. For this we will show that the successive layers in the Goodwillie cofiltration become contractible through a range. Specifically, through Theorem 2.1.9, we will show

$$\text{Conf}_i^{\text{fr}}(M_*) \otimes_{\Sigma_i \wr \mathcal{O}(n)} (LA)^{\otimes i}$$

is $(-i + \ell)$ -coconnective, where ℓ is the number of components of M_* and i is sufficiently large.

First, we show that LA is a $(-n)$ -coconnective $\mathcal{O}(n)$ -module given that A is in $\text{FPres}_n^{\leq -n}$. We prove this by induction on a resolution of A . The base case is when A is a free augmented n -disk algebra: for $V \in \text{Mod}_{\mathcal{O}(n)}(\text{Perf}_\mathbb{k}^{\leq -n})$ then $L\mathbb{F}V \simeq V$ is $(-n)$ -coconnective. To prove the inductive step, consider a pushout of $(-n)$ -coconnective algebras, $A \cong \mathbb{k} \amalg_{\mathbb{F}V} B$, where B is finitely presented $(-n)$ -coconnective and subject to the assumption that LB is $(-n)$ -coconnective. If the map $\mathbb{F}V \rightarrow B$ is not injective on homology in degree $-n$, then the pushout would have homology in degree $1 - n$. By assumption A is $(-n)$ -coconnective; consequently, the map $\mathbb{F}V \rightarrow B$ must be injective on homology in degree $-n$. Being a left adjoint, the cotangent space functor satisfies $L(\mathbb{k} \amalg_{\mathbb{F}V} B) \simeq L\mathbb{k} \amalg_{L\mathbb{F}V} LB \simeq \text{cKer}(V \rightarrow LB)$, and the cokernel is again $(-n)$ -coconnective. Consequently, LA is $(-n)$ -coconnective and, further $(LA)^{\otimes i}$ is a $(-ni)$ -coconnective $\Sigma_i \wr \mathcal{O}(n)$ -module.

Via factorization homology, the $\Sigma_i \wr \mathcal{O}(n)$ -module $(LA)^{\otimes i}$ determines a \oplus -excisive symmetric monoidal functor $\mathcal{Z}\text{Mfld}_{\Sigma_i \wr \mathcal{O}(n)}^{\text{fin}} \rightarrow \text{Ch}_\mathbb{k}^\oplus$ from conically finite zero-pointed (ni) -manifolds equipped with a $\mathcal{B}(\Sigma_i \wr \mathcal{O}(n))$ -structure on their tangent bundle. Explicitly, this functor is simply the associated

bundle construction: $W_* \mapsto \mathrm{Fr}_{W_*} \bigotimes_{\Sigma_i \wr \mathcal{O}(n)} (LA)^{\otimes i}$. Through \oplus -excision, the value of this functor on W_* is $(s - ni)$ -coconnective whenever W_* is s -coconnective. The zero-pointed (ni) -manifold $\mathrm{Conf}_i(M_*)$ is equipped with a $\mathbf{B}(\Sigma_i \wr \mathcal{O}(n))$ -structure on its tangent bundle; and Proposition 1.1.9 explains that it is conically finite and coconnectivity equal to $nl + (n - 1)(i - \ell)$, where ℓ is the number of components of M_* and we assume $i > \ell$. The coconnectivity range implying equivalence (1) now follows by addition.

Lastly, we show (4) is an equivalence. It suffices to show that $\int_{M_*^-} \mathbb{D}^n A$ is connective and has finite rank homology groups over \mathbb{k} in all dimensions. To show this, we apply the cardinality filtration of factorization homology: since $\int_{M_*^-} \mathbb{D}^n A$ is a sequential colimit of the filtration $\tau^{\leq i} \int_{M_*^-} \mathbb{D}^n A$, we further reduce to showing that the layers of the filtration are connective and grow in connectivity with i . By Theorem 2.1.9, these cardinality layers are shifts of the duals of the Goodwillie layers of $\int_{M_*} A$. Consequently, their connectivities follows from the coconnectivities of the Goodwillie layers, which were computed in the proof of equivalence (1). \square

3.4. Comparing Artin_n and FPres_n . We show that Koszul duality implements an equivalence between finitely presented $(-n)$ -coconnective algebras and connective Artin algebras. These connectivity conditions generalize classical considerations in differential graded algebra, evident in the connectivity conditions in works such as [Mo], [MM], and [Quil]. It is well known that, in general, a free coalgebra is more complicated than a free algebra due to the failure of tensor products to commute with infinite products. This failure means that the infinite product $\prod_{i \geq 0} V^{\otimes i}$ lacks a coalgebra structure in general. However, the infinite direct sum $\bigoplus_{i \geq 0} V^{\otimes i}$ does have a natural coalgebra structure: it is the free ind-nilpotent coalgebra generated by V . Consequently, whenever the natural completion map

$$\bigoplus_{i \geq 0} V^{\otimes i} \longrightarrow \prod_{i \geq 0} V^{\otimes i}$$

is an equivalence, then the infinite product does obtain a natural coalgebra structure; this is then easily seen to be free. This completion map is an equivalence under (co)connectivity hypotheses. For instance, if V is 1-connective, then $V^{\otimes i}$ is i -connective, from which the equivalence follows. Given the calculation of the bar construction on an augmented trivial associative algebra, $\mathrm{Bar}(\mathfrak{t}V) \simeq \bigoplus_{i \geq 0} (\Sigma V)^{\otimes i}$, one can summarize this discussion for Koszul duality purposes as saying that the bar construction sends augmented trivial algebras to ind-nilpotent free coaugmented coalgebras, and it sends trivial *connective* algebras to free coalgebras (which happen to ind-nilpotent).

The following lemma generalizes the preceding to n -disk algebra.

Lemma 3.4.1. *Let $V \in \mathrm{Mod}_{\mathcal{O}(n)}(\mathrm{Perf}_{\mathbb{k}}^{\geq 0})$ be an $\mathcal{O}(n)$ -module in perfect connective \mathbb{k} -modules, where \mathbb{k} is a field. The Koszul dual of the trivial algebra on V , $\mathbb{D}^n(\mathbb{k} \oplus V) \simeq \mathrm{Bar}(\mathfrak{t}V)^\vee$, is equivalent to the free augmented n -disk algebra on $V^\vee[-n]$. Likewise, $\mathrm{Bar}(\mathfrak{t}V)$ is the free augmented n -disk coalgebra on $(\mathbb{R}^n)^+ \otimes V$.*

Proof. The argument is essentially that for Lemma 2.4.3, but replacing the Goodwillie cofiltration with the cardinality filtration. For convenience, in this proof we will denote $V[n] := (\mathbb{R}^n)^+ \otimes V$ and $V[-n] := \mathrm{Map}((\mathbb{R}^n)^+, V)$. The reader should bear in mind that these notations carry the $\mathcal{O}(n)$ -action.

Consider the cardinality cofiltration for factorization cohomology with coefficients in the trivial coalgebra $\mathfrak{t}V^\vee$. There is an equivalence

$$\int_{(\mathbb{R}^n)^+} \mathbb{F}(V^\vee[-n]) \simeq \mathfrak{t}V^\vee.$$

Consequently, by Theorem 2.1.9, we have for each M_* an equivalence between the Goodwillie cofiltration of factorization homology and the cardinality cofiltration of factorization cohomology

$$P_{\leq \bullet} \int_{M_*^-} \mathbb{F}(V^\vee[-n]) \simeq \tau^{\leq \bullet} \int^{M_*} \mathfrak{t}V^\vee .$$

The 0-connectivity of V implies that $V^\vee[-n]$ is $(-n)$ -coconnective. It follows from this that each fiber of the Goodwillie cofiltration, $\text{Conf}_i^{\text{fr}}(M_*^-) \otimes_{\Sigma_i \mathcal{O}(n)} V^\vee[-n]^{\otimes i}$, is $(1-i)$ -coconnective, since $\text{Conf}_i(M_*^-)$ has nonvanishing homology in degrees at most $n + (n-1)(i-1)$, each of which is finite dimensional. Consequently, the Goodwillie cofiltration converges. Using the comparison of the Goodwillie cofiltration and the cardinality cofiltration (Theorem 2.1.9), since the Goodwillie cofiltration for a free algebra splits (Theorem 2.4.1), we obtain that the cardinality cofiltration for factorization cohomology of a trivial coalgebra also splits. Applying this result in the case of $M_* = (\mathbb{R}^n)^+$ and $M_*^- = \mathbb{R}_+^n$, we obtain the equivalence

$$\mathbb{F}(V^\vee[-n]) \simeq \int^{(\mathbb{R}^n)^+} \mathfrak{t}V^\vee .$$

As established, the finiteness condition on V gives that this \mathbb{k} -module is concentrated in negative homological degrees and is finite in each dimension. Proposition 3.3.4 therefore gives the equivalence

$$\int^{(\mathbb{R}^n)^+} \mathfrak{t}V^\vee \simeq \left(\int_{(\mathbb{R}^n)^+} \mathbb{k} \oplus V \right)^\vee .$$

The righthand side is exactly the definition of $\mathbb{D}^n(\mathbb{k} \oplus V)$, the Koszul dual of the trivial n -disk algebra on V , so the result follows. \square

Remark 3.4.2. For convenience, we have assumed \mathbb{k} is a field, but this was only necessary for the dual $\mathbb{D}^n(\mathbb{k} \oplus V)$ to be a free algebra. The coalgebra $\text{Bar}(\mathfrak{t}V)$ is free provided V and \mathbb{k} are connective as complexes, without requirement that \mathbb{k} is a field.

We now have the following important duality result.

Theorem 3.4.3. *Koszul duality restricts to a contravariant equivalence*

$$\mathbb{D}^n : \text{FPres}_n^{\leq -n} \simeq (\text{Artin}_n)^{\text{op}} : \mathbb{D}^n$$

between finitely presented $(-n)$ -coconnective augmented n -disk algebras and Artin n -disk algebras in chain complexes over a field \mathbb{k} .

Proof. Lemma 2.4.3 shows that \mathbb{D}^n sends free algebras to trivial algebras. Inspecting further, \mathbb{D}^n restricts as a functor $\text{Free}_n^{\text{perf}, \leq -n} \rightarrow \text{Triv}_n^{\geq 0}$ – this uses that the \mathbb{k} -linear dual of a coconnective object is connective, since \mathbb{k} is a field. Because $\mathbb{D}^n = (\text{Bar})^\vee : \text{Alg}_n^{\text{aug}} \rightarrow (\text{Alg}_n^{\text{aug}})^{\text{op}}$ is the composite of left adjoints, it preserves colimits. Inspecting the definitions of $\text{FPres}_n^{\leq -n}$ and Artin_n , we conclude that $\mathbb{D}^n : \text{FPres}_n^{\leq -n} \rightarrow (\text{Alg}_n^{\text{aug}})^{\text{op}}$ canonically factors through $(\text{Artin}_n)^{\text{op}}$:

$$(25) \quad \mathbb{D}^n : \text{FPres}_n^{\leq -n} \longrightarrow (\text{Artin}_n)^{\text{op}} .$$

We will now explain that this restricted functor is fully faithful.

