A STRATIFIED HOMOTOPY HYPOTHESIS

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Abstract. We show that stratified spaces, with conically smooth maps among them, embed fully faithfully into ∞-categories. This articulates a stratified generalization of the homotopy hypothesis proposed by Grothendieck. As such, each ∞-category defines a stack on stratified spaces, and we identify the descent conditions it satisfies. These include $\mathbb{R}^1$-invariance and descent for open covers and blow-ups, analogous to sheaves for the h-topology in $\mathbb{A}^1$-homotopy theory. In this way, we identify ∞-categories as *striation sheaves*, which are those sheaves on stratified spaces satisfying the indicated descent. We use this identification to construct by hand two remarkable examples of ∞-categories: $\mathcal{B}\mathsf{un}$, an ∞-category classifying constructible bundles; and $\mathcal{E}\mathsf{xit}$, the absolute exit-path ∞-category. These constructions are deeply premised on stratified geometry, the key geometric input being a characterization of stratified maps between cones.

Contents

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Introduction

The present work lies in a program whose aim is to fuse manifold topology with ∞-category theory. The following hypothesis guides our program: a manifold can be encoded as a moduli space for its stratifications. The manifold itself is a point-set geometric object, while the moduli space is ∞-categorical. The first step in our program was taken in work with Hiro Lee Tanaka in [AFT]. There, using a key notion of conical smoothness, foundations were laid for a theory of stratified spaces so as to accommodate moduli.

The present work studies the interplay of geometry and homotopy theory of stratified spaces and conically smooth maps among them. We show that the homotopy theory of conically smooth maps carries a universal property; this property forms a stratified generalization of the homotopy hypothesis put forward by Grothendieck in [Gr1], which we now recall. To a manifold, one can associate an ∞-category whose objects are points of the manifold and whose morphisms are paths between points. The homotopy hypothesis asserts that this association is fully faithful: the space of maps between manifolds is homotopy equivalent to the space of functors between associated ∞-categories, whatever model for ∞-categories one selects. The ∞-categories this association produces are ∞-groupoids: every morphism is an equivalence.

This association of an ∞-groupoid to a manifold has a stratified generalization: the exit-path ∞-category of Lurie [Lu2]. As proposed first by MacPherson, from a stratified space X one can define an entity $\mathsf{Exit}(X)$ whose objects are points of the space and whose morphisms are those paths whose direction is restricted by the stratification: the paths can exit a stratum into a less deep stratum, but they cannot return. We prove that the space of conically smooth maps between stratified spaces is homotopy equivalent to the space of functors between their exit-path ∞-categories. That is, the exit-path functor

$$\mathcal{S}\mathsf{trat} \xhookrightarrow{\mathsf{Exit}} \mathsf{Cat}_\infty$$

is a fully faithful embedding of stratified spaces into ∞-categories. This is our articulation of a stratified homotopy hypothesis. The ∞-categories produced by this association have the property that each endomorphism of an object is an equivalence.

The proof of this result follows from a formulation of dévissage of stratified structures, like that alluded to by Grothendieck in [Gr2]. In essence, ours states that one can understand stratified spaces by a combined understanding of cones, blow-ups, manifolds with corners, and induction on depth. Further, the same is true in families by the strong regularity properties afforded by conical smoothness.

As a consequence, one can view an ∞-category as a pre-stack on stratified spaces: given an ∞-category $\mathcal{C}$ and a stratified space K, a K-point of $\mathcal{C}$ is a functor

$$\mathsf{Exit}(K) \longrightarrow \mathcal{C}$$

from the exit-path ∞-category of K to $\mathcal{C}$. One can then ask what descent properties this pre-stack enjoys. The following is a list of locality properties that a space-valued presheaf on stratified spaces might or might not possess; we make these precise in §4 (see Definition 4.1.1, amplified by Remark 4.1.2).

- **Sheaf:** the presheaf satisfies descent for open covers.

- **Constructible:** the sheaf is locally constant on each stratum of each stratified space.
- **Cone-local:** the constructible sheaf satisfies descent for blow-ups along deepest strata.
- **Consecutive:** the cone-local constructible sheaf has values on iterated cones determined by values on single cones.
- **Univalent:** the consecutive cone-local constructible sheaf has *underlying* locally constant sheaf which agrees with its *maximal sub*-locally constant sheaf.

We define the collection of striation sheaves, $\mathcal{S}\mathsf{tri}$, to consist of those presheaves on stratified spaces which satisfies all of these conditions. The following is the first main theorem of this paper.

Theorem 0.0.1. *There is an equivalence*

$$\mathcal{S}\mathsf{tri} \ \simeq \ \mathsf{Cat}_\infty$$

between striation sheaves and ∞-categories. This equivalence sends an ∞-category $\mathcal{C}$ to the presheaf on stratified spaces taking values

$$K \mapsto \mathsf{Map}(\mathsf{Exit}(K), \mathcal{C})$$

where $\mathsf{Exit}(K)$ is the exit-path ∞-category of the stratified space K.

Striation sheaves offer a workable model for ∞-category theory with appealing theoretical features – for instance, the $\mathbb{Z}/2$ action on Cat_∞ sending $\mathcal{C} \mapsto \mathcal{C}^{\mathsf{op}}$ acts on $\mathcal{S}\mathsf{tri}$ by Poincaré duality of stratifications (see Remark 0.0.4). However, we do not introduce striation sheaves in order to compete with quasi-categories as a foundational setting of choice. Rather, we use them to make interesting ∞-categories by hand from geometry.

The following perspective informs this use. Quasi-categories make for such an economical model for ∞-category theory in part by allowing for coherently, rather than strictly, associative compositions of morphisms. It is technically useful to allow for coherence, but it is technically difficult to specify it. As a result, quasi-categories of interest are rarely arrived by concretely specifiying a simplicial set and verifying the inner horn filling condition. Rather, they typically result from a sequence of three steps. First, one writes down a topological, simplicial, or otherwise enriched category concretely. Second, one converts it into a quasi-category via a nerve construction. Third, one performs a formal ∞-categorical maneuver (such as taking limits/colimits, presheaves, Dwyer-Kan localization, &c) to effect a desired end result.

We posit that there are profound examples of ∞-categories which are impracticable to produce by this sequence, and we advance striation sheaves as a conduit through which manifold topology can form such profound examples. In the present work we construct several: the ∞-category $\mathcal{B}\mathsf{un}$ classifying constructible bundles, and an absolute exit-path ∞-category $\mathcal{E}\mathsf{xit}$. These build toward a third example, the earlier mentioned moduli space of stratifications on a manifold, which is the subject of a future work.

First, we give a construction of the exit-path ∞-category $\mathsf{Exit}(X)$ of a stratified space X as a striation sheaf. It is the presheaf represented by X itself. The exit-path quasi-category is one of the few quasi-categories which is actually written down by hand, and the verification of inner horn filling condition by repeated subdivisions is quite involved (see Appendix A of [Lu2]). Our verification that $\mathsf{Exit}(X)$ is a striation sheaf is perhaps easier because $\mathsf{Exit}(X)$ is so geometric. For locality in geometry, covers are often more workable than subdivisions. Specifically, one can take open covers of simplices. Such a maneuver lives outside the world of quasi-categories and simplicial sets, but within that of striation sheaves.

Our foremost example of interest ensues from the geometry of *constructible bundles*, a notion we introduce in this work. A constructible bundle $X \to K$ is a conically smooth map of stratified spaces whose restriction to each stratum $X_{|K_q} \to K_q$ is a stratified fiber bundle over an ordinary smooth manifold. We now describe the ∞-category, $\mathcal{B}\mathsf{un}$, which results from this notion. The objects are stratified spaces. A morphism in $\mathcal{B}\mathsf{un}$ between between two stratified spaces, $X_0 \to X_1$, consists of: a third stratified space X; a constructible bundle $f : X \to [0,1]$ to the stratified interval whose stratification is given by $\{0\} \subset [0,1]$; identifications of the fibers $X_0 \cong X_{|\{0\}}$ and $X_1 \cong X_{|\{1\}}$. The following commutative diagram depicts a morphism from X_0 to X_1:

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "f"', "\mathsf{cbl}"] & X_1 \arrow[l, hook'] \arrow[d] \\ \{0\} \arrow[r] & {[0,1]} & \{1\}. \arrow[l] \end{tikzcd}$$

Here cbl denotes the condition of f being constructible. This implies, in particular, that the restriction $X_{|(0,1]}$ is isomorphic to the product bundle $X_1 \times (0,1]$.

One comes to grips with the intrinsic ∞-categorical nature of $\mathcal{B}\mathsf{un}$ in attempting to compose morphisms. Given two constructible bundles $X \to [0,1]$ and $Y \to [1,2]$ with identifications $X_1 \cong Y_1$, one would like to glue the two intervals end-to-end and form a space $X \cup Y \to [0,2]$ representing a morphism in $\mathcal{B}\mathsf{un}$ from X_0 to Y_2. However, the restriction $X \cup Y_{|(0,2]}$ need not be equivalent to the product bundle $Y_2 \times (0,2]$. To solve this composition problem requires deforming $X \cup Y$ into a stratified space mapping constructibly to the stratified interval $(\{0\} \subset [0,2])$. This deformation constitutes a resolution of singularites, retracting certain floating strata to $\{0\}$. Here, one confronts an inner horn-filling problem as a fundamental feature of the geometry of constructible bundles.

Contemplating this problem might cement two notions. First, $\mathcal{B}\mathsf{un}$ does not admit any obvious manifestation in a model of ∞-categories with strictly associative compositions, such as topological categories. Second, in verifying this horn-filling condition one would want to use open covers of, for instance, the topological simplex Δ^2 as a topological space, rather than subdivisions of the simplicial set $\Delta[2]$. Our theory of striation sheaves is tailored to such situations.

One can regard $\mathcal{B}\mathsf{un}$ in a natural way as a presheaf on stratified spaces, where the value on stratified space

$$\mathcal{B}\mathsf{un}(K)$$

consists of all constructible bundles over K. Viewed in this way, the previous horn-filling condition becomes the consecutivity axiom for striation sheaves. The second main theorem of our work is the following.

Theorem 0.0.2. $\mathcal{B}\mathsf{un}$ *is a striation sheaf.*

As such, we regard $\mathcal{B}\mathsf{un}$ as an ∞-category by way of Theorem 0.0.1. A K-point of this ∞-category, namely, a functor

$$\mathsf{Exit}(K) \longrightarrow \mathcal{B}\mathsf{un} ,$$

exactly consists of a constructible bundle $X \to K$. The ∞-category $\mathcal{B}\mathsf{un}$ encodes a great deal. From it one can extract spaces of conically smooth open embeddings, the models of moduli spaces of manifolds of Hatcher [Ha] and Waldhausen [Wa], and spaces of stratifications on manifolds; combinatorially, one can extract the category of based finite sets and the simplicial category $\mathbf{\Delta}$.

Our first two examples of striation sheaves, $\mathcal{B}\mathsf{un}$ and exit-path ∞-categories, are related to one another. $\mathcal{B}\mathsf{un}$ parametrizes an absolute version of the exit-path ∞-category $\mathcal{E}\mathsf{xit}$, which we now describe. An object of $\mathcal{E}\mathsf{xit}$ is a stratified space with a point. A morphism between two pointed stratified spaces X_0 and X_1 is a third stratified space X with a constructible bundle $X \to [0,1]$, identifications of the fibers with X_0 and X_1, together with the additional datum of a stratified section $[0,1] \to X$ starting from the distinguished point of X_0 and ending at the distinguished point of X_1. This is depicted in the following diagram:

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "\mathsf{cbl}"] & X_1 \arrow[l, hook'] \arrow[d] \\ \{0\} \arrow[r] \arrow[u, bend left] & {[0,1]} \arrow[u, bend left] & \{1\} . \arrow[l] \arrow[u, bend right] \end{tikzcd}$$

We solve the existence of compositions in $\mathcal{E}\mathsf{xit}$ in the same way as for $\mathcal{B}\mathsf{un}$, by realizing $\mathcal{E}\mathsf{xit}$ as a presheaf on stratified spaces. This description is likewise very natural: an object of $\mathcal{E}\mathsf{xit}(K)$ is a constructible bundle to K together with a stratified section, $X \rightleftarrows K$. The following corollary of Theorem 0.0.2 realizes $\mathcal{E}\mathsf{xit}$ as an absolute exit-path ∞-category, tying together the three principal examples of striation sheaves introduced in this work.

Corollary 0.0.3. *$\mathcal{E}\mathsf{xit}$ is a striation sheaf. For a K-point of $\mathcal{B}\mathsf{un}$ defined by a constructible bundle $X \to K$, there is a resulting commutative diagram*

$$\begin{tikzcd}
\mathsf{Exit}(X) \arrow[r, "(X \underset{K}{\times} X \rightleftarrows X)"] \arrow[d] & \mathcal{E}\mathsf{xit} \arrow[d] \\
\mathsf{Exit}(K) \arrow[r, "(X \xrightarrow{\mathsf{cbl}} K)"] & \mathcal{B}\mathsf{un}
\end{tikzcd}$$

which is a limit diagram of ∞-categories.

In a sequel to this work, we use the theory of striation sheaves and the previous three examples to construct the aforementioned moduli space of stratifications. We postpone to that work any justification of this indeed being a beneficial approach to manifold topology.

We make several remarks before concluding this portion of the introduction.

Remark 0.0.4. The identification $\mathsf{Cat}_\infty \simeq \mathcal{S}\mathsf{tri}$ is not canonical. The space of such identifications is a torsor for $\mathsf{Aut}(\mathsf{Cat}_\infty) \simeq \mathbb{Z}/2$, with the opposite identification implemented by considering enter-paths, as opposed to exit-paths. Consequently, this description of ∞-categories as sheaves on Strat is balanced between opposites (like a one-dimensional real vector space without a preferred generator). Our equivalence $\mathsf{Cat}_\infty \simeq \mathcal{S}\mathsf{tri}$ determines a $\mathbb{Z}/2$-action on $\mathcal{S}\mathsf{tri}$. This action is not obvious, because it is not inherited from an action on Strat. Nevertheless, there are some $\mathbb{Z}/2$-orbits in $\mathcal{S}\mathsf{tri}$ consisting entirely of representables; these are the stratified spaces for which there is a Poincaré dual stratified space. An example of such is a polygonized smooth closed manifold of dimension at least 1; a non-example is an unstratified closed manifold. The involution exchanges Poincaré duals, and so one can regard this involution on $\mathcal{S}\mathsf{tri}$ as a form of Poincaré duality.