For each pair of objects $A, B \in \text{FPres}_n^{\leq -n}$ we must show that map of spaces

$$(26) \quad \text{Map}_{\text{Alg}_n^{\text{aug}}}(A, B) \xrightarrow{\mathbb{D}^n} \text{Map}_{\text{Alg}_n^{\text{aug}}}(\mathbb{D}^n B, \mathbb{D}^n A)$$

is an equivalence. Again, because \mathbb{D}^n preserves colimits, it is enough to consider the case that $A = \mathbb{F}V$ is free on a $\mathcal{O}(n)$ -module V in $\text{Perf}_{\mathbb{k}}^{\leq -n}$. For this case will explain the canonical factorization

of the map (26) through equivalences:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Alg}_n^{\mathrm{aug}}}(\mathbb{F}V, B) &\underset{(1)}{\simeq} \mathrm{Map}^{\mathrm{O}(n)}(V, B) \\
&\underset{(2)}{\simeq} \mathrm{Map}^{\mathrm{O}(n)}(((\mathbb{R}^n)^+ \otimes B)^\vee, ((\mathbb{R}^n)^+ \otimes V)^\vee) \\
&\underset{(3)}{\simeq} \mathrm{Map}^{\mathrm{O}(n)}\left(\left((\mathbb{R}^n)^+ \otimes \mathrm{Bar}(\mathbb{D}^n B)\right)^\vee, \left((\mathbb{R}^n)^+ \otimes V\right)^\vee\right) \\
&\underset{(4)}{\simeq} \mathrm{Map}^{\mathrm{O}(n)}\left(L\mathbb{D}^n B, \left((\mathbb{R}^n)^+ \otimes V\right)^\vee\right) \\
&\underset{(5)}{\simeq} \mathrm{Map}_{\mathrm{Alg}_n^{\mathrm{aug}}}(\mathbb{D}^n B, \mathfrak{t}\left(\left((\mathbb{R}^n)^+ \otimes V\right)^\vee\right)) \\
&\underset{(6)}{\simeq} \mathrm{Map}_{\mathrm{Alg}_n^{\mathrm{aug}}}(\mathbb{D}^n B, \mathbb{D}^n \mathbb{F}V)
\end{aligned}$$

The equivalence (1) is by the free-forgetful adjunction. The equivalence (2) is an application of the functor $((\mathbb{R}^n)^+ \otimes -)^\vee: \mathrm{Mod}_{\mathrm{O}(n)}(\mathrm{Perf}_{\mathbb{k}}) \rightarrow \mathrm{Mod}_{\mathrm{O}(n)}(\mathrm{Perf}_{\mathbb{k}})$, which is an equivalence, because $\mathrm{Ch}_{\mathbb{k}}$ is stable and linear dual implements an equivalence on finite chain complexes. The equivalence (3) is because of the canonical equivalence $B \simeq \mathbb{D}^n \mathbb{D}^n B$, which is Theorem 3.3.5 applied to the case of $M_* = \mathbb{R}_+^n$ and $M_*^- = (\mathbb{R}^n)^+$. The equivalence (4) is from Theorem 2.7.1. The equivalence (5) is the cotangent space-trivial adjunction. The equivalence (6) is the identification of the dual of a free algebra from Lemma 2.4.3. It is straightforward to check that this composite equivalence agrees with the map (26).

With Lemma 2.4.3, inspection of the definitions of $\mathrm{FPres}_n^{\leq -n}$ and Artin_n gives that this restricted functor (25) is essentially surjective. We conclude that (25) is an equivalence of ∞ -categories. From Lemma 3.4.1, $(\mathbb{D}^n)^{\mathrm{op}}$ restricts as an inverse to \mathbb{D}^n on $\mathrm{Triv}_n^{\geq 0}$. Because the forgetful functor $\mathrm{Alg}_n^{\mathrm{nu}}(\mathrm{Ch}_{\mathbb{k}}) \rightarrow \mathrm{Mod}_{\mathrm{O}(n)}(\mathrm{Ch}_{\mathbb{k}})$ is conservative, it follows that $(\mathbb{D}^n)^{\mathrm{op}}: (\mathrm{Artin}_n)^{\mathrm{op}} \rightarrow \mathrm{FPres}_n^{\leq -n}$ is inverse to \mathbb{D}^n . □

Corollary 3.4.4. *For $A \in \mathrm{FPres}_n^{\leq -n}$, the moduli problem MC_A is affine. Moreover, there is an equivalence*

$$\mathrm{MC}_A \simeq \mathrm{Spf}(\mathbb{D}^n A) .$$

3.5. Resolving by $\mathrm{FPres}_n^{\leq -n}$. The rest of this section is devoted to proving the last required result used in the proof of our main theorem. This is Proposition 3.5.5, which states that factorization homology with general coefficients is given by left Kan extension of factorization homology restricted to finitely presented augmented n -disk algebras whose augmentation ideal is $(-n)$ -coconnective. In order to establish this, we make use of the following notion of one ∞ -category being generated under colimits by another.

Definition 3.5.1 (Strongly generates). A functor $g: \mathcal{B} \rightarrow \mathcal{C}$ between ∞ -categories *strongly generates* if the diagram among ∞ -categories

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{g} & \mathcal{C} \\
g \downarrow & \nearrow \mathrm{id}_{\mathcal{C}} & \\
\mathcal{C} & &
\end{array}$$

exhibits the functor $\mathrm{id}_{\mathcal{C}}$ as a left Kan extension of g along g . In other words, the natural transformation $\mathrm{LKan}_g(g) \xrightarrow{\simeq} \mathrm{id}_{\mathcal{C}}$ between endofunctors on \mathcal{C} is by equivalences.

The following lemma is the main technical result of this section. The completion of its proof occupies §3.5.1 below. Recall the full ∞ -subcategories of $\mathrm{Alg}_n^{\mathrm{aug}}$ from Notation 3.3.1.

Lemma 3.5.2 (Highly coconnected free resolutions). *The inclusion from the ∞ -category of free augmented n -disk algebras on $O(n)$ -modules, whose underlying \mathbb{k} -module is $(-n)$ -coconnective and finite, into augmented n -disk algebras in \mathbb{k} -modules,*

$$\mathrm{Free}_n^{\leq -n} \hookrightarrow \mathrm{Alg}_n^{\mathrm{aug}},$$

strongly generates.

Proof. Because composition of left Kan extensions along composable functors is the left Kan extension along the composite, it suffices to show for that, for any full ∞ -subcategory $\mathcal{F} \subset \mathrm{Free}_n^{\leq -n}$, the inclusion $g_1: \mathcal{F} \hookrightarrow \mathrm{Alg}_n^{\mathrm{aug}}$ strongly generates. We do this for $\mathcal{F} = \mathrm{Free}_n^{\mathrm{perf}, \leq -n}$, the ∞ -category consisting of augmented n -disk algebras that are free on $(-n)$ -coconnective truncations of perfect $O(n)$ -modules. (Note, for example, that the trivial $O(n)$ -module $\mathbb{k}[-n] \simeq \tau^{\leq -n} C_*(O(n), \mathbb{k}[-n])$ is the $(-n)$ -coconnective truncation of a perfect $O(n)$ -module, but it is *not* itself a perfect $O(n)$ -module.)

We consider the sequence of left Kan extensions

$$\begin{array}{ccc} \mathrm{Free}_n^{\mathrm{perf}, \leq -n} & \xrightarrow{g_2 \circ g_1 \circ g_0} & \mathrm{Alg}_n^{\mathrm{aug}} \\ g_0 \downarrow & \nearrow \text{---} & \nearrow \text{---} \\ \mathrm{Free}_n^{\mathrm{all}, \leq -n} & & \\ g_1 \downarrow & \nearrow \text{---} & \nearrow \text{---} \\ \mathrm{Alg}_n^{\leq -n} & & \\ g_2 \downarrow & \nearrow \text{---} & \nearrow \text{---} \\ \mathrm{Alg}_n^{\mathrm{aug}} & & \end{array}$$

where each g_i a fully faithful inclusion with $\mathrm{Alg}_n^{\leq -n}$ consisting of those augmented n -disk algebras whose augmentation ideal is a $(-n)$ -coconnective \mathbb{k} -module. Here $\mathrm{Free}_n^{\mathrm{all}, \leq -n}$ consists of all free augmented n -disk algebras on $(-n)$ -coconnective \mathbb{k} -modules. To prove the canonical natural transformation

$$\mathrm{LKan}_g(g) \longrightarrow \mathrm{id}_{\mathrm{Alg}}$$

is an equivalence it is sufficient to show that each of the canonical natural transformations

$$(27) \quad \mathrm{LKan}_{g_0}(g_2 \circ g_1 \circ g_0) \rightarrow g_2 \circ g_1,$$

$$(28) \quad \mathrm{LKan}_{g_1}(g_2 \circ g_1) \rightarrow g_2,$$

and

$$(29) \quad \mathrm{LKan}_{g_2}(g_2) \rightarrow \mathrm{id}_{\mathrm{Alg}}$$

are equivalences.

We first prove that the arrow (27) is an equivalence, namely that the natural map

$$\mathrm{colim}_{\mathbb{F}V \in \mathrm{Free}_{n/A}^{\mathrm{perf}, \leq -n}} \mathbb{F}V \longrightarrow A$$

is an equivalence for any $A \simeq \mathbb{F}(W_0)$ a free $(-n)$ -coconnective n -disk algebra. Since $\mathrm{Mod}_{O(n)}$ is a compactly generated by the perfect $O(n)$ -modules, we can choose an expression for W_0 as filtered colimit $W_\bullet: \mathcal{J} \rightarrow \mathrm{Perf}_{O(n)}$ of perfect $O(n)$ -modules. Since the truncation functor $\tau^{\leq -n}$ preserves colimits, we obtain an expression

$$W_0 \simeq \tau^{\leq -n} W_0 \simeq \mathrm{colim}_{\mathcal{J}} \tau^{\leq -n} W_\bullet$$

for W_0 as a filtered colimit of $(-n)$ -coconnective truncations of perfect $O(n)$ -modules. Applying the free functor \mathbb{F} , which again preserves filtered colimits, we obtain a filtered diagram $A_\bullet := \mathbb{F}(\tau^{\leq -n} W_\bullet) : \mathcal{J} \rightarrow \mathbf{Free}_n^{\text{perf}, \leq -n}$ together with an identification $\text{colim}(\mathcal{J} \xrightarrow{g_2 \circ g_1 \circ g_0 \circ A_\bullet} \mathbf{Alg}_n^{\leq -n}) \simeq A$. Recall that the inclusion $\mathbf{Mod}_{O(n)}^{\leq -n} \hookrightarrow \mathbf{Mod}_{O(n)}$ preserves filtered colimits. Therefore if V is a compact of $\mathbf{Mod}_{O(n)}$ (i.e., a perfect $O(n)$ -module) then the truncation $\tau^{\leq -n} V$ is a compact object of $\mathbf{Mod}_{O(n)}^{\leq -n}$. (Note that the truncation need not be a compact object of $\mathbf{Mod}_{O(n)}$. For instance, the trivial module $\mathbb{k}[-n] \simeq \tau^{\leq -n} C_*(O(n), \mathbb{k}[-n])$ is compact as an object of $\mathbf{Mod}_{O(n)}^{\leq -n}$ but not compact as an object of $\mathbf{Mod}_{O(n)}$.) Consequently, every object of $\mathbf{Free}_n^{\text{perf}, \leq -n}$ is a compact object of $\mathbf{Alg}_n^{\leq -n}$, which is to say that the map $\text{colim}_{j \in \mathcal{J}} \text{Map}(\mathbb{F}V, A_j) \xrightarrow{\simeq} \text{Map}(\mathbb{F}V, A)$, involving spaces of morphisms in $\mathbf{Alg}_n^{\leq -n}$, is an equivalence. Consequently, we have that the natural functor between ∞ -categories

$$\text{colim}_{j \in \mathcal{J}} \mathbf{Free}_{n/A_j}^{\text{perf}, \leq -n} \longrightarrow \mathbf{Free}_{n/A}^{\text{perf}, \leq -n}$$

is an equivalence. The result is now a formal consequence of commuting colimits:

$$\text{colim}_{\mathbb{F}V \in \mathbf{Free}_{n/A}^{\text{perf}, \leq -n}} \mathbb{F}V \simeq \text{colim}_{j \in \mathcal{J}} \text{colim}_{\mathbb{F}V \in \mathbf{Free}_{n/A_j}^{\text{perf}, \leq -n}} \mathbb{F}V \simeq \text{colim}_{j \in \mathcal{J}} \text{colim}_{\mathbb{F}V \in \mathbf{Free}_{n/A_j}^{\text{perf}, \leq -n}} \mathbb{F}V \simeq \text{colim}_{j \in \mathcal{J}} A_j \simeq A.$$

We next prove that the arrow (28) is an equivalence. That is, we show that the canonical morphism

$$\text{LKan}_{g_2}(g_2 \circ g_1)(A) \simeq \text{colim}_{\mathbb{F}V \in \mathbf{Free}_{n/A}^{\leq -n}} \mathbb{F}V \longrightarrow A$$

is an equivalence for all $A \in \mathbf{Alg}_n^{\leq -n}$. This arrow fits as the lower horizontal arrow a solid diagram among augmented n -disk algebras

$$\begin{array}{ccc} \text{colim}_{\mathbb{F}V \in \mathbf{Free}_{n/A}^{\leq -n}} \text{colim}_{\bullet \in \Delta^{\text{op}}} \mathbb{F}^{\bullet+2} V & \longrightarrow & \text{colim}_{\bullet \in \Delta^{\text{op}}} \mathbb{F}^{\bullet+1} \text{Ker}(A \rightarrow \mathbb{k}) \\ \simeq \downarrow & \swarrow \text{dashed} & \downarrow \simeq \\ \text{colim}_{\mathbb{F}V \in \mathbf{Free}_{n/A}^{\leq -n}} \mathbb{F}V & \longrightarrow & A, \end{array}$$

which we now establish by way of functorial free resolutions. Namely, for each augmented n -disk algebra B there is a simplicial object

$$\mathbb{F}^{\bullet+1} \text{Ker}(B \rightarrow \mathbb{k}) : \Delta^{\text{op}} \rightarrow (\mathbf{Free}_n^{\leq -n})/B$$

given by the functorial free resolution of B ; it has the property that the canonical morphism from its colimit

$$|\mathbb{F}^{\bullet+1} \text{Ker}(B \rightarrow \mathbb{k})| \xrightarrow{\simeq} B$$

is an equivalence. This simplicial object is evidently functorial in the argument B , which is to say that it defines a functor $\mathbb{F}^{\bullet+1} \text{Ker}(- \rightarrow \mathbb{k}) : \mathbf{Alg}_n^{\leq -n} \rightarrow \mathbf{Fun}(\Delta^{\text{op}}, (\mathbf{Cat}_\infty)_{/\mathbf{Free}_n^{\leq -n}})$ to the ∞ -category of simplicial objects in ∞ -categories over $\mathbf{Free}_n^{\leq -n}$. Taking colimits establishes the square diagram and its commutativity, as well as the fact that the vertical arrows are equivalences. This also establishes the dashed arrow making the filled diagram manifestly commute.

Lastly, that (29) is an equivalence is exactly the assertion that the inclusion $\mathbf{Alg}_n^{\leq -n} \hookrightarrow \mathbf{Alg}_n^{\text{aug}}$ strongly generates. This assertion is precisely Corollary 3.5.18. \square

Lemma 3.5.3. *Let A be an augmented n -disk algebra in $\mathbf{Ch}_{\mathbb{k}}^\otimes$. Consider the ∞ -category $(\mathbf{Free}_n^{\leq -n})/A$ of augmented n -disk algebras over A which are free on $(-n)$ -coconnective $O(n)$ -modules in finite chain complexes over \mathbb{k} . This over ∞ -category*

$$(\mathbf{Free}_n^{\leq -n})/A$$

is sifted.

Proof. First, note that this ∞ -category $(\mathbf{Free}_n^{\leq -n})_{/A}$ is nonempty. We show the diagonal map $(\mathbf{Free}_n^{\text{perf}, \leq -n})_{/A} \rightarrow (\mathbf{Free}_n^{\text{perf}, \leq -n})_{/A} \times (\mathbf{Free}_n^{\text{perf}, \leq -n})_{/A}$ is final. For this, we use Quillen's Theorem A and argue that the iterated slice ∞ -category

$$((\mathbf{Free}_n^{\text{perf}, \leq -n})_{/A})_{(\mathbb{F}V \rightarrow A, \mathbb{F}W \rightarrow A)/}$$

has contractible classifying space for each pair of objects $(\mathbb{F}V \rightarrow A)$ and $(\mathbb{F}W \rightarrow A)$ of $(\mathbf{Free}_n^{\leq -n})_{/A}$. This iterated slice ∞ -category has an initial object, given by the universal arrow $\mathbb{F}(V \oplus W) \rightarrow A$ determined by the map of underlying $O(n)$ -modules $V \oplus W \rightarrow \mathbb{F}V \oplus \mathbb{F}W \rightarrow A \oplus A \rightarrow A$. \square

Through Lemma 2.5.1, the combination of Lemma 3.5.2 and Lemma 3.5.3 gives the following result.

Corollary 3.5.4. *For A an augmented n -disk algebra in $\mathbf{Ch}_{\mathbb{k}}$, the canonical arrow*

$$\text{colim}_{\mathbb{F}V \in (\mathbf{Free}_n^{\leq -n})_{/A}} \int_{M_*} \mathbb{F}V \xrightarrow{\cong} \int_{M_*} A$$

is an equivalence in $\mathbf{Ch}_{\mathbb{k}}$.

Proposition 3.5.5. *Let A be an augmented n -disk algebra in $\mathbf{Ch}_{\mathbb{k}}$. The canonical arrow*

$$\text{colim}_{F \in (\mathbf{FPres}_n^{\leq -n})_{/A}} \int_{M_*} F \xrightarrow{\cong} \int_{M_*} A$$

is an equivalence in $\mathbf{Ch}_{\mathbb{k}}$.

Proof. We explain the sequence of equivalences in $\mathbf{Ch}_{\mathbb{k}}$:

$$(30) \quad \text{colim}_{\mathbb{F}V \in (\mathbf{Free}_n^{\leq -n})_{/A}} \int_{M_*} \mathbb{F}V \longrightarrow \text{colim}_{F \in (\mathbf{FPres}_n^{\leq -n})_{/A}} \int_{M_*} F \longrightarrow \int_{M_*} A .$$

The arrows are the canonical ones, from restricting colimits. Corollary 3.5.4 states that the composite arrow is an equivalence. The result is verified upon showing that the left arrow is an equivalence.

Consider the commutative triangle of ∞ -categories

$$\begin{array}{ccc} (\mathbf{Free}_n^{\leq -n})_{/A} & \xrightarrow{\int_{M_*}} & \mathbf{Ch}_{\mathbb{k}} \\ \downarrow & \nearrow \int_{M_*} & \\ (\mathbf{FPres}_n^{\leq -n})_{/A} & & . \end{array}$$

Let $F \rightarrow A$ be an object of $(\mathbf{FPres}_n^{\leq -n})_{/A}$. From Corollary 3.5.4, the canonical map

$$\text{colim}_{\mathbb{F}V \in (\mathbf{Free}_n^{\leq -n})_{/F}} \int_{M_*} \mathbb{F}V \longrightarrow \int_{M_*} F$$

is an equivalence. It follows that the above triangle is a left Kan extension, and thereafter that the left arrow of (30) is an equivalence. \square

3.5.1. Conditions for strong generation. The following sequence of results, leading up to Corollary 3.5.18, is toward establishing the remaining step in the proof of Lemma 3.5.2 above, that the inclusion $\mathbf{Alg}_n^{\leq -n} \hookrightarrow \mathbf{Alg}_n^{\text{aug}}$ strongly generates in the sense of Definition 3.5.1.

The next formal results gives examples of functors that strongly generate.

Observation 3.5.6. For each ∞ -category \mathcal{C} , the Yoneda functor $j: \mathcal{C} \rightarrow \text{PShv}(\mathcal{C})$ strongly generates. This follows from the universal property of the Yoneda functor being the free colimit completion. For instance, the canonical natural transformation $\text{LKan}_j(j) \rightarrow \text{id}$ is by equivalences since its value on a presheaf \mathcal{F} on \mathcal{C} is the standard equivalence

$$\text{colim}(\mathcal{C}_{/\mathcal{F}} \rightarrow \mathcal{C} \xrightarrow{j} \text{PShv}(\mathcal{C})) \simeq \underset{(jc \rightarrow \mathcal{F}) \in \mathcal{C}_{/\mathcal{F}}}{\text{colim}} jc \xrightarrow{\simeq} \mathcal{F}.$$

Proposition 3.5.7. Each localization $\mathcal{B} \rightarrow \mathcal{C}$ between ∞ -categories strongly generates.