Remark 0.0.5. After Toën in [To], there is a canonical equivalence between any two ∞-categories of ∞-categories up to taking opposites. Consequently, we will typically not use notation to distinguish between an ∞-category which has first been constructed in a particular model, such as simplicial categories, and the associated ∞-category in a different model, such as quasi-categories. For instance, $\mathcal{S}\mathsf{trat}$ will stand for both a simplicial category and an ∞-category. When we make an argument using specific point-set features of a particular model (e.g., by using Kan fibrations of simplicial categories), we will use the name of that model. When we make a argument which works in every model for ∞-category theory, then we avoid use of a particular term and simply refer to ∞-categories.

Remark 0.0.6. We work throughout internal to ∞-category theory. For instance, after [To] and [BS], we employ Rezk's complete Segal spaces as an internal construction of the ∞-category of ∞-categories, rather than as a model category. As such, we do not construct a model structure for striation sheaves or transversality sheaves, it not being necessary to do so for our larger purpose. Nevertheless, such model structures could be beneficial. The interested reader could take on this question, constructing model structures for striation sheaves and transversality sheaves parallel to the complete Segal space and quasi-category model structures.

Notation 0.0.7. In the following, we encounter a number of examples of collections of topological objects which form both a discrete category of interest, such as the category Strat of stratified spaces, and a quite different higher category of interest, such the ∞-category $\mathcal{S}\mathsf{trat}$ of stratified spaces. In such cases, we will often use calligraphic typeface for the first letter of the higher category. In cases which will never be ambiguous we will stay with the plain sans serifed type (such as for the ∞-category Cat_∞ of ∞-categories).

In what remains of this introduction, we relate the present results and approach to previous work and then give a more technical linear overview of the sections of this work and the results therein.

Relation to previous works. The theory of *conically smooth* stratified spaces, introduced in [AFT] and further developed throughout this work, is designed to carry both the geometric features of the classical theory of stratifications, after Whitney and Thom-Mather, as well as have robust behavior in families, after the homotopy-theoretic stratifications of Siebenmann and Quinn. We detail these connections.

The geometric study of stratifications dates to the seminal works of Whitney, Thom, and Mather ([Wh1], [Wh2], [Wh3], [Th], and [Ma]), introduced toward the study of singular algebraic varieties and of dynamics of smooth maps. The theory of homotopy-theoretic stratifications was advanced by Siebenmann [Si] and Quinn [Qu2]. Object-wise, the geometric theory enjoys fine geometric features, such as the openness of transversality due to Trotman [Tro]. Conversely, the latter topological theories enjoy robust homotopy theoretic features: Siebenmann introduced the study of spaces of stratified maps between his locally-cone stratified sets, and showed this theory has well-behaved moduli. In particular, he proved these spaces of stratified maps are locally contractible, which can be interpreted as an isotopy extension theorem and ultimately accommodates the existence of classifying spaces for fiber bundles. In contrast, Whitney stratified spaces have not been studied in families, because the naive notion of a map of Whitney stratified spaces, one which is smooth on strata separately, leads to pathologies. Consequently, Siebenmann's results have no historical counterpart in the geometric theory after Whitney. However, the homotopy-theoretic stratifications are insufficent for more geometry. Nonexistence of tubular neighborhoods (see [RS]), absence of transversality, and difficulty in establishing the existence of pullbacks all hinder the coarser topological theory.

The notion of stratified spaces studied in this work modifies the established definitions of Whitney-Thom-Mather by adding a requirement of conically smoothness for stratifications and maps thereof. This notion is intrinsic and makes no recourse to an ambient smooth manifold. Conical smoothness ensures strong regularity properties along closed strata in a stratified space, so that there exist tubular neighborhoods along singularity locii. This regularity implies the Whitney conditions as well as many of the differing excellent properties on both the geometric and homotopy-theoretic sides. On the geometric side, Trotman's openness of transversality can be deduced from the stratified inverse function theorem of [AFT]. On the topological side, Siebenmann's stratified isotopy extension theorem follows in our theory by standard arguments on tubular neighborhoods.

Conical smoothness also cures pathologies in the geometric and homotopy-theoretic theories. It does so for a unified reason: on both sides these pathologies stem from pseudo-isotopy theory, and conical smoothness evades pseudo-isotopy. For instance, in Quinn's theory of homotopically stratified sets, there exist obstructions to the existence of tubular neighborhoods; counterexamples using pseudo-isotopy theory are constructed in [HTWW]. Hughes proves an *approximate* tubular neighborhood theorem, valid only in higher dimensions, in [Hu]; his result uses the h-cobordism theorem, which is again an application of pseudo-isotopy theory. Likewise, the same methods of [HTWW] point out pathologies in the naive notion of families of Whitney stratified spaces: a naive bundle of Whitney stratified spaces need not itself be Whitney stratified. Examples are given by cones parametrized over a circle; unless the associated pseudo-isotopy is a diffeomorphism, the link of the cone-locus of the total space lacks a smooth structure. Conical smoothness averts pseudo-isotopic difficulties because of the identification, from [AFT], between the homotopy types of the space of conically smooth automorphisms of a cone $\mathsf{C}(X)$ and the conically smooth automorphisms of the compact stratified space X which is the link of the cone-point. Without the conical smoothness, automorphisms of a cone is a pseudo-isotopy space for X, and consequently one can form pathological stratifications by gluing along pseudo-isotopies rather than along automorphisms. The authors of [HTWW] state that their work concerns the glue that holds together stratified spaces; a conclusion of their work is that pseudo-isotopy permeates this glue. With our methods, one can glue without pseudo-isotopy by requiring conical smoothness; one applies the glue in layers rather than all at once.

A main avatar of this work, the exit-path ∞-category, was previously studied in depth by Lurie in [Lu2]. This notion of exit-paths was first proposed by MacPherson, unpublished, and developed 2-categorically by Treumann in [Tre]; Treumann's approach was pushed further by Woolf [Wo]. We use our enriched category $\mathcal{S}\mathsf{trat}$ of stratified spaces and conically smooth maps to give a construction of the exit-path ∞-category of a stratified space as a complete Segal space; Lurie's construction is a quasi-category. Our proof of the fully faithfulness of the functor $\mathsf{Exit}: \mathcal{S}\mathsf{trat} \to \mathsf{Cat}_\infty$ implies, in particular, that the functor is conservative. As such, it implies a detection criterion for stratified homotopy equivalences: they are stratified maps which induce homotopy equivalences on all strata and all links of strata. This recovers a result first proved by Miller [Mi].

Our study of the descent properties of stacks on stratified spaces represented by ∞-categories bears a close analogy with Morel-Voevodsky's motivic homotopy theory ([MV]). We show that our enriched category $\mathcal{S}\mathsf{trat}$ is the $\mathbb{R}$-localization of the ordinary category of stratified spaces Strat, in clear analogy with $\mathbb{A}^1$-localization. We further prove descent for blow-ups, a topological analogue of descent for the h-topology in algebraic geometry (see [SV]). The univalence property of a striation sheaf corresponds to Rezk's completeness condition, as formulated in [Re2], and Voevodsky's univalence axiom. Our construction of striation sheaves from transversality sheaves via the topologizing diagram is a topological analogue of Joyal-Tierney's equivalence between complete Segal spaces and quasi-categories in [JT], using Rezk's classifying diagram ([Re1]). Our topologizing diagram is inspired by and generalizes the method of Hatcher and Waldhausen for constructing moduli spaces of smooth manifolds in [Ha] and [Wa]. To establish all these results makes use of a uniform system for decomposing stratified spaces in terms of links, blow-ups, and induction on depth of strata, which we conceive of as a dévissage for stratified structures like that continually mentioned by Grothendieck in his Esquisse d'un Programme [Gr2].

Lastly, our notion of a constructible bundle is a geometric refinement of the stratified systems of fibrations introduced by Quinn in [Qu1] for the study of h-cobordisms in families. (The constructible bundle is a further refinement of this notion, as there exists an additional stratification on the total space.) In Quinn's theory, strata have regular neighborhoods which are open mapping cylinders; it is shown that demanding such for each pair of strata grants as much for links of links. This composability feature foreshadows the consecutivity of $\mathcal{B}\mathsf{un}$, the paramount technical property which allows for composing morphisms in $\mathcal{B}\mathsf{un}$ and so used to prove that it indeed forms an ∞-category. Lastly, the verification of univalence for $\mathcal{B}\mathsf{un}$ is a stratified generalization of the homotopy equivalence $\mathsf{Diff}(M,M) \simeq \mathsf{Emb}^{\sim}(M,M)$ between diffeomorphisms and self-embeddings of a manifold isotopic to a diffeomorphism.

Linear overview.

Section 1 recapitulates the fundamental features of conically smooth stratified spaces, including links and the unzipping construction, as developed in [AFT]. Constructible closed stratified subspaces have regular neighborhoods; we make essential use of this technical feature throughout.

Section 2 studies $\mathcal{S}\mathsf{trat}$, a simplicial enrichment of the discrete category of stratified spaces Strat. The construction of $\mathcal{S}\mathsf{trat}$ is an application of the *topologizing diagram*, which produces space-valued presheaves from groupoid-valued presheaves on stratified spaces. We introduce isotopy sheaves as a class of groupoid-valued presheaves which become constructible sheaves after applying the topologizing diagram. From this, we deduce that $\mathcal{S}\mathsf{trat}$ is the Dwyer-Kan localization of Strat with respect to stratified homotopy equivalences. We proceed to study homotopy colimits in $\mathcal{S}\mathsf{trat}$. The key technical input is a description of the homotopy type of spaces of conically smooth maps between cones. Using this, we identify the following distinguished classes of homotopy colimit diagrams: open covering sieves, blow-ups for deepest strata; and double cone gluing diagrams.

Section 3 concerns the interaction of $\mathcal{S}\mathsf{trat}$ with Rezk's theory of complete Segal spaces. We show that the standard stratification of the n-simplices defines a fully faithful functor

$$\mathbf{\Delta} \overset{\mathsf{st}}{\hookrightarrow} \mathcal{S}\mathsf{trat}$$

from the simplicial indexing category to the simplicial category of stratified spaces; this sends $[p]$ to the topological p-simplex Δ^p equipped with the standard stratification. We define the simplicial space $\mathsf{Exit}(X)$ as the composite,

$$\mathsf{Exit}(X)\colon \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{st}} \mathcal{S}\mathsf{trat}^{\mathsf{op}} \xrightarrow{\mathcal{S}\mathsf{trat}(-,X)} \mathcal{S}\mathsf{paces}~,$$

given by restricting to $\mathbf{\Delta}^{\mathsf{op}}$ the representable presheaf X. Using the analysis of homotopy colimits in $\mathcal{S}\mathsf{trat}$ from §2, we prove that $\mathsf{Exit}(X)$ is a complete Segal space, hence an ∞-category. The equivalence between quasi-categories and complete Segal spaces exchanges $\mathsf{Exit}(X)$ with Lurie's exit-path quasi-category of X. We show that the resulting functor $\mathsf{Exit} : \mathcal{S}\mathsf{trat} \to \mathsf{Cat}_\infty$ preserves a distinguished classes of colimit diagrams.

Section 4 introduces striation sheaves. These are constructible sheaves on stratified spaces that send the distinguished classes of colimit diagrams to limit diagrams in $\mathcal{S}\mathsf{paces}$. We prove that the ∞-category of simplicial spaces $\mathsf{PShv}(\mathbf{\Delta})$ is equivalent to the ∞-category $\mathsf{Shv}^{\mathsf{cone,cbl}}(\mathsf{Strat})$ of those constructible sheaves on stratified spaces which satisfy descent for blow-ups. Using this, we prove the further equivalent equivalence between the ∞-category of striation sheaves $\mathfrak{S}\mathsf{tri}$ and the ∞-category of ∞-categories Cat_∞. These proofs uses all of our analysis of homotopy colimits of stratified spaces to show that $\mathcal{S}\mathsf{trat}$ is generated under homotopy colimits of distinguished diagrams by the three element ∞-subcategory $\{\emptyset, *, \Delta^1\}$ consisting of the empty set, a point, and the standardly stratified 1-simplex. As a corollary, we deduce that the exit-path functor $\mathsf{Exit} : \mathcal{S}\mathsf{trat} \to \mathsf{Cat}_\infty$ is fully faithful:

$$\mathcal{S}\mathsf{trat}(X, Y) \ \simeq\ \mathsf{Map}\big(\mathsf{Exit}(X), \mathsf{Exit}(Y)\big)~.$$

Section 5 introduces transversality sheaves as an efficient point-set means for constructing striation sheaves. We prove that the topologizing diagram sends transversality sheaves to striation sheaves; this elaborates on the construction of constructible sheaves from isotopy sheaves from §2. For transversality sheaves, the topologizing diagram agrees with the classifying diagram of Rezk. This connects our passage between transversality sheaves and striation sheaves to that between quasi-categories and complete Segal spaces of Joyal-Tierney [JT].

Section 6 constructs Bun and Exit using the notion of a constructible bundle of stratified spaces. Toward this study, we establish a number of basic crucial properties of constructible bundles of stratified spaces. By extensive use of the results and techniques of [AFT] we prove: that constructible bundles pull back; that constructible bundles compose; that constructible bundles can be recognized stratum locally in the source. These lemmas are used to prove that Bun and Exit are transversality sheaves, and thus that their topologizing diagrams $\mathcal{B}\mathsf{un}$ and $\mathcal{E}\mathsf{xit}$ form striation sheaves, hence ∞-categories. The most technically involved verification is the consecutivity condition, which relies on a classification of isomorphism classes of constructible bundles over a cone. In doing so, we prove a conceptually appealing description of morphisms in $\mathcal{B}\mathsf{un}$. Namely, there is a surjection from the set of isomorphism classes of spans

$$X_0 \xleftarrow{\mathsf{p.cbl}} L \xrightarrow{\mathsf{open}} X_1$$

to the set of isomorphism classes of constructible bundles $X \to \Delta^1$, and thereafter a surjection to $\pi_0\,\mathcal{B}\mathsf{un}(\Delta^1)$. As such, $\mathcal{B}\mathsf{un}$ receives an essentially surjective functor from a manner of Burnside ∞-category formed from the classes of proper constructible and open maps in $\mathcal{S}\mathsf{trat}$.

We single out special ∞-subcategories of $\mathcal{B}\mathsf{un}$, consisting of closed, creation, refinement, and embedding morphisms. We exhibit a factorization system on $\mathcal{B}\mathsf{un}$ in terms of these morphisms. There are two cylinder functors, the open cylinder Cylo and the reversed cylinder Cylr, which define monomorphisms $\mathcal{S}\mathsf{trat}^{\mathsf{open}} \hookrightarrow \mathcal{B}\mathsf{un}$ and $(\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}})^{\mathsf{op}} \hookrightarrow \mathcal{B}\mathsf{un}$ from the ∞-categories of stratified spaces with open maps and from the ∞-category of stratified spaces with proper constructible bundles. Lastly, we identify the distinguished ∞-subcategories of $\mathcal{B}\mathsf{un}$ in terms of these cylinder functors.

- Embeddings: $\mathcal{B}\mathsf{un}^{\mathsf{emb}}$ is equivalent to the ∞-subcategory of $\mathcal{S}\mathsf{trat}^{\mathsf{open}}$ consisting of those maps which are stratified open embeddings.
- Refinements: $\mathcal{B}\mathsf{un}^{\mathsf{ref}}$ is equivalent to the ∞-subcategory of $\mathcal{S}\mathsf{trat}^{\mathsf{open}}$ consisting of those maps which are homeomorphisms of underlying topological spaces.
- Closed morphisms: $\mathcal{B}\mathsf{un}^{\mathsf{cls}}$ is contravariantly equivalent to the ∞-subcategory of $\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}}$ consisting of those proper constructible bundles which are injective.
- Creation morphisms: $\mathcal{B}\mathsf{un}^{\mathsf{creat}}$ is contravariantly equivalent to the ∞-subcategory of $\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}}$ consisting of those proper constructible bundles which are surjective.

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1. Stratified spaces

We recall some relevant notions among stratified spaces as developed in [AFT]. We emphasize that this section is very much an overview, and we point the unfamiliar reader to that reference for precise definitions and details.

1.1. **Stratified spaces.** A *stratified space* is a paracompact Hausdorff topological space X, equipped with continuous map $X \to P$ to a poset (with the poset topology), that is equipped with a *conically smooth* atlas

$$\Big\{\mathbb{R}^{i_\alpha} \times \mathsf{C}(Z_\alpha) \hookrightarrow X\Big\}_\alpha$$

by *basics*. Let us divulge some of these terms. Typically, we denote a stratified space simply with its *underlying topological space* X, or with its *underlying stratified topological space* $X = (X \to P)$ when the stratification requires notation; and for each subset $Q \subset P$ we denote the preimage $X_Q \subset X$, which we refer to as the *Q-stratum (of X)*. Each letter Z is understood as another stratified space, which we require to be compact. For $Z \to P$ a stratified space, there is its *open* cone and its *closed* cone

$$\mathsf{C}(Z) := \ast \coprod_{Z\times\{0\}} Z\times[0,1) \quad \underset{\text{open}}{\subset} \quad \overline{\mathsf{C}}(Z) := \ast \coprod_{Z\times\{0\}} Z\times[0,1] \quad \longrightarrow \quad P^{\triangleleft} := \ast \underset{P\times\{0\}}{\amalg} P\times\{0<1\}$$

each of which is stratified by the corresponding push out of posets. So the $\ast$-stratum is precisely the cone-point $\ast = \mathsf{C}(Z)_\ast = \overline{\mathsf{C}}(Z)_\ast$. A *basic* is a stratified space of the form $\mathbb{R}^i \times \mathsf{C}(Z) = \big(\mathbb{R}^i\times\mathsf{C}(Z) \to \mathsf{C}(Z) \to P^{\triangleleft}\big)$ with $Z = (Z \to P)$ a compact stratified space. Each basic has a *cone-locus* as well as an *origin*:

$$\mathbb{R}^i \subset \mathbb{R}^i\times\mathsf{C}(Z) \qquad \text{and} \qquad 0 \in \mathbb{R}^i \subset \mathbb{R}^i \times \mathsf{C}(Z)~.$$

A map between basics $f\colon \mathbb{R}^i\times\mathsf{C}(Y) \to \mathbb{R}^j\times\mathsf{C}(Z)$ is *conically smooth* if either the image of the source cone-locus does not intersect the target cone-locus and does so conically smoothly, or the map carries the cone-locus to the cone-locus and, for each $(p,s,z)\in\mathbb{R}^i\times\mathsf{C}(Y)$, and each $v\in\mathbb{R}^i$, the limit

$$\lim_{t\to 0}\frac{f(p+tv,ts,z)}{t} \quad \in \quad \mathbb{R}^j\times\mathsf{C}(Z) \tag{1}$$

exists and is again *conically smooth* in the arguments (p,s,z) and v. (While this notion seems circular, the topological dimension of Z is necessarily strictly less than that of X, so one can induct on dimension to establish a true definition.) An *atlas* is a jointly surjective collection of open embeddings from basics into X in which each transition map is conically smooth, and each cone-locus $\mathbb{R}^i\subset\mathbb{R}^i\times\mathsf{C}(Z)\hookrightarrow X\to P$ is indexed by a single element of P. Provided an atlas exists on a stratified topological space, it is contained in a unique *maximal* atlas. For a general continuous map $f\colon X\to Y$ between stratified topological spaces, we say f is *conically smooth* if it is locally conically smooth with respect to a choice of charts.

We distinguish the following important classes of conically smooth maps:

- **Embedding:** f is an *embedding* if f is an isomorphism onto its image.
- **Open embedding:** f is an *open embedding* if it is open as well as an embedding.

- **Refinement:** We say f is a *refinement* if it is a homeomorphism of underlying topological spaces, and, for each stratum $X_p \subset X$, the restriction $f_| \colon X_p \to Y$ is an embedding.
- **Open:** f is *open* if it is an open embedding on underlying topological spaces, and f is a refinement onto its image.
- **Proper:** f is *proper* if $f^{-1}C \subset X$ is compact for each compact subspace $C \subset Y$.
- **Fiber bundle:** f is a *fiber bundle* if the collection of images $\phi(O) \subset Y$, indexed by pullback diagrams

$$\begin{tikzcd} F \times O \arrow[r] \arrow[d] & X \arrow[d] \\ O \arrow[r, "\phi"] & Y \end{tikzcd}$$

in which the horizontal maps are open embeddings, is a basis for the topology of Y.
- **Submersion:** f is a *submersion* if the collection of images $\psi(U \times O) \subset X$, indexed by diagrams

$$\begin{tikzcd} U \times O \arrow[r, "\psi"] \arrow[d] & X \arrow[d] \\ O \arrow[r] & Y \end{tikzcd}$$

in which the horizontal maps are open embeddings, is a basis for the topology of X.
- **Constructible:** f is a *constructible bundle* if, for each stratum $Y_q \subset Y$, the restriction $f_| \colon f^{-1}Y_q \to Y_q$ is a fiber bundle.

Conically smooth maps compose, as do each of the named classes of maps. (See Proposition 6.1.7 for the **constructible** case.) This yields the following variety of subcategories

$$\mathsf{Strat}^{\mathsf{cbl}} \subset \mathsf{Strat} \supset \mathsf{Strat}^{\mathsf{open}} \supset \mathsf{Strat}^{\mathsf{emb}} , \mathsf{Strat}^{\mathsf{ref}}$$

where the superscripts indicate an aforementioned class of maps, except emb which signifies *open* embeddings. The category Strat admits finite products.

As with manifolds with boundary, there are stratified spaces with boundary. A stratified space *with boundary* is a stratified space $\overline{X}$ together with a sub-stratified space $\partial\overline{X} \subset \overline{X}$ for which there is a conically smooth open embedding $\partial\overline{X} \times \mathbb{R}_{\geq 0} \hookrightarrow \overline{X}$ where here $\mathbb{R}_{\geq 0}$ is the open cone on a point. The existence of collar-neighborhoods verifies that smooth manifolds with boundary are examples of stratified spaces with boundary. Note that, unlike the situation in classical manifold theory, a stratified space might have many, or no, *boundary* structures.

Example 1.1.1. The closed cone on a compact stratified space $\overline{\mathsf{C}}(L)$ is naturally a stratified space with boundary, as seen by the inclusion $L \times \mathbb{R}_{\geq 0} \cong L \times (0,1] \hookrightarrow \overline{\mathsf{C}}(L)$.

For $\partial\overline{X} \times \mathbb{R}_{\geq 0} \hookrightarrow \overline{X}$ a stratified space with boundary, we refer to $\partial\overline{X}$ as the *boundary* of $\overline{X}$, and $\mathsf{int}(\overline{X}) := \overline{X} \smallsetminus \partial\overline{X}$ as the *interior.* We use the notation $\mathsf{Strat}^{\partial}$ for the category for which an object is a stratified space with boundary, and for which a morphism from $(\partial\overline{X} \subset \overline{X})$ to $(\partial\overline{Y} \subset \overline{Y})$ is a conically smooth map $\overline{X} \to \overline{Y}$ (so the morphisms do not make reference to the boundary structure). There is an apparent forgetful functor $\mathsf{Strat}^{\partial} \to \mathsf{Strat}$, $(\partial\overline{X} \subset \overline{X}) \mapsto \overline{X}$. Taking interiors gives a functor

$$\mathsf{int} \colon \mathsf{Strat}^{\partial} \longrightarrow \mathsf{Strat} ,$$

and there is a natural transformation from int to the forgetful functor. A stratified space X is *finitary* if it is the interior of a compact stratified space with boundary.

1.2. **Links and regular neighborhoods.** For L a compact stratified space, there is a pushout diagram in Strat

$$\begin{tikzcd} L \arrow[r, "\{0\}"] \arrow[d] & L \times [0,1) \arrow[d] \\ \ast \arrow[r] & \mathsf{C}(L) \end{tikzcd} \tag{2}$$

There is a likewise pushout diagram upon applying $\mathbb{R}^i \times -$, thereby witnessing each basic as a pushout. Say a stratum $X_0 \subset X$ is a *deepest* stratum if $X_0 \subset X$ is closed as a subspace. By design, the conical smoothness of an atlas for X yields, for each deepest stratum $X_0 \subset X$, a pushout diagram in Strat

$$\begin{tikzcd} \mathsf{Link}_{X_0}(X) \arrow[r] \arrow[d] & \mathsf{Unzip}_{X_0}(X) \arrow[d] \\ X_0 \arrow[r] & X \end{tikzcd}$$

which restricts over each basic neighborhood $\mathbb{R}^i \times \mathsf{C}(L) \hookrightarrow X$ of a point $x \in X_0$ as the square (2). By iterating this construction, we have a likewise pushout diagram for $X_0 \subset X$ not just a single stratum, but a constructible closed sub-stratified space of X. We refer to such a pushout square as a *blow-up square (along $X_0 \subset X$)*; and we refer to the upper left stratified space as the *link (of X_0 in X)*, and the upper right stratified space as the *unzip (of X along X_0)*. Because it is the case locally, notice that each map in such a blow-up square is proper and constructible.

The existence of a collar-neighborhood of each face of a smooth manifold with corners ultimately grants the existence of a conically smooth tubular neighborhood of $X_0 \subset X$:

$$X_0 \subset \mathsf{C}(\pi) \xhookrightarrow{\mathsf{open\ emb}} X$$

where $\mathsf{C}(\pi) \to X_0$ is the fiberwise open cone on the link $\mathsf{Link}_{X_0}(X) \xrightarrow{\pi} X$. This map from the fiberwise cone is an open embedding of stratified spaces. Thereafter this gives the existence of a conically smooth regular neighborhood of each stratum $X_p \subset X$ of an arbitrary stratified space. Consequently, for each stratified space $X = (X \to P)$ and each downward closed subset $Q \subset P$, there exists a conically smooth map

$$X_Q \subset \mathsf{C}(\pi_Q) \xhookrightarrow{\mathsf{open}} X$$

where $\mathsf{Link}_{X_Q}(X) \xrightarrow{\pi_Q} X_Q$ is the projection from the link, and $\mathsf{C}(\pi_Q)$ is the fiberwise open cone. This map from the fiberwise cone is a refinement onto its image, which is an open subspace of X. In particular, the existence of conically smooth bump functions gives the existence of a conically smooth map $X \to \mathbb{R}_{\geq 0}$ extending the projection $\mathsf{C}(\pi_Q) \to [0,1) \hookrightarrow \mathbb{R}_{\geq 0}$ followed by the standard inclusion.

This discussion prompts the following notion.

Definition 1.2.1 (Weakly regular subspace)**.** A subspace $K_0 \subset K$ of a stratified space is *weakly regular* if there is a refinement $\widetilde{K} \to K$ with respect to which the subspace K_0 is the image of a proper constructible embedding $\widetilde{K}_0 \to \widetilde{K}$.

Example 1.2.2. A finite union of affine planes $\bigcup_{i \in I} V_i \subset \mathbb{R}^n$ is a weakly regular subspace.

Example 1.2.3. Consider a basic $\mathbb{R}^i \times \mathsf{C}(L)$. Scaling down by, say, 50% witnesses a conically smooth map $\mathbb{D}^i \times \overline{\mathsf{C}}(L) \hookrightarrow \mathbb{R}^i \times \mathsf{C}(L)$. The image of this map is a weakly regular subspace.

Lemma 1.2.4 (Shrinking Lemma)**.** *Each open cover $\mathcal{U}_0$ of a stratified space K admits an open refinement $\mathcal{V}_0$ for which each $V \in \mathcal{V}_0$ is the interior of its closure $\overline{V} \subset K$ which is a weakly regular subspace of K.*

Proof. By design, conically smooth open embeddings from basics form a basis for the topology of K. Therefore each such $\mathcal{U}_0$ admits a refinement by basics. Simply by scaling, conically smooth embeddings $\mathbb{R}^i\times\mathsf{C}(L)\hookrightarrow\mathbb{R}^i\times\mathsf{C}(L)$ that factor as the interior through a conically smooth embedding $\mathbb{D}^i\times\overline{\mathsf{C}}(L)\to\mathbb{R}^i\times\mathsf{C}(L)$, form a local base for the topology about the origin $0\in\mathbb{R}^i\times\mathsf{C}(L)$. The result follows because the image of each conically smooth embedding $\mathbb{D}^i\times\overline{\mathsf{C}}(L)\hookrightarrow\mathbb{R}^i\times\mathsf{C}(L)$ is a weakly regular subspace.