Proof. Let $g: \mathcal{B} \rightarrow \mathcal{C}$ be a localization. We must show that the canonical natural transformation $\text{LKan}_g(g) \rightarrow \text{id}_{\mathcal{C}}$ is by equivalences in \mathcal{C} . From the standard formula for the values of a left Kan extension, this is to show that, for each object $c \in \mathcal{C}$, the morphism in \mathcal{C}

$$\text{colim}(\mathcal{B}_{/c} \xrightarrow{g} \mathcal{C}_{/c} \rightarrow \mathcal{C}) \longrightarrow \text{colim}(\mathcal{C}_{/c} \rightarrow \mathcal{C}) \simeq c$$

is an equivalence. To show this it is sufficient to show that the functor $\mathcal{B}_{/c} \rightarrow \mathcal{C}_{/c}$ is final for each object $c \in \mathcal{C}$. Through Quillen's Theorem A, this is the problem of showing that, for each morphism $c' \xrightarrow{f} c$ in \mathcal{C} , the iterated slice ∞ -category $(\mathcal{B}_{/c})^{f/}$ has contractible classifying space: $\text{B}((\mathcal{B}_{/c})^{f/}) \simeq *$.

Let $c' \xrightarrow{f} c$ be a morphism in \mathcal{C} . Because $\mathcal{B} \xrightarrow{g} \mathcal{C}$ is a localization, the restriction functor $\text{Fun}(\mathcal{C}, \text{Spaces}) \xrightarrow{g^*} \text{Fun}(\mathcal{B}, \text{Spaces})$ is fully faithful. Therefore, for each functor $\mathcal{C} \xrightarrow{\mathcal{F}} \text{Spaces}$, the value of the counit of the left Kan extension-restriction adjunction

$$g_! g^* \mathcal{F} \xrightarrow{\simeq} \mathcal{F}$$

is an equivalence in $\text{Fun}(\mathcal{C}, \text{Spaces})$. Applying this to the representable copresheaf $\mathcal{F} := \mathcal{C}(c', -)$, then evaluating on c , we obtain an equivalence of spaces

$$\text{B}\left(\mathcal{B}_{/c} \times_{\mathcal{C}} \mathcal{C}^{c'/}$$

where the second equivalence is the standard formula computing the values of a left Kan extension and the first equivalence is through the straightening-unstraightening construction of §2.2 of [Lu1]. Quillen's Theorem A identifies the fiber over $f \in \mathcal{C}(c', c)$ of this map as the classifying space $\text{B}((\mathcal{B}_{/c})^{f/})$. We conclude that this classifying space is contractible, as desired. \square

Remark 3.5.8. A functor $\mathcal{B} \rightarrow \mathcal{C}$ that strongly generates is final with respect any functor $\mathcal{C} \rightarrow \mathcal{E}$ which preserves colimits, in that there is a natural equivalence $\text{colim}(\mathcal{C} \rightarrow \mathcal{E}) \simeq \text{colim}(\mathcal{B} \rightarrow \mathcal{E})$. Strong generation of $\mathcal{B} \rightarrow \mathcal{C}$ is thus a weakening of requiring both the conditions that \mathcal{B} generate \mathcal{C} under colimits and that the functor $\mathcal{B} \rightarrow \mathcal{C}$ being final.

Lemma 3.5.9. A functor $g: \mathcal{B} \rightarrow \mathcal{C}$ between ∞ -categories strongly generates if and only if the restricted Yoneda functor

$$\mathcal{C} \longrightarrow \text{PShv}(\mathcal{B})$$

is fully faithful.

Proof. By definition, g strongly generates if and only if the canonical natural transformation

$$\text{LKan}_g(g) \longrightarrow \text{id}_{\mathcal{C}}$$

between endofunctors on \mathcal{C} is by equivalences. This is to say that, for each object $c \in \mathcal{C}$, the canonical morphism $\text{LKan}_g(g)(c) \rightarrow c$ in \mathcal{C} is an equivalence. Via the standard formula computing the values of left Kan extensions, this is to say that, for each object $c \in \mathcal{C}$, the canonical morphism

$$\text{colim}(\mathcal{B}_{/c} \longrightarrow \mathcal{B} \xrightarrow{g} \mathcal{C}) \rightarrow c$$

in \mathcal{C} is an equivalence. By the defining universal property of colimits in an ∞ -category, this is equivalent to the assertion that, for each pair of objects $c, c' \in \mathcal{C}$, the canonical map between spaces

$$\mathcal{C}(c, c') \longrightarrow \lim((\mathcal{B}_{/c})^{\text{op}} \rightarrow \mathcal{B}^{\text{op}} \xrightarrow{g^{\text{op}}} \mathcal{C}^{\text{op}} \xrightarrow{\mathcal{C}(-, c')} \text{Spaces})$$

is an equivalence. Through the straightening-unstraightening equivalence (§2.2 of [Lu1]), the right-hand limit space is canonically identified an end; namely, as the space of functors over \mathcal{B} from $\mathcal{B}_{/c}$ to $\mathcal{B}_{/c'}$. As so, the above map of spaces is canonically identified as the map on morphism spaces

$$\mathcal{C}(c, c') \longrightarrow \text{Map}_{/\mathcal{B}}(\mathcal{B}_{/c}, \mathcal{B}_{/c'}) \simeq \text{Map}_{\text{PShv}(\mathcal{B})}(\mathcal{C}(g-, c), \mathcal{C}(g-, c'))$$

induced by the restricted Yoneda functor $\mathcal{C} \rightarrow \text{PShv}(\mathcal{B})$. We conclude that g strongly generates if and only if this restricted Yoneda functor is fully faithful. \square

Lemma 3.5.10. *Let $g: \mathcal{C}_0 \hookrightarrow \mathcal{C}$ be a fully faithful functor between ∞ -categories, and suppose \mathcal{C} is presentable. Let \mathcal{K} be an ∞ -category whose classifying space $\text{BK} \simeq *$ is terminal. If the composite functor $\mathcal{C}_0^{\mathcal{K}} \xrightarrow{g^{\mathcal{K}}} \mathcal{C}^{\mathcal{K}} \xrightarrow{\text{colim}} \mathcal{C}$ is a localization, then g strongly generates.*

Proof. Through Lemma 3.5.9, we seek to prove that the restricted Yoneda functor $\mathcal{C} \rightarrow \text{PShv}(\mathcal{C}_0)$ is fully faithful. This restricted Yoneda functor is accessible and preserves limits. Because \mathcal{C} is assumed presentable, this restricted Yoneda functor admits a left adjoint $L: \text{PShv}(\mathcal{C}_0) \rightarrow \mathcal{C}$. Through universal properties of presheaf ∞ -categories as free colimit completions, this left adjoint fits into a commutative diagram among ∞ -categories

$$\begin{array}{ccccc} \mathcal{C}_0^{\mathcal{K}} & \xrightarrow{g^{\mathcal{K}}} & \mathcal{C}^{\mathcal{K}} & \xrightarrow{\text{colim}} & \mathcal{C} \\ & \searrow^{j^{\mathcal{K}}} & & & \uparrow L \\ & & \text{PShv}(\mathcal{C}_0)^{\mathcal{K}} & \xrightarrow{\text{colim}} & \text{PShv}(\mathcal{C}_0) \end{array}$$

in which $j: \mathcal{C}_0 \rightarrow \text{PShv}(\mathcal{C}_0)$ denotes the Yoneda functor. The restricted Yoneda functor $\mathcal{C} \rightarrow \text{PShv}(\mathcal{C}_0)$ being fully faithful is equivalent to its left adjoint L being a localization. We prove that L is a localization by proving the following two points.

- (1) The composite functor $\mathcal{C}_0^{\mathcal{K}} \xrightarrow{j^{\mathcal{K}}} \text{PShv}(\mathcal{C}_0)^{\mathcal{K}} \xrightarrow{\text{colim}} \text{PShv}(\mathcal{C}_0)$ strongly generates.
- (2) Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{C} & \mathcal{Z} \\ & \searrow J & \nearrow L \\ & & \mathcal{Y} \end{array}$$

be a diagram among ∞ -categories. Suppose both \mathcal{Y} and \mathcal{Z} are presentable. Suppose C is a localization, J strongly generates, and L is accessible and preserves colimits. Supposing these conditions, the functor L is a localization.

Our result follows by applying (2) to the case of $\mathcal{X} := \mathcal{C}_0^{\mathcal{K}}$, $\mathcal{Z} := \mathcal{C}$, and $\mathcal{Y} := \text{PShv}(\mathcal{C}_0)$, with $C := \text{colim} \circ g^{\mathcal{K}}$, $J := \text{colim} \circ j^{\mathcal{K}}$, and $L := L$.

We prove (1). We must prove that the canonical natural transformation $\text{LKan}_{\text{colim} \circ j^{\mathcal{K}}}(\text{colim} \circ j^{\mathcal{K}}) \rightarrow \text{id}$ between endofunctors on $\text{PShv}(\mathcal{C}_0)$ is by equivalences. We explain that this natural transformation

factors as the sequence of natural transformations

$$\begin{aligned}
\mathrm{LKan}_{\mathrm{colim} \circ j^{\mathcal{K}}}(\mathrm{colim} \circ j^{\mathcal{K}}) &\xrightarrow{(a)} \mathrm{LKan}_{\mathrm{colim}}(\mathrm{LKan}_{j^{\mathcal{K}}}(\mathrm{colim} \circ j^{\mathcal{K}})) \\
&\xrightarrow{(b)} \mathrm{LKan}_{\mathrm{colim}}(\mathrm{colim} \circ \mathrm{LKan}_{j^{\mathcal{K}}}(j^{\mathcal{K}})) \\
&\xrightarrow{(c)} \mathrm{LKan}_{\mathrm{colim}}(\mathrm{colim} \circ \mathrm{LKan}_j(j)^{\mathcal{K}}) \\
&\xrightarrow{(d)} \mathrm{LKan}_{\mathrm{colim}}(\mathrm{colim}) \\
&\xrightarrow{(e)} \mathcal{BK} \otimes \mathrm{id} \\
&\xrightarrow{(f)} \mathrm{id}
\end{aligned}$$

each of which is by equivalences. The universal property of left Kan extensions gives that a left Kan extension of a composition is an iteration of left Kan extensions, which gives the identity (a). The identity (b) follows from the fact that, from the universal property of colimits, the colimit functor preserves colimits. The identity (c) is valid because colimits in functor categories are computed pointwise. The identity (d) is valid because the Yoneda functor strongly generating (Observation 3.5.6). The identity (e), involving tensoring with the classifying space of \mathcal{K} , is verified from the standard expression computing the values of left Kan extensions. The identity (f) invokes the assumption that the classifying space of \mathcal{K} is terminal. This completes the proof of (1).

We now prove (2). For $W := L^{-1}(\mathcal{Z}^\sim) \subset \mathcal{Y}$, we must verify that the canonical functor $\mathcal{Y}[W^{-1}] \rightarrow \mathcal{Z}$ under \mathcal{Y} is an equivalence. The assumption that L preserve colimits implies this functor $\mathcal{Y}[W^{-1}] \rightarrow \mathcal{Z}$ preserves colimits. The assumption that J strongly generates implies composition with the localization $\mathcal{X} \xrightarrow{J} \mathcal{Y} \rightarrow \mathcal{Y}[W^{-1}]$ also strongly generates. We are therefore reduced to the case that the functor $\mathcal{Y} \xrightarrow{L} \mathcal{Z}$ is conservative; for this case the problem is to verify that this functor $\mathcal{Y} \xrightarrow{L} \mathcal{Z}$ is an equivalence.