□

1.3. **Sheaves.** There is a Grothendieck topology on Strat induced from the standard Grothendieck topology on topological spaces via the *underlying topological space* functor $\mathsf{Strat}\to\mathsf{Top}$. A sieve $\mathcal{U}\subset\mathsf{Strat}_{/K}$ on a stratified space K is a *covering sieve* if it has the following property.

- **Open:** For each $(U\to K)\in\mathcal{U}$ there is a morphism $(U\to K)\to(U_0\to K)$ in $\mathcal{U}$ with $U_0\to K$ an open embedding of stratified spaces.
- **Surjective:** For each $x\in K$, the $(\{x\}\to K)$ belongs to $\mathcal{U}$.

The ∞-category of *sheaves* on Strat is the full ∞-subcategory

$$\mathsf{Shv}(\mathsf{Strat})\ \subset\ \mathsf{PShv}(\mathsf{Strat})$$

consisting of those presheaves $\mathcal{F}$ for which, for each covering sieve $\mathcal{U}$ of a stratified space K, the restriction of $\mathcal{F}$ along the adjoint diagram

$$(\mathcal{U}^{\mathsf{op}})^{\triangleleft}\cong(\mathcal{U}^{\triangleright})^{\mathsf{op}}\longrightarrow\mathsf{Strat}^{\mathsf{op}}\xrightarrow{\mathcal{F}}\mathcal{S}\mathsf{paces}$$

is a limit diagram. Equivalently, the canonical map $\mathcal{F}(K)\xrightarrow{\simeq}\lim\limits_{U\in\mathcal{U}}\mathcal{F}(U)$ is an equivalence of spaces for each covering sieve $\mathcal{U}$ of a stratified space K. Being defined in terms of limits, the inclusion $\mathsf{Shv}(\mathsf{Strat})\to\mathsf{PShv}(\mathsf{Strat})$ preserves limits, and from presentability considerations the adjoint functor theorem can be applied, thereby producing a left adjoint

$$\mathsf{PShv}(\mathsf{Strat})\longrightarrow\mathsf{Shv}(\mathsf{Strat})$$

which is *sheafification.* The restriction of the forgetful functor $\mathsf{Strat}_{|\mathsf{Top}^{\mathsf{open}}}\to\mathsf{Top}^{\mathsf{open}}$, to *open embeddings* of topological spaces, is a Cartesian fibration; the Cartesian morphisms are *open embeddings* of stratified spaces. It follows that this collection of covering sieves on Strat is a Grothendieck topology. Therefore the sheafification functor is left exact, and the aforementioned adjunction is a morphism of ∞-topoi; see [Lu1].

In a standard manner, we will extend a presheaf on Strat to subspaces of stratified spaces. Namely, for K a stratified space, consider the poset $\mathsf{Sub}(K)$ of subspaces of the underlying topological space of K, ordered by inclusion. Taking images defines a functor $\mathsf{Strat}_{/K}\to\mathsf{Sub}(K)$. This functor restricts as an equivalence of categories $\mathsf{Strat}^{\mathsf{emb}}_{/K}\to\mathsf{Sub}^{\mathsf{open}}(K)$ from the subcategory of *open embeddings* to K, to the subposet of *open* subspaces of K. We thus have the composite functor

$$\mathsf{PShv}(\mathsf{Strat})\xrightarrow{|K}\mathsf{PShv}(\mathsf{Strat}^{\mathsf{emb}}_{/K})\ \simeq\ \mathsf{PShv}\big(\mathsf{Sub}^{\mathsf{opn}}(K)\big)\xrightarrow{\mathsf{LKan}}\mathsf{PShv}\big(\mathsf{Sub}(K)\big)$$

where LKan is given by left Kan extension. Explicitly, for $\mathcal{F}$ a presheaf on Strat, the value of this composite functor on $\mathcal{F}$ evaluates on a subspace $K_0\subset K$ as

$$\mathcal{F}\mapsto\Big((K_0\subset K)\mapsto\underset{K_0\subset O\underset{\mathsf{open}}{\subset}K}{\mathsf{colim}}\mathcal{F}(O)\ =:\ \mathcal{F}(K_0)\Big)$$

where the colimit is over the poset of open subspaces of K, each of which canonically inherits the structure of a stratified space. We will not distinguish in notation between a presheaf $\mathcal{F}$ on Strat and its image under this composite functor. This way we can simply write $\mathcal{F}(K_0)$ for value of the above assignment, as indicated. The definition of a topological space is just so that the category indexing this colimit is filtered (as an ∞-category). For $\overline{\mathcal{U}}_0$ a collection of subspaces of K whose collections of interiors cover K with $\overline{\mathcal{U}} \subset \mathsf{Strat}_{/K}$ the sieve on K consisting of those conically smooth maps $X \to K$ that factor through some member of $\overline{\mathcal{U}}_0$, then paracompactness of K implies the diagram

$$(\overline{\mathcal{U}}^{\mathsf{op}})^{\triangleleft} \xrightarrow{\mathsf{image}} \mathsf{Sub}(K)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathcal{S}\mathsf{paces}$$

is a limit diagram for each sheaf $\mathcal{F}$ on Strat.

2. Constructible sheaves

We examine constructible sheaves on the site of stratified spaces. These are sheaves on Strat that restrict to locally constant sheaves on each stratum of each stratified space.

2.1. **Stratified homotopy.** We give a stratified analogue of (smooth) homotopy.

Definition 2.1.1. Let $f, g\colon X \to Y$ be two conically smooth maps among stratified spaces. A *stratified homotopy* from f to g is a conically smooth map $H : X \times \mathbb{R} \to Y \times \mathbb{R}$ over $\mathbb{R}$ whose restrictions are identified as $f = H_{|\{0\}}$ and $g = H_{|\{1\}}$. We write $f \simeq g$ if there exists a stratified homotopy from f to g. A conically smooth map $f : X \to Y$ is a *stratified homotopy equivalence* if there exists a conically smooth map $Y \xrightarrow{g} X$ for which $1_X \simeq gf$ and $gf \simeq 1_Y$; g is a *stratified homotopy inverse* of f.

Observation 2.1.2. By induction on $i \geq 0$, the projection $X \times \mathbb{R}^i \to X$ is a stratified homotopy equivalence.

Lemma 2.1.3. *For each stratified space with boundary $\overline{X}$, the inclusion of the interior $\mathsf{int}(\overline{X}) \to \overline{X}$ is a stratified homotopy equivalence.*

Proof. We first consider the case $\overline{X} = [0, 1)$. Choose a smooth function $\phi\colon [0, 1) \to [0, 1]$ satisfying

- $\phi(0) > 0$,
- $\phi(s) = 0$ for $s \geq \frac{1}{2}$,
- $s + \phi(s) \leq 1$.

The map $[0, 1) \xrightarrow{g} [0, 1)$, given by $g(s) = s+\phi(s)$, is a homotopy inverse to the inclusion of the interior $(0, 1) \to [0, 1)$. Relevant homotopies are given by $H\colon [0, 1)\times[0, 1) \to [0, 1)$ given by $H_t(s) = s+t\phi(s)$ and $H'\colon (0, 1) \times [0, 1) \to (0, 1)$ given by $H'_t(s) = s + (1 - t)\phi(s)$.

Immediately after the above case, we see the result for the case $\overline{X} = \partial\overline{X} \times [0, 1)$. In the above argument, f and g and H and H' restrict as the identity map on $[\frac{1}{2}, 1)$. Consequently, this product case extends to the general case.

□

Observation 2.1.4. For each pair of stratified spaces X and Y, stratified homotopy defines an equivalence relation on the set $\mathsf{Strat}(X, Y)$ of conically smooth maps. Moreover, should two of the arrows in a commutative diagram in Strat

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "g"'] & & Z \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

be stratified homotopy equivalences, then the third, too, is a stratified homotopy equivalence.

Observation 2.1.5. A conically smooth map $X \xrightarrow{f} Y$ is a stratified homotopy equivalence if and only if it fits into a diagram in Strat

$$\begin{tikzcd}
& X \arrow[rr, "1_X"] \arrow[d, "\{0\}"'] & & X \arrow[d, "\{0\}"] \\
& X \times \mathbb{R} \arrow[rr, "H"] & & X \times \mathbb{R} \\
Y \arrow[r, "g"] \arrow[d, "\{0\}"'] & X \arrow[r, "f"] \arrow[u, "\{1\}"'] & Y \arrow[r, "g"] \arrow[d, "\{0\}"] & X \arrow[u, "\{1\}"] \\
Y \times \mathbb{R} \arrow[rr, "H'"] & & Y \times \mathbb{R} & \\
Y \arrow[rr, "1_Y"] \arrow[u, "\{1\}"'] & & Y \arrow[u, "\{1\}"] &
\end{tikzcd}$$

in which both H and H' lie over $\mathbb{R}$.

Notation 2.1.6. We use the notation $\mathcal{J}$ for the collection of those morphisms in Strat which are stratified homotopy equivalences. We use the notation $\mathcal{R}$ for the collection of those morphisms in Strat of the form $X \times \mathbb{R} \xrightarrow{\mathsf{pr}} X$.

Observation 2.1.4 grants that $\mathcal{J}$ forms a subcategory of Strat, and Observation 2.1.2 gives an inclusion $\mathcal{R} \subset \mathcal{J}$.

Lemma 2.1.7. *The canonical map of localizations*

$$\mathsf{Strat}[\mathcal{R}^{-1}] \longrightarrow \mathsf{Strat}[\mathcal{J}^{-1}]$$

is an equivalence of ∞-categories.

Proof. Consider a functor $\mathsf{Strat} \to \mathcal{C}$ to an ∞-category for which the restriction to $\mathcal{R}$ factors through $\mathcal{C}^{\sim} \subset \mathcal{C}$, the equivalences of $\mathcal{C}$. Consider a morphism $X \to Y$ for which there is a stratified homotopy inverse. Examine the diagram of Observation 2.1.5. By assumption, the vertical arrows are carried to isomorphisms in $\mathcal{C}$, and necessarily the upper and the lower horizontal arrows are as well. It follows that $X \to Y$ is carried to an equivalence in $\mathcal{C}$. $\square$

2.2. **Constructible sheaves.** We define *constructibility* for sheaves.

Let Z be a space. Cotensoring with Z gives the presheaf

$$Z^{-}\colon\ \mathsf{Man}^{\mathsf{op}} \longrightarrow \mathcal{S}\mathsf{paces}^{\mathsf{op}} \xrightarrow{\mathcal{S}\mathsf{paces}(-,Z)} \mathcal{S}\mathsf{paces}$$

where the first functor is the underlying space functor, and the second is Yoneda.

Lemma 2.2.1. *Let $\mathcal{F}$ be a sheaf on Strat. The following conditions on $\mathcal{F}$ are equivalent.*

(1) *Each stratified homotopy $f\colon X \to Y$ induces an equivalence of spaces*

$$f^*\colon \mathcal{F}(Y) \xrightarrow{\simeq} \mathcal{F}(X)\ .$$

(2) *For each stratified space K, the projection $K \times \mathbb{R} \to K$ induces an equivalence of spaces*

$$\mathcal{F}(K) \xrightarrow{\simeq} \mathcal{F}(K \times \mathbb{R})\ .$$

(3) *For each stratified space K, the restriction $\mathcal{F}_{|K\times -}$ of $\mathcal{F}$ along $\mathsf{Man} \xrightarrow{K\times -} \mathsf{Strat}$ is a locally constant sheaf.*

(4) *For each stratified space K, there is a canonical equivalence of presheaves on Man:*

$$\mathcal{F}_{|K\times -}\ \simeq\ \mathcal{F}(K)^{-}\ .$$

(5) *The functor $\mathcal{F}\colon \mathsf{Strat}^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}$ factors through $\mathsf{Strat}[\mathcal{J}^{-1}]^{\mathsf{op}}$.*

(6) *The functor* $\mathcal{F}\colon \mathsf{Strat}^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}$ *factors through* $\mathsf{Strat}[\mathcal{R}^{-1}]^{\mathsf{op}}$.

Proof. Definitionally, (1) is equivalent to (5), and (2) is equivalent to (6). The equivalence of (1) and (2) follows immediately from Lemma 2.1.7.

We now prove that, for each space Z, the presheaf $Z^{-} := \mathcal{S}\mathsf{paces}(-, Z)$ on Man is a sheaf. Let $\mathcal{U}$ be a covering sieve of a smooth manifold S. Consider the resulting composite functor $\mathcal{U}^{\triangleright} \to \mathsf{Man} \to \mathcal{S}\mathsf{paces}$. Because $\mathcal{U}$ is in particular a hypercover of S, then by [DI], this composite functor is a colimit diagram. Being in terms of a cotensor, it follows that $(\mathcal{U}^{\triangleright})^{\mathsf{op}} \to \mathsf{Man}^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}^{\mathsf{op}} \xrightarrow{Z^{-}} \mathcal{S}\mathsf{paces}$ is a limit diagram. This shows that Z^{-} is a sheaf on Man.

Let us now argue that Z^{-} is locally constant. Let S be a smooth manifold and choose a hypercover $\mathcal{U}$ of S comprised of Euclidean spaces. From the previous paragraph, we know that the canonical map $Z^{S} \xrightarrow{\simeq} \lim\limits_{U\in\mathcal{U}} Z^{U}$ is an equivalence of spaces. Because each U is contractible, we see that $Z^{U} \simeq Z$, and we conclude that Z^{-} is locally constant.

The full subcategory $\mathsf{Man}^{\mathsf{Euc}} \subset \mathsf{Man}$ consisting of Euclidean spaces forms a hyperbasis for the Grothendieck topology of Man whose associated ∞-topos of sheaves is hypercomplete. There is then a natural equivalence $\mathcal{G} \xrightarrow{\simeq} \mathcal{G}'$ of sheaves on Man if and only if there is a natural equivalence of restrictions $\mathcal{G}_{|\mathsf{Euc}} \xrightarrow{\simeq} \mathcal{G}'_{|\mathsf{Euc}}$ to $\mathsf{Man}^{\mathsf{Euc}}$. Because $\mathbb{R}^n$ is contractible, then the restriction of $\mathcal{F}(K)^{-}$ to $\mathsf{Man}^{\mathsf{Euc}}$ is canonically identified as the constant sheaf at $\mathcal{F}(K)$. And so (4) is true if and only if the restriction of $\mathcal{F}_{|K\times -}$ to $\mathsf{Man}^{\mathsf{Euc}}$ is constant.

Likewise, (3) is true if and only if the restriction of $\mathcal{F}_{|K\times -}$ to $\mathsf{Man}^{\mathsf{Euc}}$ is locally constant. Because each object of $\mathsf{Man}^{\mathsf{Euc}}$ is contractible, this is the case if and only if $\mathcal{F}_{|K\times -}$ is constant. We conclude that (3) is equivalent to (4).

If (2) is true, a quick induction argument on $n \geq 0$ implies that the map of spaces $\mathcal{F}(K) \xrightarrow{\simeq} \mathcal{F}(X \times \mathbb{R}^n)$, induced by the projection, is an equivalence; so (2) implies (4). Conversely, if (4) is true, then the map induced from the projection $\mathcal{F}(K) \to \mathcal{F}(K \times \mathbb{R})$ is necessarily an equivalence, and we see that (4) implies (2).

□

Remark 2.2.2. Lemma 2.2.1 fundamentally relies on the existence of conically smooth bump functions, and thereafter on the existence of conically smooth partitions of unity. This contrasts with $\mathbb{A}^1$-homotopy theory.

Definition 2.2.3. The ∞-category of *constructible sheaves on* Strat is the full ∞-subcategory

$$\mathsf{Shv}^{\mathsf{cbl}}(\mathsf{Strat}) \ \subset \ \mathsf{Shv}(\mathsf{Strat})$$

consisting of those sheaves that satisfy the equivalent conditions of Lemma 2.2.1.

Observation 2.2.4. The ∞-category of constructible sheaves fits into a pullback diagram among ∞-categories

$$\begin{tikzcd}
\mathsf{Shv}^{\mathsf{cbl}}(\mathsf{Strat}) \arrow[r] \arrow[d] & \mathsf{PShv}(\mathsf{Strat}[\mathcal{R}^{-1}]) \arrow[d] \\
\mathsf{Shv}(\mathsf{Strat}) \arrow[r] & \mathsf{PShv}(\mathsf{Strat})
\end{tikzcd}$$

which is comprised of fully faithful functors.

2.3. **Isotopy sheaves.**

Here we give a useful method of obtaining constructible sheaves from point-set data.

The adjoint functor theorem grants the existence of a left adjoint to the inclusion

$$L\colon \mathsf{Shv}(\mathsf{Strat}) \longrightarrow \mathsf{Shv}^{\mathsf{cbl}}(\mathsf{Strat})\ .$$

The next section explicates the values of L on a certain class of sheaves: *isotopy sheaves*. In the present section, we define isotopy sheaves and give examples.

Definition 2.3.1. The category of *isotopy sheaves*

$$\mathsf{Isot} \;\subset\; \mathsf{Cat}_{/\,\mathsf{Strat}}$$

is the full subcategory of categories $\mathfrak{I}$ over Strat which satisfy the following conditions.

- **Right fibration:** The functor $\mathfrak{I} \to \mathsf{Strat}$ is a right fibration.
- **Sheaf:** For each covering sieve $\mathcal{U} \subset \mathsf{Strat}_{/K}$, the restriction of $\mathfrak{I}$ along the adjoint diagram $\mathcal{U}^{\triangleright} \to \mathsf{Strat}$ is a limit diagram of groupoids: for each covering sieve $\mathcal{U}$ of K, the canonical map of groupoids
$$\mathfrak{I}(K) \xrightarrow{\simeq} \lim_{U \in \mathcal{U}} \mathfrak{I}(U)$$
is an equivalence.
- **Isotopy extension:** For each weakly regular subspace $K_0 \subset K$, the canonical functor of groupoids
$$\mathfrak{I}(K) \longrightarrow \mathfrak{I}(K_0)$$
is surjective on Hom-sets.
- **Isotopy equivalence:** For each stratified space K, the projection $K \times \mathbb{R} \to K$ induces a bijection of isomorphism classes
$$\pi_0\mathfrak{I}(K) \xrightarrow{\cong} \pi_0\mathfrak{I}(K \times \mathbb{R}) ~.$$

The terminology of Definition 2.3.1 as *sheaves* is justified through the following observation.

Observation 2.3.2. Manifestly, isotopy sheaves are a full subcategory of the ordinary category of right fibrations over Strat. This category of right fibrations has a natural simplicial enrichment, and thus yields an ∞-category $\mathsf{RFib}_{\mathsf{Strat}}$. The straightening functor of [Lu1] gives an equivalence of ∞-categories $\mathsf{RFib}_{\mathsf{Strat}} \xrightarrow{\simeq} \mathsf{PShv}(\mathsf{Strat})$ to that of (space-valued) presheaves on Strat. The defining properties of an isotopy sheaf grant that the composite functor $\mathsf{Isot} \to \mathsf{RFib}_{\mathsf{Strat}} \xrightarrow{\simeq} \mathsf{PShv}(\mathsf{Strat})$ factors through $\mathsf{Shv}(\mathsf{Strat})$, (space-valued) sheaves on the site Strat.

Below we introduce construction for turning isotopy sheaves into constructible sheaves. It makes use of the *extended simplices*, which we now define.

Definition 2.3.3 ($\Delta^\bullet_e$)**.** The *extended* cosimplicial smooth manifold

$$\Delta^\bullet_e \colon \mathbf{\Delta} \longrightarrow \mathsf{Man}~, \qquad [p] \mapsto \bigl\{\{0,\dots,p\} \xrightarrow{t} \mathbb{R} \mid \sum_i t_i = 1\bigr\} \text{ and } ([p] \xrightarrow{\rho} [q]) \mapsto \bigl(t \mapsto (j \mapsto \sum_{\rho(i)=j} t_i)\bigr)$$

has values which are affine hyperplanes in $\mathbb{R}^{\{0,\dots,p\}}$. We regard $\Delta^\bullet_e$ as a cosimplicial stratified space by way of the standard inclusion $\mathsf{Man} \hookrightarrow \mathsf{Strat}$.

Definition 2.3.4 (Topologizing diagram)**.** The *topologizing diagram* functor is the composite

$$\mathsf{PShv}(\mathsf{Strat}) \longrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{PShv}(\mathsf{Strat})) \longrightarrow \mathsf{PShv}(\mathsf{Strat})$$

of restriction along $\mathsf{Strat} \times \mathbf{\Delta} \xrightarrow{-\times\Delta^\bullet_e} \mathsf{Strat}$ with left Kan extension along the projection $\mathsf{Strat} \times \mathbf{\Delta} \to \mathsf{Strat}$.

Because the projection $\mathsf{Strat} \times \mathbf{\Delta} \to \mathsf{Strat}$ is Cartesian, the values of the topologizing diagram can be expressed as the assignment

$$\mathfrak{F} ~\mapsto \Bigl(K \mapsto |\mathfrak{F}(K \times \Delta^\bullet_e)|\Bigr)$$

where $|-|$ denotes geometric realization.

Remark 2.3.5. The terminology is intended to evoke the *classifying diagram* functor $\mathsf{PShv}(\mathbf{\Delta}) \to \mathsf{PShv}(\mathbf{\Delta})$ given by $\mathcal{C} \mapsto |\mathcal{C}(\Delta[-] \times E[\bullet])|$ where $E[p]$ is the nerve of the connected groupoid whose set of objects is $\{0,\dots,p\}$. (See [Re1] and [JT]). See Remark 5.1.3.

Lemma 2.3.6. *The topologizing diagram functor factors through* $\mathsf{PShv}(\mathsf{Strat}[\mathcal{J}^{-1}])$*. That is, for each presheaf* $\mathfrak{F}$ *on* Strat*, and each stratified space* K*, projection induces an equivalence of spaces*

$$|\mathfrak{F}(K \times \Delta_e^\bullet)| \xrightarrow{\simeq} |\mathfrak{F}(K \times \mathbb{R} \times \Delta_e^\bullet)| \ .$$

Proof. Recognize an identification $\Delta_e^1 \cong \mathbb{R}$, after which there results an identification of simplicial objects $\mathfrak{F}(K \times \mathbb{R} \times \Delta_e^\bullet) \simeq \mathfrak{F}(K \times \Delta_e^\bullet)^{\Delta[1]}$. Thereafter, stratified homotopies induce simplicial homotopies. Therefore stratified homotopy equivalences induce simplicial homotopy equivalences. □

Theorem 2.3.7. *The restriction of the topologizing diagram functor to isotopy sheaves factors through constructible sheaves:*

$$\mathsf{Isot} \longrightarrow \mathsf{Shv}^{\mathsf{cbl}}(\mathsf{Strat}) \ , \qquad \mathfrak{I} \mapsto |\mathfrak{I}(- \times \Delta_e^\bullet)| \ .$$

Proof. We realize the assignment $K \mapsto |\mathfrak{I}(K \times \Delta_e^\bullet)|$ as given by the composite

$$\mathsf{Strat}^{\mathsf{op}} \xrightarrow{\mathfrak{I}} \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Gpd}) \longrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}} \times \mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set}) \xrightarrow{\delta^*} \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set}) \longrightarrow \mathcal{S}\mathsf{paces} \tag{3}$$

where the second functor is given by taking the nerve of the groupoid; the third functor is restriction along the diagonal $\delta : \mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$; the last functor from the ordinary category of simplicial sets to the ∞-category of spaces is given by localizing with respect to weak homotopy equivalences. In the following, we denote the bisimplicial set given by taking the nerve as $\mathfrak{I}(K \times \Delta_e^\bullet)_\star$.

We divide the proof into steps. In **Step 1** we show that the simplicial set produced above is a Kan complex, and that the restriction maps induced by regular embeddings are Kan fibrations. This will imply strong homotopy invariance properties which are then used in **Step 2**, where we show that $|\mathfrak{I}(- \times \Delta_e^\bullet)|$ defines a sheaf valued in the ∞-category of spaces.

Step 0: Straighten the right fibration $\mathfrak{I} \to \mathsf{Strat}$ to a functor $\mathfrak{I}\colon \mathsf{Strat}^{\mathsf{op}} \to \mathsf{Gpd}$ to groupoids, that we give the same notation. Replace this functor, up to equivalence, by one for which, for each weakly regular subspace $K_0 \subset K$, the restriction functor $\mathfrak{I}(K) \to \mathfrak{I}(K_0)$ among groupoids is an **isofibration**.

Recall that a functor $\mathcal{G} \to \mathcal{G}'$ between groupoids is an *isofibration* if every isomorphism in $\mathcal{G}'$ with a lift to $\mathcal{G}$ of its target extends as a lift of the isomorphism. This replacement can be done, for instance, by first endowing Gpd with the restriction of the canonical model structure, then by performing a fibrant replacement in a projective model structure on the category of functors $\mathsf{Strat}^{\mathsf{op}} \to \mathsf{Gpd}$.

Step 1: We show that the isotopy conditions imply that for every regularly embedded subspace $K_0 \subset K$, the map of simplicial sets

$$\delta^*\mathfrak{I}(K \times \Delta_e^\bullet)_\star \longrightarrow \delta^*\mathfrak{I}(K_0 \times \Delta_e^\bullet)_\star$$

is a Kan fibration; as the case that $\emptyset = K_0$, this will imply that each $\delta^*\mathfrak{I}(K \times \Delta^\bullet)_\star$ is a Kan complex. This is equivalent to showing that the map of bisimplicial sets

$$\mathfrak{I}(K \times \Delta_e^\bullet)_\star \longrightarrow \mathfrak{I}(K_0 \times \Delta_e^\bullet)_\star$$

is a fibration with respect to the diagonal model structure of [Moe]. By [Ja], fibrations for the diagonal model structures are detected by having the right lifting property with respect to the maps

$$\partial\Delta[p] \boxtimes \Delta[q] \coprod_{\partial\Delta[p]\boxtimes\Lambda_l[q]} \Delta[p] \boxtimes \Lambda_l[q] \ \hookrightarrow \ \Delta[p] \boxtimes \Delta[q] \quad (0 \leq k \leq p \ , \ 0 \leq l \leq q \neq 0) \tag{4}$$

and

$$\Lambda_k[p] \boxtimes \Delta[q] \coprod_{\Lambda_k[p]\boxtimes\partial\Delta[q]} \Delta[p] \boxtimes \partial\Delta[q] \ \hookrightarrow \ \Delta[p] \boxtimes \Delta[q] \quad (0 \leq k \leq p \neq 0 \ , \ 0 \leq l \leq q) \tag{5}$$

where $\boxtimes : \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set}) \times \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set}) \to \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}} \times \mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set})$ is the external product of simplicial sets. In the case at hand, for each stratified space Z and each $p \geq 0$, the simplicial set $\mathfrak{I}(Z \times \Delta_e^p)_\star$

is 1-coskeletal, for it is the nerve of a groupoid. Thus, it suffices to only verify the aforementioned right lifting property with respect to (4) and (5) in the cases $q = 0, 1$.

Lifting for (4): Again, $\mathfrak{I}(Z \times \Delta^{\bullet}_e)_\star$ being the nerve of a groupoid, the two cases $(l, q) = (0, 1), (1, 1)$ are equivalent to each other; so we will only concern ourselves with the $(l, q) = (1, 1)$ case, for which $\Lambda_l[q] = \Delta[\{0\}] \subset \Delta[1]$. This is the problem of finding a filler in the diagram of bisimplicial sets

$$\begin{tikzcd} \partial\Delta[p] \boxtimes \Delta[1] \coprod_{\partial\Delta[p]\boxtimes\Delta[\{0\}]} \Delta[p] \boxtimes \Delta[\{0\}] \arrow[r] \arrow[d] & \mathfrak{I}(K \times \Delta^{\bullet}_e)_\star \arrow[d] \\ \Delta[p] \boxtimes \Delta[1] \arrow[r] \arrow[ur, dashed] & \mathfrak{I}(K_0 \times \Delta^{\bullet}_e)_\star . \end{tikzcd}$$

Unwinding, this is the problem of showing the restriction functor among groupoids

$$\mathfrak{I}(K \times \Delta^p_e) \longrightarrow \mathfrak{I}\big(K_0 \times \Delta^p_e \underset{K_0 \times \partial\Delta^p_e}{\cup} K \times \partial\Delta^p_e\big)$$

is a Cartesian fibration. This functor is restriction along a weakly regular subspace. With the isofibration assumption on $\mathfrak{I}$, that this restriction functor is Cartesian follows directly from the **isotopy extension** condition.