Invoking the universal property of localizations, conservativity of L implies the existence of a unique functor $\mathcal{Z} \xrightarrow{!} \mathcal{Y}$ under \mathcal{X} . Because localizations are epimorphisms, the resulting composite functor $\mathcal{Z} \xrightarrow{!} \mathcal{Y} \xrightarrow{L} \mathcal{Z}$ under \mathcal{X} is canonically equivalent to the identity functor on \mathcal{Z} ; in symbols $L \circ ! \simeq \mathrm{id}_{\mathcal{Z}}$. It remains to verify that the composite functor $\mathcal{Y} \xrightarrow{L} \mathcal{Z} \xrightarrow{!} \mathcal{Y}$ is equivalent to the identity functor on \mathcal{Y} . To do this we explain the equivalences among endofunctors on \mathcal{Y}

$$\begin{aligned}
! \circ L &\xleftarrow{(i)} ! \circ L \circ \mathrm{LKan}_J(J) \\
&\xleftarrow{(ii)} \mathrm{LKan}_J(! \circ L \circ J) \\
&\xleftarrow{(iii)} \mathrm{LKan}_J(! \circ L \circ ! \circ C) \\
&\xleftarrow{(iv)} \mathrm{LKan}_J(! \circ C) \\
&\xleftarrow{(v)} \mathrm{LKan}_J(J) \\
&\xleftarrow{(vi)} \mathrm{id}_{\mathcal{Y}} .
\end{aligned}$$

The identifications (i) and (vi) follow directly from the assumption that the functor $\mathcal{X} \xrightarrow{J} \mathcal{Y}$ strongly generates. The equivalences (iii) and (v) follow from the definition of the factorization $J: \mathcal{X} \xrightarrow{C} \mathcal{Z} \xrightarrow{!} \mathcal{Y}$. The equivalence (iv) follows directly from the equivalence $L \circ ! \simeq \mathrm{id}_{\mathcal{Z}}$ argued above.

To establish the identity (ii) it is enough to explain why the functor $! \circ L$ preserves colimits. Because L is assumed to do as much, it is enough to explain why the functor $\mathcal{Z} \xrightarrow{!} \mathcal{Y}$ preserves colimits. That is, for each functor $\mathcal{J} \rightarrow \mathcal{Z}$, we must explain why the canonical morphism $\mathrm{colim}(\mathcal{J} \rightarrow \mathcal{Z} \xrightarrow{!} \mathcal{Y}) \rightarrow !(\mathrm{colim}(\mathcal{J} \rightarrow \mathcal{Z}))$ in \mathcal{Y} is an equivalence. Using that L is conservative, it is enough to verify that the morphism $L(\mathrm{colim}(\mathcal{J} \rightarrow \mathcal{Z} \xrightarrow{!} \mathcal{Y})) \rightarrow L \circ !(\mathrm{colim}(\mathcal{J} \rightarrow \mathcal{Z}))$ in \mathcal{Z} is an equivalence. Using the

assumption that L preserves colimits, it is enough to verify that the morphism $\text{colim}(\mathcal{J} \rightarrow \mathcal{Z} \xrightarrow{!} \mathcal{Y} \xrightarrow{L} \mathcal{Z}) \rightarrow L \circ !(\text{colim}(\mathcal{J} \rightarrow \mathcal{Z}))$ in \mathcal{Z} is an equivalence. This follows using the equivalence $L \circ ! \simeq \text{id}_{\mathcal{Z}}$ argued above. We conclude that $\mathcal{Z} \xrightarrow{!} \mathcal{Y}$ preserves colimits, thereby completing the proof of this lemma. \square

For the next two results, we make use of the following notation.

Notation 3.5.11. Let \mathcal{C} be an ∞ -category and let $\mathcal{C}^\sim \subset \mathcal{W} \subset \mathcal{C}$ be an ∞ -subcategory containing the maximal ∞ -subgroupoid. For each ∞ -category \mathcal{K} , we denote the pullback

$$\text{Fun}_{\mathcal{W}}(\mathcal{K}, \mathcal{C}) := \text{Fun}(\mathcal{K}^\sim, \mathcal{W}) \times_{\text{Fun}(\mathcal{K}^\sim, \mathcal{C})} \text{Fun}(\mathcal{K}, \mathcal{C}),$$

which is the ∞ -subcategory of $\text{Fun}(\mathcal{K}, \mathcal{C})$ consisting of those natural transformations by \mathcal{W} .

Lemma 3.5.12. *Let \mathcal{C} be an ∞ -category and let $\mathcal{C}^\sim \subset \mathcal{W} \subset \mathcal{C}$ be an ∞ -subcategory containing the maximal ∞ -subgroupoid. Consider the simplicial space*

$$\mathbf{B} \text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})$$

which is the classifying space of the functor category of Notation 3.5.11. There is a canonical identification between the complete Segal localization of this simplicial space

$$\mathbf{B} \text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})_{\text{cpt.Seg}}^\wedge \simeq \mathcal{C}[\mathcal{W}^{-1}]$$

and the complete Segal presentation of the ∞ -categorical localization of \mathcal{C} by \mathcal{W} .

Proof. This is a routine argument among universal properties. We use Rezk's complete Segal space presentation of ∞ -categories: restriction along the standard functor $\mathbf{\Delta} \hookrightarrow \text{Cat}_\infty$ induces a fully faithful embedding $\text{Cat}_\infty \hookrightarrow \text{Fun}(\mathbf{\Delta}^{\text{op}}, \text{Spaces})$ whose essential image is characterized by the Segal and completeness localities.

We make two observations about the simplicial ∞ -category $\text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})$, both of which are manifest. First, this simplicial ∞ -category lies under the simplicial space $\text{Map}([\bullet], \mathcal{C}) \simeq \text{Fun}_{(\mathcal{C}^\sim)}([\bullet], \mathcal{C})$, which is the complete Segal presentation of \mathcal{C} . Second, for each functor $\mathcal{C} \rightarrow \mathcal{Z}$ that carries each morphism in \mathcal{W} to an equivalence in \mathcal{Z} , there is a unique simplicial functor $\text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C}) \rightarrow \text{Map}([\bullet], \mathcal{Z})$ to the complete Segal presentation of \mathcal{Z} under the induced map between simplicial spaces $\text{Map}([\bullet], \mathcal{C}) \rightarrow \text{Map}([\bullet], \mathcal{Z})$.

Consider the simplicial space $\mathbf{B} \text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})$, which is the object-wise classifying space. Thereafter, consider its complete and Segal localization, which we denote as $\mathbf{B} \text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})_{\text{cpt.Seg}}^\wedge$. This complete Segal simplicial space presents an ∞ -category, that we give the same notation, under \mathcal{C} . The second observation above, applied to the case $\mathcal{Z} \simeq \mathcal{C}[\mathcal{W}^{-1}]$ under \mathcal{C} , offers a canonical functor

$$(31) \quad \mathbf{B} \text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})_{\text{cpt.Seg}}^\wedge \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$$

under \mathcal{C} . By construction, the composite functor $\mathcal{W} \rightarrow \mathcal{C} \rightarrow \mathbf{B} \text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})_{\text{cpt.Seg}}^\wedge$ factors through the classifying space $\mathcal{W} \rightarrow \mathbf{B}\mathcal{W}$. As so, there is a unique functor

$$(32) \quad \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathbf{B} \text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})_{\text{cpt.Seg}}^\wedge$$

under \mathcal{C} . We assert that the functors (31) and (32) are mutual inverses, thereby demonstrating the equivalence

$$\mathbf{B} \text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})_{\text{cpt.Seg}}^\wedge \simeq \mathcal{C}[\mathcal{W}^{-1}]$$

between ∞ -categories under \mathcal{C} . The composition of (32) followed by (31) is indeed equivalent to the identity functor since $\mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is an epimorphism. To see that the composition of (31) followed by (32) is indeed equivalent to the identity functor, apply the above discussion to $\mathcal{Z} \simeq \mathbf{B} \text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})_{\text{cpt.Seg}}^\wedge$ under \mathcal{C} ; the result is a unique endofunctor of $\mathbf{B} \text{Fun}_{\mathcal{W}}([\bullet], \mathcal{C})_{\text{cpt.Seg}}^\wedge$ under \mathcal{C} . \square

Lemma 3.5.13. *Let \mathcal{C} be an ∞ -category that contains finite limits and geometric realizations, and such that finite limits commute with geometric realizations. Let $\mathcal{C}_0 \subset \mathcal{C}$ be a full ∞ -subcategory. The composite functor*

$$\mathcal{C}_0^{\Delta^{\text{op}}} \longrightarrow \mathcal{C}^{\Delta^{\text{op}}} \xrightarrow{\text{colim}} \mathcal{C},$$

is a localization if the following condition is satisfied.

- *For each simplicial object $A_\bullet \in \mathcal{C}^{\Delta^{\text{op}}}$, there exists a simplicial object $F_\bullet \in \mathcal{C}_0^{\Delta^{\text{op}}}$ and a map between simplicial objects in $\mathcal{C}^{\Delta^{\text{op}}}$,*

$$F_\bullet \longrightarrow A_\bullet,$$

that induces an equivalence $|F_\bullet| \xrightarrow{\sim} |A_\bullet|$ in \mathcal{C} between their colimits.

Proof. We apply Lemma 3.5.12 to the ∞ -category $\mathcal{C}_0^{\Delta^{\text{op}}}$ and the ∞ -subcategory $\mathcal{W} := \text{colim}^{-1}(\mathcal{C}^\sim) \subset \mathcal{C}_0^{\Delta^{\text{op}}}$, which is the ∞ -subcategory of $\mathcal{C}_0^{\Delta^{\text{op}}}$ consisting of those morphisms that become equivalences upon geometric realization in \mathcal{C} . The effect is that, to prove the result it suffices to show that, for each $[p] \in \Delta$, the canonical functor

$$\text{Fun}_{\mathcal{W}}([p], \mathcal{C}_0^{\Delta^{\text{op}}}) \rightarrow \text{Map}([p], \mathcal{C})$$

induces an equivalence between the space of p -simplices of \mathcal{C} and the classifying space of $\text{Fun}_{\mathcal{W}}([p], \mathcal{C}_0^{\Delta^{\text{op}}})$. To do this it suffices to show that for each p -simplex $A : [p] \rightarrow \mathcal{C}$, the classifying space of the fiber over A

$$\begin{array}{ccc} \text{Fun}_{\mathcal{W}}([p], \mathcal{C}_0^{\Delta^{\text{op}}})_{|A} & \longrightarrow & \text{Fun}_{\mathcal{W}}([p], \mathcal{C}_0^{\Delta^{\text{op}}}) \\ \downarrow & & \downarrow \\ \{A\} & \longrightarrow & \text{Map}([p], \mathcal{C}) \end{array}$$

is contractible.