Lifting for (5): These lifting problems reduce similarly as the following problems. We must show the restriction functor between groupoids

$$\mathfrak{I}(K \times \Delta^p_e) \longrightarrow \mathfrak{I}\big(K_0 \times \Delta^p_e \underset{K_0 \times (\Lambda^p_k)_e}{\cup} K \times (\Lambda^p_k)_e\big)$$

is surjective on objects ($q = 0$), and is a Cartesian fibration ($q = 1$). With the isofibration assumption on $\mathfrak{I}$, the surjectivity on objects follows from the **isotopy equivalence** condition because the subspace

$$K_0 \times \Delta^p_e \underset{K_0 \times (\Lambda^p_k)_e}{\cup} K \times (\Lambda^p_k)_e \subset K \times \Delta^p_e$$

is a weakly regular subspace, and is in fact a regular neighborhood. Also with the isofibration assumption on $\mathfrak{I}$, the Cartesian fibration property follows directly from the **isotopy extension** contition.

Step 2: We show that the presheaf $|\mathfrak{I}(- \times \Delta^{\bullet}_e)| : \mathsf{Strat}^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}$ is a sheaf. Let $\mathcal{U} \subset \mathsf{Strat}_{/K}$ be a covering sieve. We must show that the diagram

$$(\mathcal{U}^{\mathsf{op}})^{\triangleleft} \longrightarrow \mathsf{Strat}^{\mathsf{op}} \xrightarrow{|\mathfrak{I}(- \times \Delta^{\bullet}_e)|} \mathcal{S}\mathsf{paces} \tag{6}$$

is a limit diagram. Consider the subcategory $\mathcal{U}^{\mathsf{opn}} \subset \mathcal{U} \subset \mathsf{Strat}_{/K}$ consisting of those $(U \to K) \in \mathcal{U}$ that are inclusions of open subspaces. By definition of the Grothendieck topology on Strat, the functor (6) is a limit diagram if and only if its restriction

$$\big((\mathcal{U}^{\mathsf{opn}})^{\mathsf{op}}\big)^{\triangleleft} \longrightarrow (\mathcal{U}^{\mathsf{op}})^{\triangleleft} \to \mathsf{Strat}^{\mathsf{op}} \xrightarrow{|\mathfrak{I}(- \times \Delta^{\bullet}_e)|} \mathcal{S}\mathsf{paces} \tag{7}$$

is a limit diagram.

Now consider the poset $\mathsf{Sub}(K)$ of subspaces of K, ordered by inclusion. Note the full inclusion $\mathcal{U}^{\mathsf{opn}} \subset \mathsf{Sub}(K)$. Consider the subposet $\mathcal{U}^{\mathsf{cpt,reg}} \subset \mathsf{Sub}(K)$ of the compact weakly regular subspaces $K_0 \subset K$ for which the interior $\mathsf{int}(K_0)$ belongs to $\mathcal{U}^{\mathsf{opn}}$ and the closure of the interior $\overline{\mathsf{int}}(K_0)$ is K_0 itself. Lemma 1.2.4 gives that the functor (7) is a limit diagram if and only if its restriction

$$\big((\mathcal{U}^{\mathsf{cpt,reg}})^{\mathsf{op}}\big)^{\triangleleft} \xrightarrow{\mathsf{int}} \big((\mathcal{U}^{\mathsf{opn}})^{\mathsf{op}}\big)^{\triangleleft} \longrightarrow (\mathcal{U}^{\mathsf{op}})^{\triangleleft} \to \mathsf{Strat}^{\mathsf{op}} \xrightarrow{|\mathfrak{I}(- \times \Delta^{\bullet}_e)|} \mathcal{S}\mathsf{paces} \tag{8}$$

is a limit diagram. Left Kan extension along the inclusion $\mathcal{U}^{\mathsf{opn}} \hookrightarrow \mathsf{Sub}(K)$ gives the composite functor

$$|\mathfrak{I}\big(\overline{(-)} \times \Delta^{\bullet}_e\big)| \colon \big((\mathcal{U}^{\mathsf{cpt,reg}})^{\mathsf{op}}\big)^{\triangleleft} \longrightarrow (\mathsf{Sub}(K)^{\mathsf{op}})^{\triangleleft} \xrightarrow{|\mathfrak{I}(- \times \Delta^{\bullet}_e)|_!} \mathcal{S}\mathsf{paces} \ . \tag{9}$$

The inclusion $\mathsf{int}(K_0) \hookrightarrow K_0$ for each subspace $K_0 \subset K$ gives the natural transformation from the functor (9) to the functor (8):

$$\begin{tikzcd} \mathcal{U}^{\mathsf{cpt,reg}} \arrow[r, bend left, "|\mathfrak{I}\left(\overline{(-)}\times\Delta^{\bullet}_e\right)|"{name=U}] \arrow[r, bend right, "|\mathfrak{I}(-\times\Delta^{\bullet}_e)|"'{name=D}] & \mathcal{S}\mathsf{paces} \arrow[from=U, to=D, phantom, "\Downarrow"] \end{tikzcd} . \tag{10}$$

Lemma 2.1.3 gives that each inclusion $\mathsf{int}(K_0) \to K_0$ is a stratified homotopy equivalence for each $(K_0 \subset K) \in \mathcal{U}^{\mathsf{cpt,reg}}$. Lemma 2.3.6 thereafter implies then that the natural transformation (10) is an equivalence of functors. We conclude that (8) is a limit diagram if and only if (9) is a limit diagram.

To conclude that (9) is a limit diagram, we employ our chosen point-set model for $|\mathfrak{I}(- \times \Delta^{\bullet}_e)|$ given by (3); which is to say we work with the point-set expression $\delta^*\mathfrak{I}(-\times\Delta^{\bullet}_e)_\star$, the diagonal of the bisimplicial set obtained from the nerve of the groupoid. The previous step verifies that this functor takes values in Kan complexes. Therefore, it suffices to show that this functor from stratified spaces to the category of Kan complexes

$$|\mathfrak{I}\big(\overline{(-)} \times \Delta^{\bullet}_e\big)|\colon \big((\mathcal{U}^{\mathsf{cpt,reg}})^{\mathsf{op}}\big)^{\triangleleft} \longrightarrow \mathsf{Kan} \tag{11}$$

is a homotopy limit diagram in Quillen's model. The sheaf $\mathfrak{I}$ being an isotopy sheaf, we have that, for each $(K_0 \subset K) \subset (K_0' \subset K)$ in $\mathcal{U}^{\mathsf{cpt,reg}}$, the restriction $\mathfrak{I}(K_0') \to \mathfrak{I}(K_0)$ is a Kan fibration. Therefore, to show that (11) is a homotopy limit diagram in Quillen's model it is enough to verify that it is a limit diagram. The definition of a topological space gives that, with respect to this functor, for each subspace $K_0 \subset K$ the slice category $(\mathcal{U}_0^{\mathsf{opn}})_{/K_0}$ is filtered (as an ∞-category). With the paracompactness of K, another application of Lemma 1.2.4 gives that (11) is a limit diagram in Kan because $\mathfrak{I}$ is a sheaf.

□

Remark 2.3.8. Following up on Observation 2.3.2, we invite a development of a model structure on categories over Strat for which isotopy sheaves, or some minor variation thereof, form the fibrant objects and for which the topologizing diagram functor to constructible sheaves is fully faithful.

From the defining property of L as a left adjoint in a localization, after Theorem 2.3.7 there exists a canonical natural transformation from the restriction of L to Isot to the topologizing diagram functor. The next result verifies that this is an equivalence.

Lemma 2.3.9. *For each isotopy sheaf $\mathfrak{I}$, the canonical natural transformation*

$$L\mathfrak{I} \longrightarrow |\mathfrak{I}(- \times \Delta^{\bullet}_e)|$$

is an equivalence.

Proof. There is a unit in the adjunction $\mathfrak{I} \to L\mathfrak{I}$. The topologizing diagram applied to the unit gives a natural transformation

$$|\mathfrak{I}(- \times \Delta^{\bullet}_e)| \longrightarrow |L\mathfrak{I}(- \times \Delta^{\bullet}_e)| \xrightarrow{\simeq} L\mathfrak{I}$$

where the right arrow is induced from the zero-section. This is an equivalence because L takes values in *constructible* sheaves. By inspection, this natural transformation factors the unit $\mathfrak{I} \to L\mathfrak{I}$. From the universal property of L as a left adjoint in a localization, this natural transformation is a left inverse to the one in the statement of the lemma. We now verify that this natural transformation is also a right inverse. By inspection, the resulting endo-transformation $|\mathfrak{I}(- \times \Delta^{\bullet}_e)| \to L\mathfrak{I} \to |\mathfrak{I}(- \times \Delta^{\bullet}_e)|$ factors as as a composition

$$|\mathfrak{I}(- \times \Delta^{\bullet}_e)| \longrightarrow |\mathfrak{I}\big((- \times \Delta^{\bullet}_e) \times \Delta^{\bullet'}_e\big)| \longrightarrow |\mathfrak{I}(- \times \Delta^{\bullet'}_e)|$$

of that induced by projection off the second cosimplicial factor, followed by that induced by the zero-section of the first factor. Because $|\mathfrak{I}(- \times \Delta^{\bullet}_e)|$ is constructible, for each fixed $\bullet$ the left arrow is an equivalence; for the same reason the second arrow is an equivalence for each fixed $\bullet$.

□

2.4. **The ∞-category $\mathcal{S}\mathsf{trat}$.** The identification $L\mathfrak{I} \simeq |\mathfrak{I}(-\times\Delta^\bullet_e)|$ of Lemma 2.3.9 gives an explicit Kan-enrichment of Strat whose associated ∞-category agrees with the localization $\mathsf{Strat}[\mathcal{J}^{-1}]$.

Lemma 2.4.1. *For each stratified space K, the projection from the slice category $\mathsf{Strat}_{/K} \to \mathsf{Strat}$ is an isotopy sheaf.*

Proof. The functor $\mathsf{Strat}_{/K} \to \mathsf{Strat}$ is manifestly a right fibration – it is the unstraightening of the representable presheaf $\mathsf{Strat}(-,K)\colon \mathsf{Strat}^{\mathsf{op}} \to \mathsf{Set}$. This functor $\mathsf{Strat}(-,K)$ carries colimit diagrams in Strat to limit diagrams in Set. Because covering sieves in Strat give colimit diagrams, the sheaf property follows. The isotopy conditions follow directly after the existence of regular neighborhoods of constructible closed subspaces.

□

Lemma 2.4.1 informs us that the Yoneda functor $K \mapsto (\mathsf{Strat}_{/K} \to \mathsf{Strat})$ factors through isotopy sheaves: $\mathsf{Strat} \longrightarrow \mathsf{Isot}$. Through Theorem 2.3.7, this leaves us with a functor

$$\mathsf{Strat} \longrightarrow \mathsf{Shv}^{\mathsf{cbl}}(\mathsf{Strat})~, \qquad K \mapsto L\,\mathsf{Strat}(-,K)~. \tag{12}$$

Definition 2.4.2. The ∞-category of *stratified spaces* is the essential image

$$\mathsf{Strat} \longrightarrow \mathcal{S}\mathsf{trat} ~\subset~ \mathsf{Shv}^{\mathsf{cbl}}(\mathsf{Strat})$$

of the functor (12). We denote the defining functor as $c\colon \mathsf{Strat} \to \mathcal{S}\mathsf{trat}$.

There is an immediate consequence of Lemma 2.3.9.

Corollary 2.4.3. *The ∞-category $\mathcal{S}\mathsf{trat}$ has the following model as a Kan-enriched category. An object of $\mathcal{S}\mathsf{trat}$ is a stratified space. The simplicial set of morphisms from X to Y is $\mathsf{Strat}(X\times\Delta^\bullet_e, Y)$. Composition is given by the assignment*

$$(X\times\Delta^q_e \xrightarrow{f} Y)\circ(Y\times\Delta^q_e \xrightarrow{g} Z) = (X\times\Delta^q_e \xrightarrow{(x,t)\mapsto g(f(x,t),t)} Z)~.$$

As so, the given functor $\mathsf{Strat} \to \mathcal{S}\mathsf{trat}$ is Hom-wise inclusion of the zero-simplices.

Observation 2.4.4. For each stratified space X, the projection $X\times\mathbb{R} \to X$ is an equivalence in the ∞-category $\mathcal{S}\mathsf{trat}$.

Theorem 2.4.5. *The natural functor $\mathsf{Strat} \longrightarrow \mathcal{S}\mathsf{trat}$ induces an equivalence between ∞-categories*

$$\mathsf{Strat}[\mathcal{J}^{-1}] \xrightarrow{\simeq} \mathcal{S}\mathsf{trat}$$

from the localization of the ordinary category Strat with respect to stratified homotopy equivalences and the ∞-category associated to the Kan-enriched category $\mathcal{S}\mathsf{trat}$.

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathsf{Strat} \arrow[r] \arrow[d, hook] & \mathsf{Strat}[\mathcal{J}^{-1}] \arrow[d, hook] \\
\mathsf{Shv}(\mathsf{Strat}) \arrow[r, "L"] & \mathsf{Shv}^{\mathsf{cbl}}(\mathsf{Strat})
\end{tikzcd}$$

where the vertical functors are Yoneda functors; these are fully faithful.

□

The defining functor $c\colon \mathsf{Strat} \to \mathcal{S}\mathsf{trat}$ yields an adjunction

$$c^*\colon~ \mathsf{PShv}(\mathcal{S}\mathsf{trat}) \rightleftarrows \mathsf{PShv}(\mathsf{Strat})\colon c_* \tag{13}$$

given by restriction and right Kan extension.