We prove this contractibility by showing that each such fiber ∞ -category $\text{Fun}_{\mathcal{W}}([p], \mathcal{C}_0^{\Delta^{\text{op}}})_{|A}$ is cofiltered; this is to say that each solid diagram among ∞ -categories

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \text{Fun}_{\mathcal{W}}([p], \mathcal{C}_0^{\Delta^{\text{op}}})_{|A} \\ \downarrow & \dashrightarrow & \nearrow \\ \mathcal{K}^\triangleleft & & \end{array}$$

can be filled, provided \mathcal{K} is finite – here, $\mathcal{K}^\triangleleft$ denote the left-cone on \mathcal{K} . Let \mathcal{K} be a finite ∞ -category. We proceed by induction on p , and first consider the base case of $p = 0$. Given such a \mathcal{K} -point G , we can find a filler in the diagram among ∞ -categories

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{G} & \text{Fun}_{\mathcal{W}}([0], \mathcal{C}_0^{\Delta^{\text{op}}})_{|A} \\ \downarrow & & \downarrow \\ \mathcal{K}^\triangleleft & \dashrightarrow \lim_{\mathcal{K}} G & \dashrightarrow \mathcal{C}_0^{\Delta^{\text{op}}} \times_{\mathcal{C}} \mathcal{C}_{/A} \end{array}$$

defined as a limit diagram of the upper right composite. This limit indeed exists since \mathcal{C} is assumed to admit finite limits. The assumption that geometric realizations in \mathcal{C} commute with finite limits grants that the canonical morphism in \mathcal{C} from the geometric realization $| \lim_{\mathcal{K}} G | \rightarrow A$ in \mathcal{C} is an equivalence. We conclude that the bottom horizontal arrow factors through the ∞ -subcategory $\mathcal{C}_0^{\Delta^{\text{op}}}|_A := \mathcal{C}_0^{\Delta^{\text{op}}} \times_{\mathcal{C}} \{A \xrightarrow{\sim} A\} \subset \mathcal{C}_0^{\Delta^{\text{op}}} \times_{\mathcal{C}} \mathcal{C}_{/A}$. The bulleted condition applied to $A_\bullet := \lim_{\mathcal{K}} G \in \mathcal{C}_0^{\Delta^{\text{op}}}$ grants the existence of $F_\bullet \in \mathcal{C}_0^{\Delta^{\text{op}}}$ and a map $F_\bullet \rightarrow \lim_{\mathcal{K}} G$ that induces an equivalence on geometric

realizations. This defines an extension

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{G} & \mathrm{Fun}_{\mathcal{W}}([0], \mathcal{C}_0^{\Delta^{\mathrm{op}}})|_A \\ \downarrow & \dashrightarrow & \nearrow \\ \mathcal{K}^{\triangleleft} & \xrightarrow{F_{\bullet} \rightarrow G} & \end{array}$$

where the functor $\mathcal{K}^{\triangleleft} \rightarrow \mathcal{W}|_A$ assigns to the cone-point the aforementioned F_{\bullet} .

We now show the inductive step. Suppose $p > 0$ and consider the inclusion $[p-1] \cong \{1 < \dots < p\} \subset [p]$. Let $A : [p] \rightarrow \mathcal{C}$ be a p -simplex of \mathcal{C} , and choose an arbitrary \mathcal{K} -point G as before. We first argue the existence of a filler among the diagram of ∞ -categories

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{G} & \mathrm{Fun}_{\mathcal{W}}([p], \mathcal{C}_0^{\Delta^{\mathrm{op}}})|_A \\ \downarrow & & \downarrow \\ \mathcal{K}^{\triangleleft} & \dashrightarrow & \mathrm{Fun}_{\mathcal{W}}([p-1], \mathcal{C}_0^{\Delta^{\mathrm{op}}})|_{A|_{[p-1]}} \times \mathrm{Fun}_{\mathcal{W}}([p], \mathcal{C}_0^{\Delta^{\mathrm{op}}})|_A . \end{array}$$

The projection to first factor, $\mathrm{Fun}_{\mathcal{W}}([p], \mathcal{C}_0^{\Delta^{\mathrm{op}}})|_A$, is the limit diagram of the evident functor induced by the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$. The projection to the first factor $\mathrm{Fun}_{\mathcal{W}}([p-1], \mathcal{C}_0^{\Delta^{\mathrm{op}}})|_{A|_{[p-1]}}$ is by induction. Inspecting the inductive construction of such fillers reveals that the resulting two projections to $\mathrm{Fun}_{\mathcal{W}}([p-1], \mathcal{C}_0^{\Delta^{\mathrm{op}}})|_{A|_{[p-1]}}$ canonically agree, since finite limits are assumed to commute with geometric realizations. This establishes the dashed functor making the diagram commute.

Restricting to the initial object of $\mathcal{K}^{\triangleleft}$ gives a functor $[p] \rightarrow \mathcal{C}^{\Delta^{\mathrm{op}}}$ whose colimit is identified as A , and for which the restriction to $\{1 < \dots < p\} \subset [p]$ lies in the essential image of $\mathcal{C}_0^{\Delta^{\mathrm{op}}}$. Let A_{\bullet}^0 denote the restriction $\{0\} \subset [p] \xrightarrow{A_{\bullet}} \mathcal{C}^{\Delta^{\mathrm{op}}}$. We again apply the bulleted condition to find an object $F_{\bullet} \in \mathcal{C}_0^{\Delta^{\mathrm{op}}}$ and a morphism $F_{\bullet} \rightarrow A_{\bullet}^0$ of simplicial objects in \mathcal{C} that induces an equivalence $|F_{\bullet}| \xrightarrow{\simeq} |A_{\bullet}^0|$ between geometric realizations. Concatenating with F_{\bullet} gives the requisite lift $\mathcal{K}^{\triangleleft} \rightarrow \mathrm{Fun}_{\mathcal{W}}([p], \mathcal{C}_0^{\Delta^{\mathrm{op}}})|_A$. \square

We now aim to check the bulleted condition of Lemma 3.5.13 for our case at hand. This is Lemma 3.5.16 below. We first establish a few intermediate results.

Observation 3.5.14. *Each of the ∞ -categories*

$$\mathrm{Ch}_{\mathbb{k}} \quad \text{and} \quad \mathrm{Mod}_{\mathrm{O}(n)}(\mathrm{Ch}_{\mathbb{k}}) \quad \text{and} \quad \mathrm{Alg}_n^{\mathrm{aug}}$$

has the property that finite limits commute with geometric realizations.

Proof. Because $\mathrm{Ch}_{\mathbb{k}}$ is a stable ∞ -category, finite limits commute with geometric realizations. Because limits and colimits are computed object-wise in $\mathrm{Mod}_{\mathrm{O}(n)}(\mathrm{Ch}_{\mathbb{k}}) := \mathrm{Fun}(\mathrm{BO}(n), \mathrm{Ch}_{\mathbb{k}})$, this ∞ -category possesses this property as well. Because the forgetful functor $\mathrm{Alg}_n^{\mathrm{aug}} \rightarrow \mathrm{Mod}_{\mathrm{O}(n)}(\mathrm{Ch}_{\mathbb{k}})$ preserves and creates limits as well as sifted colimits, the ∞ -category $\mathrm{Alg}_n^{\mathrm{aug}}$ too possesses this property. \square

The given proof of the following lemma was suggested by the referee.

Lemma 3.5.15. *Let W be a finite $\mathrm{O}(n)$ -module over a field \mathbb{k} . Let $W_{\bullet} \rightarrow W$ be an augmented simplicial $\mathrm{O}(n)$ -module demonstrating a colimit. For any N , there exists a simplicial object W'_{\bullet} in finite $(-N)$ -coconnective $\mathrm{O}(n)$ -modules and a morphism of simplicial objects*

$$W'_{\bullet} \longrightarrow W_{\bullet}$$

that induces an equivalence $|W'_{\bullet}| \simeq |W_{\bullet}| \simeq W$ between their colimits.

Proof. We will employ the ∞ -categorical Dold–Kan correspondence, see §1.2.4 of [Lu2]. This asserts that for \mathcal{C} a stable ∞ -category, there is an equivalence between simplicial objects and sequential objects

$$\mathcal{C}^{\Delta^{\text{op}}} \longrightarrow \mathcal{C}^{\mathbb{N}}$$

which sends a simplicial object X_{\bullet} to its skeletal filtration, $i \mapsto \text{Sk}_i X_{\bullet}$. Let $W_{\bullet} \rightarrow W$ be as above, and consider the map from the skeletal filtration $\text{Sk}_{\bullet} W_{\bullet} \rightarrow W$. Since W is a compact object of $\text{Mod}_{\text{O}(n)}$, the identity map $W \xrightarrow{\sim} \varinjlim_p \text{Sk}_p W_{\bullet}$ lifts through some finite stage of the filtration, $W \rightarrow \text{Sk}_q W_{\bullet}$ for some sufficiently large q . Let m be the maximum of q and $N + \ell$, where ℓ is the largest degree in which $H_{\ell} W$ is nonzero. Given $m \geq q$, there then exists a map of sequential objects

$$\begin{array}{ccccccc} 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & W \xrightarrow{=} W \xrightarrow{=} \dots \\ \downarrow & & & & \downarrow & & \downarrow \\ \text{Sk}_0 W_{\bullet} & \longrightarrow & \dots & \longrightarrow & \text{Sk}_{m-1} W_{\bullet} & \longrightarrow & \text{Sk}_m W_{\bullet} \longrightarrow \text{Sk}_{m+1} W_{\bullet} \longrightarrow \dots \end{array}$$

which induces the equivalence $W \simeq \varinjlim_p \text{Sk}_p W_{\bullet}$ on taking sequential colimits. Using the inverse equivalence in the ∞ -categorical Dold–Kan correspondence, the top sequential object maps to $S^m \otimes W[-m]$, the tensor of the desuspension of W with the simplicial set $S^m := \Delta[m]/\partial\Delta[m]$. Note that this is a simplicial object in $(-N)$ -coconnective chain complexes due to the inequality $m \geq N + \ell$. We thus obtain a map of simplicial objects

$$S^m \otimes W[-m] \longrightarrow W_{\bullet}$$

from an $(-N)$ -coconnective simplicial object to W_{\bullet} which induces an equivalence of colimits. \square

Lemma 3.5.16. *For each simplicial object in $\text{Alg}_n^{\text{aug}}$, there exists a simplicial object F_{\bullet} in $\text{Alg}_n^{\leq -n}$ and a morphism of simplicial objects*

$$F_{\bullet} \longrightarrow A_{\bullet}$$

that induces an equivalence $|F_{\bullet}| \simeq |A_{\bullet}|$ between their colimits.

Proof. Let $\mathcal{M} \subset \text{Alg}_n^{\text{aug}}$ be the full ∞ -subcategory consisting of those augmented n -disk algebras A with the following property.

Let $N \geq n$. Let A_{\bullet} be a simplicial augmented n -disk algebra equipped with an identification $|A_{\bullet}| \simeq A$ of its geometric realization. There exists a simplicial object F_{\bullet} in augmented n -disk algebras whose augmentation ideal is $(-N)$ -coconnective, together with a morphism $F_{\bullet} \rightarrow A_{\bullet}$ of simplicial augmented n -disk algebras, that induces an equivalence $|F_{\bullet}| \simeq |A_{\bullet}|$ between their colimits.

We prove that the inclusion $\mathcal{M} \hookrightarrow \text{Alg}_n^{\text{aug}}$ is an equivalence between ∞ -categories, which clearly implies the result.

We prove this in two steps:

- (1) \mathcal{M} contains every free augmented n -disk algebra $\mathbb{F}W$ generated by a finite $\text{O}(n)$ -module W .
- (2) \mathcal{M} contains every augmented n -disk algebra.

We prove (1), that \mathcal{M} contains each free augmented n -disk algebra $\mathbb{F}W$ on a finite $\text{O}(n)$ -module. Given an augmented simplicial object $A_{\bullet} \rightarrow \mathbb{F}W$ in augmented n -disk algebras demonstrating a colimit, consider the augmented simplicial $\text{O}(n)$ -module

$$W_{\bullet} := A_{\bullet} \times_{\mathbb{F}W} W \longrightarrow W$$

constructed by pulling back the augmented simplicial $\text{O}(n)$ -module $A_{\bullet} \rightarrow \mathbb{F}W$ along the map between the augmentation $\text{O}(n)$ -modules $W \rightarrow \mathbb{F}W$. Using that the forgetful functor $\text{Alg}_n^{\text{aug}} \rightarrow \text{Mod}_{\text{O}(n)}(\text{Ch}_{\mathbb{k}})$ preserves sifted colimits, Observation 3.5.14 gives that the augmented simplicial $\text{O}(n)$ -module $W_{\bullet} \rightarrow W$ demonstrates a colimit. By Lemma 3.5.15 there exists a simplicial $(-N)$ -coconnective and finite $\text{O}(n)$ -module W'_{\bullet} with a map $W'_{\bullet} \rightarrow W_{\bullet}$ inducing an equivalence $|W'_{\bullet}| \xrightarrow{\simeq}$

$|W_\bullet| \simeq W$ between their colimits, for any $N \geq n$. Applying the free-forgetful adjunction, we have a map of simplicial augmented n -disk algebras

$$\mathbb{F}(W'_\bullet) \longrightarrow A_\bullet .$$

Being a left adjoint, the free functor \mathbb{F} preserves geometric realizations. Therefore, the induced map between colimits $|\mathbb{F}(W'_\bullet)| \rightarrow |A_\bullet| \simeq \mathbb{F}W$ is an equivalence between augmented n -disk algebras. This completes the first step.

We now prove (2). Let $A_\bullet \rightarrow A$ be an augmented simplicial object witnessing a colimit among augmented n -disk algebras. Given $N \geq n$, we construct a simplicial augmented n -disk algebra F_\bullet whose augmentation ideal is $(-N)$ -coconnective, together with a map $F_\bullet \rightarrow A_\bullet$ between simplicial objects that induces an equivalence $|F_\bullet| \xrightarrow{\simeq} |A_\bullet|$ between their colimits. We construct F_\bullet as a sequential colimit $F_\bullet := \varinjlim_{\ell \geq 0} F_\bullet^\ell$, each cofactor in which is a simplicial augmented n -disk algebra whose augmentation ideal is $(-N)$ -coconnective. We construct this sequential diagram

$$\operatorname{colim}_{\ell \geq 0} \mathbb{N}_{\leq \ell} \simeq \mathbb{N} \rightarrow \operatorname{Fun}(\Delta^{\operatorname{op}}, \operatorname{Alg}_n^{\operatorname{aug}, \leq -N})_{/A_\bullet} , \quad \ell \mapsto F_\bullet^\ell ,$$

by induction on ℓ .

As the base case we construct F_\bullet^0 . Choose a collection

$$\left\{ [\mathbb{k}[k_j] \xrightarrow{x_j} \operatorname{Ker}(A \rightarrow \mathbb{1})] \mid j \in J_0 \right\} \subset H_*(\operatorname{Ker}(A \rightarrow \mathbb{1}))$$

of generators for the homology groups of the augmentation ideal of A . For this collection, the natural morphism $\coprod_{j \in J_0} \mathbb{F}[k_j] \rightarrow A$ between augmented n -disk algebras is surjective on homology groups, where $\mathbb{F}[x_j]$ is the free augmented n -disk algebra generated by the free $O(n)$ -module on $\mathbb{k}[k_j]$. For each $j \in J_0$ consider the augmented simplicial object $\mathbb{F}[x_j] \times_{A_\bullet} \rightarrow \mathbb{F}[x_j]$ among augmented n -disk algebras. Observation 3.5.14 grants that this augmented simplicial object witnesses a colimit. As so we can apply (1) to obtain, for each $j \in J_0$, a simplicial augmented n -disk algebra $F_\bullet^{0,j}$ whose augmentation ideal is $(-N)$ -coconnective, as it fits into a diagram among augmented n -disk algebras

$$\begin{array}{ccc} F_\bullet^{0,j} & \longrightarrow & A_\bullet \\ \downarrow & & \downarrow \\ \mathbb{F}[x_j] & \longrightarrow & A \end{array}$$

for which the vertical arrows induce equivalences on geometric realizations. We now set

$$F_\bullet^0 := \coprod_{j \in J_0} F_\bullet^{0,j}$$

to be the coproduct in simplicial objects among augmented n -disk algebras. By construction, there is a canonical morphism $F_\bullet^0 \rightarrow A_\bullet$ which induces a surjection on the homology groups of the geometric realizations $|F_\bullet^0| \rightarrow |A_\bullet| \simeq A$.

We now establish the inductive step. Assume we have constructed the diagram $F_\bullet^0 \rightarrow \dots \rightarrow F_\bullet^\ell$ over A_\bullet . Consider the pullback simplicial n -disk algebra $A_\bullet^\ell := \mathbb{1} \times_{A_\bullet} F_\bullet^\ell$. Set $A^\ell := |A_\bullet^\ell|$ to be the augmented n -disk algebra which is the geometric realization of this simplicial one. Through Observation 3.5.14 we identify this geometric realization as the pullback $A^\ell \simeq \mathbb{1} \times_{A_\bullet} |F_\bullet^\ell|$, the augmentation ideal of which is the kernel $\operatorname{Ker}(|F_\bullet^\ell| \rightarrow A)$. As in the base case, choose a collection

$$\left\{ [\mathbb{k}[k_j] \xrightarrow{x_j} \operatorname{Ker}(A^\ell \rightarrow \mathbb{1})] \mid j \in J_\ell \right\} \subset H_*(\operatorname{Ker}(A^\ell \rightarrow \mathbb{1}))$$

of this augmentation ideal. As in the base case, we can choose, for each $j \in J_\ell$, a simplicial object $F_\bullet^{\ell,j}$ whose augmentation ideal is $(-N-1)$ -coconnective, as it fits into a diagram among augmented

n -disk algebras

$$\begin{array}{ccc} F_{\bullet}^{\ell,j} & \longrightarrow & A_{\bullet}^{\ell} \\ \downarrow & & \downarrow \\ \mathbb{F}[x_j] & \longrightarrow & A^{\ell} \end{array}$$

for which the vertical arrows induce equivalences on geometric realizations. We now set $F_{\bullet}^{\ell+1}$ to be the pushout among simplicial objects in augmented n -disk algebras:

$$\begin{array}{ccc} \coprod_{j \in J_{\ell}} F_{\bullet}^{\ell,j} & \longrightarrow & F_{\bullet}^{\ell} \\ \downarrow & & \downarrow \\ \mathbb{k} & \longrightarrow & F_{\bullet}^{\ell+1}. \end{array}$$

Lemma 3.3.3 guarantees that this pushout indeed has the property that its augmentation ideal is a simplicial $(-N)$ -coconnective \mathbb{k} -module. By construction there is a canonical morphism $F_{\bullet}^{\ell+1} \rightarrow A_{\bullet}$ between simplicial objects that induces a surjection between homology groups of the geometric realization $|F_{\bullet}^{\ell+1}| \rightarrow A$. This concludes the construction of the sequential diagram $\mathbb{N} \xrightarrow{\ell \mapsto F_{\bullet}^{\ell}} \text{Fun}(\Delta^{\text{op}}, \text{Alg}_n^{\text{aug}, \leq -N})/A_{\bullet}$.

Set $F_{\bullet} := \varinjlim_{\ell} F_{\bullet}^{\ell}$ to be the sequential colimit among augmented n -disk algebras. It remains to check that the canonical morphism from the geometric realization $|F_{\bullet}| \rightarrow |A_{\bullet}| \simeq A$ is an equivalence. The morphism $|F_{\bullet}^0| \rightarrow A$ is a surjection on homology groups. Given the factorization $|F_{\bullet}^0| \rightarrow |F_{\bullet}| \rightarrow A$, we conclude that the morphism $|F_{\bullet}| \rightarrow A$ is a surjection on homology groups. We must check that this morphism $|F_{\bullet}| \rightarrow A$ is an injection on homology groups. Let $\mathbb{k}[k] \rightarrow |F_{\bullet}|$ represent an element of the kernel of $H_*(|F_{\bullet}|) \rightarrow H_*(A)$. Because \mathbb{k} , and therefore $\mathbb{k}[k]$, is compact, there is an $\ell \geq 0$ for which this representative factors as $\mathbb{k}[k] \rightarrow |F_{\bullet}^{\ell}| \rightarrow |F_{\bullet}|$. Because the map $\mathbb{k}[k] \rightarrow |F_{\bullet}^{\ell}| \rightarrow A$ factors through 0, the construction of $F_{\bullet}^{\ell+1}$ is just so that the composite map $\mathbb{k}[k] \rightarrow |F_{\bullet}^{\ell}| \rightarrow |F_{\bullet}^{\ell+1}|$ factors through 0. Therefore, the map $\mathbb{k}[k] \rightarrow |F_{\bullet}|$ represents zero in the homology $H_*(|F_{\bullet}|)$. We conclude that the kernel of the homomorphism $H_*(|F_{\bullet}|) \rightarrow H_*(A)$ is zero, which completes this proof. \square

Corollary 3.5.17. *The composite functor*

$$(\text{Alg}_n^{\leq -n})^{\Delta^{\text{op}}} \hookrightarrow (\text{Alg}_n^{\text{aug}})^{\Delta^{\text{op}}} \xrightarrow{\text{colim}} \text{Alg}_n^{\text{aug}}$$

is a localization.

Proof. We check the bulleted condition of Lemma 3.5.13. First, Observation 3.5.14 states that finite limits and geometric realizations commute in the ∞ -category $\text{Alg}_n^{\text{aug}}$. The second condition of the Lemma 3.5.13 is Lemma 3.5.16. \square

The next result completes the proof of Lemma 3.5.2.