Definition 2.4.6. The ∞-category of *sheaves on* Strat is the pullback

$$\begin{tikzcd} \mathsf{Shv}(\mathcal{S}\mathsf{trat}) \arrow[r] \arrow[d] & \mathsf{PShv}(\mathcal{S}\mathsf{trat}) \arrow[d, "c^*"] \\ \mathsf{Shv}(\mathsf{Strat}) \arrow[r] & \mathsf{PShv}(\mathsf{Strat}). \end{tikzcd}$$

In other words, a presheaf $\mathcal{F}$ on $\mathcal{S}\mathsf{trat}$ is a sheaf if, for each covering sieve $\mathcal{U} \subset \mathsf{Strat}_{/K}$, the diagram $(\mathcal{U}^{\mathsf{op}})^{\triangleleft} \to \mathsf{Strat}^{\mathsf{op}} \to \mathcal{S}\mathsf{trat}^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathcal{S}\mathsf{paces}$ is a limit diagram.

Theorem 2.4.7. *The adjunction (13) restricts as an equivalence of ∞-categories*

$$\mathsf{Shv}(\mathcal{S}\mathsf{trat}) \;\simeq\; \mathsf{Shv}^{\mathsf{cbl}}(\mathsf{Strat})$$

between constructible sheaves on Strat *and sheaves on* $\mathcal{S}\mathsf{trat}$*.*

Proof. This is immediate after Theorem 2.4.5, in light of Observation 2.2.4. □

2.5. **Distinguished colimits in** $\mathcal{S}\mathsf{trat}$**.** We name some colimits in the ∞-category $\mathcal{S}\mathsf{trat}$, most of which are colimits in Strat as well. The *new* colimits that we examine arise from a classification of maps among basic singularity types, up to $\mathbb{R}$-invariance. This clasification is not available in the non-$\mathbb{R}$-invariant situation.

2.5.1. *Open covers and blow-ups.* We show that an open cover gives a colimit diagram in $\mathcal{S}\mathsf{trat}$, and also that a blow-up diagram along a deepest stratum is a colimit diagram in $\mathcal{S}\mathsf{trat}$.

Lemma 2.5.1. *For each covering sieve $\mathcal{U}^{\triangleright} \to \mathsf{Strat}$ of a stratified space X, the composite functor*

$$\mathcal{U}^{\triangleright} \longrightarrow \mathsf{Strat} \longrightarrow \mathcal{S}\mathsf{trat}$$

is a colimit diagram.

Proof. By definition of the ∞-category $\mathcal{S}\mathsf{trat}$, for each stratified space Z, the presheaf

$$\mathsf{Strat}^{\mathsf{op}} \xrightarrow{\mathcal{S}\mathsf{trat}(-,Z)} \mathcal{S}\mathsf{paces}$$

is a sheaf. Since this is in fact constructible, the result follows. □

Lemma 2.5.2. *For each stratified space X with deepest stratum $X_d \subset X$, the blow-up square*

$$\begin{tikzcd} \mathsf{Link}_{X_d}(X) \arrow[r] \arrow[d] & \mathsf{Unzip}_{X_d}(X) \arrow[d] \\ X_d \arrow[r] & X \end{tikzcd}$$

is a pushout diagram in $\mathcal{S}\mathsf{trat}$*.*

Proof. Fix a stratified space Z. We must show that the diagram of spaces and restriction maps among them

$$\begin{tikzcd} \mathcal{S}\mathsf{trat}\big(\mathsf{Link}_{X_d}(X), Z\big) & \mathcal{S}\mathsf{trat}\big(\mathsf{Unzip}_{X_d}(X), Z\big) \arrow[l] \\ \mathcal{S}\mathsf{trat}(X_d, Z) \arrow[u] & \mathcal{S}\mathsf{trat}(X, Z) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pullback. Results in [AFT] grant that the inclusions $X_d \hookrightarrow X$ and $\mathsf{Link}_{X_d} \hookrightarrow \mathsf{Unzip}_{X_d}(X)$ each have stratified regular neighborhoods. It follows that the horizontal maps are Kan fibrations. So it is enough to verify that each map of fiber Kan complexes is an isomorphism. This follows because the blow-up diagram in question is a pushout in Strat, and $- \times \Delta^q_e$ preserves colimits in Strat. □

Corollary 2.5.3. *For each compact stratified space L, the diagram*

$$\begin{tikzcd} L \arrow[r, "\{0\}"] \arrow[d] & L \times [0,1) \arrow[d] \\ \ast \arrow[r] & \mathsf{C}(L) \end{tikzcd}$$

is a pushout in $\mathcal{S}\mathsf{trat}$.

2.5.2. *Double-cones.* We show that double closed cones are pushouts in $\mathcal{S}\mathsf{trat}$ in terms of single closed cones.

Lemma 2.5.4. *For each stratified space X and each compact stratified space Z, there is a canonical identification of the space of maps to the cone*

$$\mathcal{S}\mathsf{trat}\big(X, \mathsf{C}(Z)\big) \;\simeq\; \coprod_{X_0 \underset{\mathsf{cbl,clsd}}{\subset} X} \mathcal{S}\mathsf{trat}\big(X \smallsetminus X_0, Z\big)$$

with the coproduct indexed by sub-stratified spaces $X_0 \hookrightarrow X$ whose inclusion is constructible and closed.

Proof. Let $X \times \Delta^q_e \xrightarrow{f} \mathsf{C}(Z)$ be a conically smooth map. Consider the preimage $f^{-1}(\ast) \subset X \times \Delta^q_e$. Because $\ast \hookrightarrow \mathsf{C}(Z)$ is constructible and closed, then so is the inclusion of this preimage. Because Δ^q_e is trivially stratified, then the sub-stratified space $f^{-1}(\ast) \subset X \times \Delta^q_e$ is of the form $X_0 \times \Delta^q_e \subset X \times \Delta^q_e$ for a unique sub-stratified space $X_0 \subset X$ whose inclusion is constructible and closed. It is straightforward to notice that the assignment $f \mapsto X_0$ defines a map from the Kan complex $\mathcal{S}\mathsf{trat}\big(X, \mathsf{C}(Z)\big)$ to the indexing set of the coproduct. In particular, we recognize

$$\mathcal{S}\mathsf{trat}\big(X, \mathsf{C}(Z)\big) \;\simeq\; \coprod_{X_0 \underset{\mathsf{cbl,clsd}}{\subset} X} \mathcal{S}\mathsf{trat}_{X_0}\big(X, \mathsf{C}(Z)\big)$$

as a coproduct, where the X_0-cofactor is the space of those maps $X \times \Delta^q_e \xrightarrow{f} \mathsf{C}(Z)$ for which $X_0 \times \Delta^q_e = f^{-1}(\ast)$.

It remains to show that the composite map

$$\mathcal{S}\mathsf{trat}_{X_0}\big(X, \mathsf{C}(Z)\big) \longrightarrow \mathcal{S}\mathsf{trat}\big(X \smallsetminus X_0, Z \times (0,1)\big) \xrightarrow[\text{Lem 2.4.4}]{\simeq} \mathcal{S}\mathsf{trat}(X \smallsetminus X_0, Z)$$

is an equivalence of spaces. Fix a compact smooth manifold S with boundary, and a union of components of its boundary $S_0 \subset \partial S$. Fix a conically smooth map $X \times S_0 \xrightarrow{f_0} \mathsf{C}(Z)$ for which $X_0 \times S_0 = f_0^{-1}(\ast)$. Use the notation $\overline{f_0}_{|}\colon (X \smallsetminus X_0) \times S_0 \to Z$ for the projection of the restriction, and the notation $\underline{f_0}\colon X \times S_0 \xrightarrow{f_0} \mathsf{C}(Z) \to [0,1)$ for the projection. We must show that the map of path components of spaces of maps relative to $f_0 \mapsto \overline{f_0}_{|}$

$$\pi_0\Big(\mathcal{S}\mathsf{trat}^{\mathsf{rel} f_0}_{X_0}\big(X \times S, \mathsf{C}(Z)\big)\Big) \longrightarrow \pi_0\Big(\mathcal{S}\mathsf{trat}^{\mathsf{rel} \overline{f_0}_{|}}\big((X \smallsetminus X_0) \times S, Z\big)\Big) \tag{14}$$

is surjective. So fix a conically smooth map $(X \smallsetminus X_0) \times S \xrightarrow{\overline{f}} Z$ extending $\overline{f_0}_{|}$.

Let us first prove the desired surjectivity of (14) for the case $Z = \ast$. Because the codomain of (14) is terminal in this case, the problem is to show that the domain is not empty. This is to say that each conically smooth map $X \times S_0 \xrightarrow{f_0} [0,1)$ for which $X_0 \times S_0 = f_0^{-1}\{0\}$ can be extended to a conically smooth map $X \times S \xrightarrow{f} [0,1)$ for which $X_0 \times S_0 = f^{-1}\{0\}$. Choose a collar-neighborhood $S_0 \times [0,1) \subset S$, and choose a smooth partition of unity $\{\phi_0, \phi_1\}$ subordinate to the open cover $S_0 \times [0,1) \cup S \smallsetminus S_0$ of the smooth manifold S. Choose a conically smooth map $X \xrightarrow{\alpha} [0,1)$ for which $X_0 = \alpha^{-1}\{0\}$ – see §1.2. By design, the conically smooth map

$$f\colon X \times S \longrightarrow [0,1)~, \qquad f(x,s) = \phi_0(s)\alpha(x) + \phi_1(s) f_0(x,s)$$

is defined, and has the property that $X_0 \times S = f^{-1}\{0\}$. This concludes the verification that (14) is surjective for the case $Z = *$.

Knowing the case of $Z = *$, we can choose a conically smooth extension $\underline{f}\colon X \times S \to [0,1)$ of $\underline{f_0}$. The resulting conically smooth map

$$f\colon X \times S \longrightarrow \mathsf{C}(Z)~, \qquad (x,s) \mapsto \big(\overline{f}(x,s), \underline{f}(x,s)\big) ~\in~ * \underset{Z\times\{0\}}{\amalg} Z \times [0,1)$$

is a lift of f, thereby implying the desired surjectivity for the case of a general compact Z.

$\square$

Corollary 2.5.5. *For L and Z compact stratified spaces, there is a canonical identification of the space of based morphisms*

$$\mathcal{S}\mathsf{trat}_*\big(\mathbb{R}^i \times \mathsf{C}(L), \mathbb{R}^j \times \mathsf{C}(Z)\big) ~\simeq \coprod_{L' \underset{\mathsf{cbl,opn}}{\subset} L} \mathcal{S}\mathsf{trat}\big(L', Z\big)$$

with the coproduct indexed by sub-stratified spaces $L' \hookrightarrow L$ whose inclusion is constructible and open.

Proof. After Observation 2.4.4, we can assume $i = 0 = j$. Note that constructible closed sub-stratified spaces are exactly the complements of constructible open sub-stratified spaces. The result now follows from Lemma 2.5.4, upon noticing that constructible closed sub-stratified spaces of $\mathsf{C}(L)$ that include the cone-point are in bijection with those of L.

$\square$

Corollary 2.5.6. *For each compact stratified space L, and each stratified space X, the fiber over $x \in X$ of the map of spaces*

$$\mathsf{ev}_0\colon~ \mathcal{S}\mathsf{trat}\big(\mathbb{R}^i \times \mathsf{C}(L), X\big) \longrightarrow X~,$$

given by restriction along the origin is canonically identified as the space

$$\coprod_{L' \underset{\mathsf{cbl,opn}}{\subset} L} \mathcal{S}\mathsf{trat}\big(L', Z\big)$$

where $\big(\mathbb{R}^i \times \mathsf{C}(Z), 0\big) \hookrightarrow (X, x)$ is a basic neighborhood.

Proof. After Observation 2.4.4, we can take $i = 0$. We will prove that, for each neighborhood $x \in O \subset X$, the canonical map of spaces of based maps

$$\mathcal{S}\mathsf{trat}_*\big(\mathsf{C}(L), O\big) \longrightarrow \mathcal{S}\mathsf{trat}_*\big(\mathsf{C}(L), X\big)$$

is an equivalence. Given that links are well-defined (a result of [AFT]), the corollary follows from the identification of Corollary 2.5.5.

Fix a pair $(S, S_0 \subset \partial S)$ consisting of a compact smooth manifold with boundary and a union of components of its boundary, equipped with a conically smooth map $\mathsf{C}(L) \times S_0 \xrightarrow{f_0} O$ whose restriction to $\{0\} \times S_0$ is constantly x. We must show that the map of sets of path components of spaces of based maps relative to f_0

$$\pi_0\Big(\mathcal{S}\mathsf{trat}_*^{\mathsf{rel} f_0}\big(\mathsf{C}(L) \times S, O\big)\Big) \longrightarrow \pi_0\Big(\mathcal{S}\mathsf{trat}_*^{\mathsf{rel} f_0}\big(\mathsf{C}(L) \times S, X\big)\Big)$$

is surjective. Let $\mathsf{C}(L) \times S \xrightarrow{f} X$ be a conically smooth map extending f_0 whose restriction to $\{0\} \times S_0$ is constantly x. In [AFT], it is explained that there exists a conically smooth map $\mathsf{C}(L) \times \mathbb{R} \xrightarrow{\phi} \mathsf{C}(L)$ for which

- $\phi_t = 1_{\mathsf{C}(L)}$ for each $t \leq 0$,
- $\phi_t(0) = 0$ for all $t \in \mathbb{R}$,
- $\phi_s(\mathsf{C}(L)) \subset \phi_t(\mathsf{C}(L))$ whenever $s > t \geq 0$,
- the collection $\big\{\phi_t\big(\mathsf{C}(L)\big) \subset \mathsf{C}(L) \mid t \in \mathbb{R}\big\}$ is a basis for the topology about $0 \in \mathsf{C}(L)$.