Corollary 3.5.18. *The inclusion $\text{Alg}_n^{\leq -n} \hookrightarrow \text{Alg}_n^{\text{aug}}$ strongly generates.*

Proof. By Corollary 3.5.17, the functor

$$(\text{Alg}_n^{\leq -n})^{\Delta^{\text{op}}} \hookrightarrow (\text{Alg}_n^{\text{aug}})^{\Delta^{\text{op}}} \xrightarrow{\text{colim}} \text{Alg}_n^{\text{aug}}$$

is a localization. Consider Lemma 3.5.10 applied to the case that $\mathcal{K} = \Delta^{\text{op}}$, and that $\mathcal{C} = \text{Alg}_n^{\text{aug}}$ with $\mathcal{C}_0 = \text{Alg}_n^{\leq -n}$. We have shown that the hypotheses of this lemma are satisfied, therefore the inclusion $\text{Alg}_n^{\leq -n} \hookrightarrow \text{Alg}_n^{\text{aug}}$ strongly generates. \square

4. HOCHSCHILD HOMOLOGY OF ASSOCIATIVE AND ENVELOPING ALGEBRAS

In the remainder of this paper, we detail the meaning and consequences of our main theorem in the 1-dimensional case, where it becomes a statement about usual Hochschild homology and where the Maurer–Cartan functor MC reduces to the familiar Maurer–Cartan functors for associative and Lie algebras. We will first describe the general case of an associative algebra, then we will further specialize to the case of enveloping algebras of Lie algebras in characteristic zero.

4.1. Case $n = 1$. For an augmented associative algebra A , consider the Maurer–Cartan functor MC_A from Definition 3.1.8, i.e., by considering A as a framed 1-disk algebra. Our main theorem has the following consequence in dimension 1. This generalizes an essentially equivalent result for cyclic homology due to Feigin & Tsygan in [FT2]; see also the operadic generalization of Getzler & Kapranov in [GeK].

Corollary 4.1.1. *Let A be an augmented associative algebra over a field \mathbb{k} . There is an equivalence*

$$\text{HH}_*(A)^\vee \simeq \text{HH}_*(\text{MC}_A)$$

between the dual of the Hochschild homology of A and the Hochschild homology of the moduli functor MC_A . If A is either connected and degreewise finite, or (-1) -coconnective and finitely presented, then there is an equivalence

$$\text{HH}_*(A)^\vee \simeq \text{HH}_*(\mathbb{D}A)$$

between the linear dual of the Hochschild homology of A and the Hochschild homology of the Koszul dual of A .

Proof. The result follows immediately from Theorem 3.2.4 together with the equivalence $\int_{S^1} A \simeq \text{HH}_*(A)$ (see Theorem 3.19 of [AF1]). \square

Specializing further to the case where the associative algebra is the enveloping algebra of a Lie algebra.

Corollary 4.1.2. *Let \mathfrak{g} be Lie algebra over a field \mathbb{k} which is degreewise finite and connective. There is an equivalence*

$$\text{HH}_*(\text{U}\mathfrak{g})^\vee \simeq \text{HH}_*(\text{C}^*\mathfrak{g})$$

between the dual of the Hochschild homology of the enveloping algebra of \mathfrak{g} and the Hochschild homology of the Lie algebra cochains of \mathfrak{g} .

Proof. We will apply Corollary 4.1.1 to the case $A = \text{U}\mathfrak{g}$. We first make a standard identification of the Koszul dual:

$$\mathbb{D}(\text{U}\mathfrak{g}) := \text{Hom}_{\mathbb{k}}\left(\int_{\mathbb{R}^+} \text{U}\mathfrak{g}, \mathbb{k}\right) \simeq \text{Hom}_{\mathbb{k}}\left(\mathbb{k} \otimes_{\text{U}\mathfrak{g}} \mathbb{k}, \mathbb{k}\right) \simeq \text{Hom}_{\text{U}\mathfrak{g}}(\mathbb{k}, \mathbb{k}) =: \text{C}^*\mathfrak{g}.$$

The first and last equivalences are definitional; the second equivalence above is \otimes -excision for the closed interval; the third equivalence is the Hom-tensor adjunction.

By the Poincaré–Birkhoff–Witt filtration, $\text{U}\mathfrak{g} \simeq \text{Sym}(\mathfrak{g})$, with the given finiteness and connectivity conditions on \mathfrak{g} imply them for $\text{U}\mathfrak{g}$. Therefore, the conditions of Corollary 4.1.1 apply and so give the result. \square

Remark 4.1.3. A different proof of Corollary 4.1.1 can be given via Morita theory, after results of Lurie, using that Koszul duality interchanges quasi-coherent sheaves and ind-coherent sheaves; a treatment along these lines has been given by Campbell in [Ca]. We briefly summarize: by Theorem 3.5.1 of [Lu4], Koszul duality for associative algebras extends to a Koszul duality for modules and an equivalence $\text{Perf}_A^R \simeq \text{Coh}_{\text{MC}_A}^L$ between perfect right A -modules and left coherent sheaves on MC_A . Coherent sheaves are, by definition, the dual to perfect sheaves; Hochschild homology is a symmetric monoidal functor and therefore preserves duals. Consequently, the Hochschild homology $\text{HH}_*(\text{Perf}_{\text{MC}_A}^L)$ is the dual of $\text{HH}_*(\text{Perf}_A^R)$. Corollary 4.1.1 then follows from showing formal descent

for Hochschild homology of formal moduli problems, implying that the Hochschild homology of $\text{Perf}_{\text{MC}_A}^{\mathbb{L}}$ agrees with that of MC_A .

4.2. Lie algebras. We will now specialize the preceding results in order to offer an interesting interpretation of our results in Lie theory over a field \mathbb{k} of characteristic zero. Our main duality result, Theorem 3.2.4, specializes to one in Lie theory, Corollary 4.2.2, which makes no reference to either n -disk algebras or factorization homology: there is a duality between the Hochschild homology of an enveloping algebra and the Hochschild homology of the associated Maurer–Cartan space.

Convention (Characteristic zero and framings). Henceforward, we make the following choices to more conveniently make the connection with Lie algebras. We fix a field \mathbb{k} of characteristic zero. We work with (augmented) \mathcal{E}_n -algebras in place of (augmented) Disk_n -algebras; and likewise with *framed* zero-pointed n -manifolds in place of zero-pointed n -manifolds. These easy substitutions do not affect the preceding arguments.

We consider the enveloping n -disk algebra $U_n\mathfrak{g}$ of a Lie algebra \mathfrak{g} . Maintaining the assumption of \mathbb{k} being characteristic zero, we will make use of the following sequence of adjunctions

$$(33) \quad \begin{array}{c} \text{Alg}_{\text{Lie}} \xleftarrow{\quad U_n \quad} \text{Alg}_{\mathcal{E}_n}^{\text{aug}} \xleftarrow[\Omega]{\quad \text{Bar} \quad} \text{Alg}_{\mathcal{E}_{n-1}}^{\text{aug}} \xleftarrow[\Omega]{\quad \text{Bar} \quad} \dots \\ \text{Alg}_{\text{Lie}} \xleftarrow{\quad \dots \quad} \text{Alg}_{\mathcal{E}_n}^{\text{aug}} \xleftarrow[\Omega]{\quad \text{Bar} \quad} \text{Alg}_{\mathcal{E}_{n-1}}^{\text{aug}} \xleftarrow[\Omega]{\quad \text{Bar} \quad} \dots \\ \text{Alg}_{\text{Lie}} \xleftarrow{\quad U_{n-1} \quad} \text{Alg}_{\mathcal{E}_n}^{\text{aug}} \end{array}$$

where the curved arrows are left adjoints. The right adjoints consist in shifting the augmentation ideal down a single homological degree. The bar-loop adjunction is constructed in §5 of [AF1]. A shifted version of the Poincaré–Birkhoff–Witt filtration applies to these n -disk enveloping algebra: there is an equivalence

$$(34) \quad U_n\mathfrak{g} \simeq \text{Sym}(\mathfrak{g}[n-1])$$

as \mathbb{k} -modules. A similar picture in the setting of D-modules, including the Poincaré–Birkhoff–Witt filtration, is given in [FG]. We establish (33) and (34) in [AFK], but will use them in the following corollary.

Remark 4.2.1. Parts of the preceding picture of (33) and (34) can likely also be deduced from the formality of the \mathcal{E}_n operad in characteristic zero.

The following result and proof follows that of Corollary 4.1.2.

Corollary 4.2.2. *Let \overline{M} be a framed compact n -manifold with partitioned boundary $\partial\overline{M} = \partial_{\mathbb{L}} \sqcup \partial_{\mathbb{R}}$. Let \mathfrak{g} be a Lie algebra over a field \mathbb{k} of characteristic zero. Suppose \mathfrak{g} is either*

- *n -connective and finite in each degree, or*
- *(-1) -coconnective and finite as a \mathbb{k} -module.*

There is a natural equivalence of chain complexes over \mathbb{k}

$$\left(\int_{\overline{M} \setminus \partial_{\mathbb{R}}} U_n\mathfrak{g} \right)^{\vee} \simeq \int_{\overline{M} \setminus \partial_{\mathbb{L}}} \mathbb{C}^*\mathfrak{g}$$

between the linear dual of the factorization homology with coefficients in the enveloping n -disk algebra of \mathfrak{g} and the factorization homology with coefficients in the Lie algebra cochains of \mathfrak{g} .

Proof. We apply the simple version of Theorem 3.2.4, where the algebra A is either connected or sufficiently coconnected so that the factorization homology on the dual side is in terms of \mathbb{D}^n rather than MC_A . That is, the combination of Theorem 2.1.9, Theorem 2.1.7, and Proposition 3.3.4 (summarized in the introduction as Theorem 0.0.3), gives an equivalence

$$\left(\int_{\overline{M} \setminus \partial_{\mathbb{R}}} A \right)^{\vee} \simeq \int_{\overline{M} \setminus \partial_{\mathbb{L}}} \mathbb{D}^n A$$

provided that A is either connected or $(-n)$ -coconnective.

We now complete the proof in two steps. Firstly, we show the equivalence $\mathbb{D}^n(\mathbf{U}_n\mathfrak{g}) \simeq \mathbf{C}^*\mathfrak{g}$. This is dual to the equivalence $\int_{(\mathbb{R}^n)^+} \mathbf{U}_n\mathfrak{g} \simeq \mathbf{C}_*\mathfrak{g}$, which follows from (33) and the identification of $\int_{(\mathbb{R}^n)^+}$ as the n -times iterated bar construction in §5 of [AF1].

Lastly, we show the given conditions on \mathfrak{g} give the required conditions on $A = \mathbf{U}_n\mathfrak{g}$. This follows from the identification $\mathbf{U}_n\mathfrak{g} \simeq \mathbf{Sym}(\mathfrak{g}[n-1])$ and the fact that, in characteristic zero, symmetric powers preserve coconnectivity. □

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