By design, the preimage $f^{-1}O\subset \mathsf{C}(L)\times S$ contains a neighborhood of $\mathsf{C}(L)\times S_0\bigcup\{0\}\times S$. So there is a conically smooth map $S\xrightarrow{\epsilon}\mathbb{R}$ taking values in the non-negatives, whose restriction to S_0 is constantly 0, for which the composition $\mathsf{C}(L)\times S\xrightarrow{\phi_\epsilon}\mathsf{C}(L)\times S\xrightarrow{f} X$ factors through O. The family $[0,1]\ni t\mapsto f\circ\phi_{t\epsilon}$ witnesses a stratified homotopy from f to $f\circ\phi_\epsilon$, which verifies the surjectivity. $\square$

Lemma 2.5.7. *For each compact stratified space L, the diagram in* $\mathcal{S}\mathsf{trat}$

$$\begin{tikzcd}
\overline{\mathsf{C}}(\emptyset) \arrow[r, "\overline{\mathsf{C}}(\emptyset\hookrightarrow L)"] \arrow[d, "\{1\}"'] & \overline{\mathsf{C}}(L) \arrow[d, "\{1\}"] \\
\overline{\mathsf{C}}^2(\emptyset) \arrow[r, "\overline{\mathsf{C}}^2(\emptyset\hookrightarrow L)"] & \overline{\mathsf{C}}^2(L)
\end{tikzcd}$$

is a pushout.

Proof. Fix a stratified space Z. We must show that the diagram of spaces

$$\begin{tikzcd}
\mathcal{S}\mathsf{trat}\big(\overline{\mathsf{C}}(\emptyset),Z\big) & \mathcal{S}\mathsf{trat}\big(\overline{\mathsf{C}}(L),Z\big) \arrow[l] \\
\mathcal{S}\mathsf{trat}\big(\overline{\mathsf{C}}^2(\emptyset),Z\big) \arrow[u] & \mathcal{S}\mathsf{trat}\big(\overline{\mathsf{C}}^2(L),Z\big) \arrow[l] \arrow[u]
\end{tikzcd}$$

is a pullback. In the diagram of the statement of the lemma, the horizontal maps have conically smooth regular neighborhoods, manifestly. It follows that the horizontal maps in the above diagram are Kan fibrations, and therefore the underlying spaces of the point-set fibers of the horizontal maps agree with the fibers. Fix a conically smooth map $\gamma\colon \Delta^1\cong\overline{\mathsf{C}}^2(\emptyset)\to Z$, and denote the restriction $z_1\colon *=\overline{\mathsf{C}}(\emptyset)\xrightarrow{\{1\}}\overline{\mathsf{C}}^2(\emptyset)\xrightarrow{\gamma} Z$. We will argue that the map of relative mapping spaces

$$\mathcal{S}\mathsf{trat}^{\mathsf{rel}\gamma}\big(\overline{\mathsf{C}}^2(L),Z\big)\longrightarrow \mathcal{S}\mathsf{trat}^{\mathsf{rel}z_1}\big(\overline{\mathsf{C}}(L),Z\big) \tag{15}$$

is an equivalence. We will do this by repeatedly applying Corollary 2.5.6 to identify various spaces of relative maps from cones.

Denote the restriction $z_0\colon *=\overline{\mathsf{C}}(\emptyset)\xrightarrow{\{0\}}\overline{\mathsf{C}}^2(\emptyset)\xrightarrow{\gamma} Z$, and choose a basic neighborhood $z_0\in\mathbb{R}^i\times\mathsf{C}(K_0)\subset Z$. Corollary 2.5.6 gives the canonical identification

$$\mathcal{S}\mathsf{trat}^{\mathsf{rel}z_0}\big(\overline{\mathsf{C}}^2(\emptyset),Z\big)\ \simeq\ \mathcal{S}\mathsf{trat}\big(\overline{\mathsf{C}}(\emptyset),K_0\big)\amalg\mathcal{S}\mathsf{trat}(\emptyset,K_0)\ .$$

Therefore, the conically smooth map $\gamma\colon\Delta^1=\overline{\mathsf{C}}^2(\emptyset)\to Z$ is classified either by a map $\widetilde{z}_1\colon *=\overline{\mathsf{C}}(\emptyset)\to K_0$, or by the unique map $\emptyset\to K_0$. We examine these two cases separately.

Suppose γ is classified by $\emptyset\to K_0$. In this case, the composite map

$$\mathcal{S}\mathsf{trat}^{\mathsf{rel}\gamma}\big(\overline{\mathsf{C}}^2(L),Z\big)\longrightarrow\mathcal{S}\mathsf{trat}^{\mathsf{rel}z_0}\big(\overline{\mathsf{C}}^2(L),Z\big)\underset{\text{Cor 2.5.6}}{\simeq}\coprod_{C'\underset{\mathsf{cbl,opn}}{\subset}\overline{\mathsf{C}}(L)}\mathcal{S}\mathsf{trat}(C',K_0)$$

factors through those cofactors indexed by those $C'\subsetneq\overline{\mathsf{C}}(L)$ that do not contain the cone-point. Each such constructible open is of the form $C'=L'\times(0,1]\subset\overline{\mathsf{C}}(L)$ for a unique constructible open $L'\subset L$. Furthermore, this factorized map is an equivalence:

$$\mathcal{S}\mathsf{trat}^{\mathsf{rel}\gamma}\big(\overline{\mathsf{C}}^2(L),Z\big)\xrightarrow{\simeq}\coprod_{L'\underset{\mathsf{cbl,opn}}{\subset}L}\mathcal{S}\mathsf{trat}(L'\times(0,1),K_0)\underset{\text{Lem 2.4.4}}{\simeq}\coprod_{L'\underset{\mathsf{cbl,opn}}{\subset}L}\mathcal{S}\mathsf{trat}(L',K_0)\ .$$

That (15) is an equivalence in this case follows immediately from Corollary 2.5.6.

Suppose γ is classified by a map $\widetilde{z}_1 \colon * \to K_0$. In this case, the composite map

$$\mathcal{S}\mathsf{trat}^{\mathsf{rel}\gamma}\big(\overline{\mathsf{C}}^2(L), Z\big) \longrightarrow \mathcal{S}\mathsf{trat}^{\mathsf{rel}z_0}\big(\overline{\mathsf{C}}^2(L), Z\big) \underset{\text{Cor 2.5.6}}{\simeq} \coprod_{C' \underset{\mathsf{cbl,opn}}{\subset} \overline{\mathsf{C}}(L)} \mathcal{S}\mathsf{trat}(C', K_0)$$

factors through the cofactor indexed by $\overline{\mathsf{C}}(L) \subset \overline{\mathsf{C}}(L)$, for this is the only constructible open sub-stratified space that contains the cone-point. Furthermore, this factorized map recognizes $\mathcal{S}\mathsf{trat}^{\mathsf{rel}\gamma}\big(\overline{\mathsf{C}}^2(L), Z\big)$ as the fiber:

$$\begin{tikzcd}
\mathcal{S}\mathsf{trat}^{\mathsf{rel}\gamma}\big(\overline{\mathsf{C}}^2(L), Z\big) \arrow[r] \arrow[d] & \mathcal{S}\mathsf{trat}\big(\overline{\mathsf{C}}(L), K_0\big) \arrow[d, "\mathsf{ev}_*"] \\
* \arrow[r, "\widetilde{z}_1"] & \mathcal{S}\mathsf{trat}(*, K_0);
\end{tikzcd}$$

which is the assertion that the canonical map of spaces

$$\mathcal{S}\mathsf{trat}^{\mathsf{rel}\gamma}\big(\overline{\mathsf{C}}^2(L), Z\big) \xrightarrow{\simeq} \mathcal{S}\mathsf{trat}^{\mathsf{rel}\widetilde{z}_1}\big(\overline{\mathsf{C}}(L), K_0\big)$$

is an equivalence. Choose a basic neighborhood $\widetilde{z}_1 \in \mathbb{R}^j \times \mathsf{C}(K_1) \hookrightarrow K_0$. Such a choice determines a basic neighborhood $z_1 \in \mathbb{R}^{i+j+1} \times \mathsf{C}(K_0) \subset Z$. Corollary 2.5.6 then gives the two canonical identifications

$$\mathcal{S}\mathsf{trat}^{\mathsf{rel}\widetilde{z}_1}\big(\overline{\mathsf{C}}(L), K_0\big) \;\simeq\; \coprod_{L' \underset{\mathsf{cbl,opn}}{\subset} L} \mathcal{S}\mathsf{trat}(L', K_1) \;\simeq\; \mathcal{S}\mathsf{trat}^{\mathsf{rel}z_1}\big(\overline{\mathsf{C}}(L), Z\big)\ .$$

which respect the map (15). So (15) is an equivalence in this case as well.

□

Remark 2.5.8. While the diagram of Lemma 2.5.7 exists in Strat, it is not a pushout therein. In fact, this homotopy colimit has the unusual feature that it is not a equivalent to a point-set colimit in any readily apparent model of spaces. We therefore see Lemma 2.5.7 as emphasizing the role of the localization $\mathsf{Strat}[\mathcal{J}^{-1}] \simeq \mathcal{S}\mathsf{trat}$ in this ultimate characterization of ∞-categories in terms of stratified spaces.

3. Exit paths

We define a functor $\mathsf{Exit} \colon \mathcal{S}\mathsf{trat} \to \mathsf{Cat}_\infty$. For this, we make use of an important cosimplicial stratified space, and use the incarnation of ∞-categories as *complete Segal spaces.*

3.1. Complete Segal spaces.

Definition 3.1.1 (After [Re1])**.** The ∞-category of *complete Segal spaces* is the full ∞-subcategory

$$\mathsf{PShv}^{\mathsf{Segal,cplt}}(\boldsymbol{\Delta}) \;\subset\; \mathsf{PShv}(\boldsymbol{\Delta})$$

consisting of those presheaves $\mathcal{C}$ that satisfy the following two conditions.

(1) **Segal:** For each $0 \leq k \leq p$, $\mathcal{C}$ carries the diagram in $\boldsymbol{\Delta}$

$$\begin{tikzcd}
\{k\} \arrow[r] \arrow[d] & \{k < \cdots < p\} \arrow[d] \\
\{0 < \cdots < k\} \arrow[r] & \{0 < \cdots < p\}
\end{tikzcd}$$

to a pullback diagram of spaces.

(2) **Complete:** The functor $\mathsf{Map}(-,\mathcal{C})\colon \mathsf{PShv}(\mathbf{\Delta})^{\mathsf{op}} \longrightarrow \mathcal{S}\mathsf{paces}$ carries the diagram

$$\begin{tikzcd}
\Delta^{\{0<2\}} \amalg \Delta^{\{1<3\}} \arrow[r] \arrow[d] & \Delta^{\{0<1<2<3\}} \arrow[d] \\
\Delta^{\{0=2\}} \amalg \Delta^{\{1=3\}} \arrow[r] & \ast
\end{tikzcd}$$

to a pullback diagram of spaces.

Regarding each finite non-empty linearly ordered set as a category in a standard manner, and thereafter as an ∞-category, gives a functor

$$\mathbf{\Delta} \longrightarrow \mathsf{Cat}_\infty \ .$$

There results the restricted Yoneda functor

$$\mathsf{Cat}_\infty \longrightarrow \mathsf{PShv}(\mathbf{\Delta}) \ . \tag{16}$$

Theorem 3.1.2 ([Re1])**.** *The functor (16) factors as an equivalence of ∞-categories*

$$\mathsf{Cat}_\infty \xrightarrow{\simeq} \mathsf{PShv}^{\mathsf{Segal,cplt}}(\mathbf{\Delta}) \ .$$

Remark 3.1.3. While we attribute Theorem 3.1.2 to [Re1], that work served as a foundational definition for ∞-*categories* and made use of model categories. Our phrasing of Theorem 3.1.2 is along the lines of [BS], which works internal to quasi-categories.

3.2. **Standard simplices.** Recall the standard cosimplicial topological space $\mathbf{\Delta} \to \mathsf{Top}$ given on objects, and on morphisms, as

$$[p] \mapsto \Delta^p := \Big\{\{0,\dots,p\} \xrightarrow{t} [0,1] \mid \sum_i t_i = 1\Big\} \ , \qquad ([p] \xrightarrow{\rho} [q]) \mapsto \big(t \mapsto (j \mapsto \sum_{\rho(i)=j} t_i)\big)$$

where the rightmost sum is understood to take the value 0 should the indexing set be empty.

Definition 3.2.1 ($\Delta^\bullet$)**.** The *standard* cosimplicial stratified space

$$\mathsf{st}\colon \mathbf{\Delta} \longrightarrow \mathsf{Strat}$$

is given as

$$[p] \mapsto \big(\Delta^p \to [p] \ , \quad t \mapsto \mathsf{Max}\{i \mid t_i \neq 0\}\big) \ .$$

Observation 3.2.2. For each $p \geq 0$, there is an identification $\overline{\mathsf{C}}(\Delta^{p-1}) \cong \Delta^p$ between the closed cone and the standardly stratified simplex.

Corollary 3.2.3. *For each $0 \leq k \leq p$ the commutative diagram in $\mathcal{S}\mathsf{trat}$*

$$\begin{tikzcd}
\Delta^{\{k\}} \arrow[r] \arrow[d] & \Delta^{\{k<\cdots<p-1\}} \arrow[d] \\
\Delta^{\{0<\cdots<k\}} \arrow[r] & \Delta^{\{0<\cdots<p\}}
\end{tikzcd}$$

is a pushout.

Proof. Make use of Observation 3.2.2. Use induction on k. Apply Lemma 2.5.7 for the base case $k = 1$.

□

Lemma 3.2.4. *The composite functor $\mathbf{\Delta} \xrightarrow{\mathsf{st}} \mathsf{Strat} \xrightarrow{c} \mathcal{S}\mathsf{trat}$ is fully faithful.*

Proof. We must show that the map of spaces $\mathbf{\Delta}\big([p],[q]\big) \xrightarrow{\simeq} \mathcal{S}\mathsf{trat}\big(\Delta^p,\Delta^q\big)$ is an equivalence. We do this by induction on p. The assertion is clear for $p=0$, by inspection. Assume $p>0$ and consider the diagram of spaces

$$\begin{tikzcd}
\mathbf{\Delta}\big([p],[q]\big) \arrow[r] \arrow[d] & \mathcal{S}\mathsf{trat}\big(\Delta^p,\Delta^q\big) \arrow[d] \\
\mathbf{\Delta}\big(\{0\},[q]\big) \arrow[r] & \mathcal{S}\mathsf{trat}\big(\Delta^{\{0\}},\Delta^q\big)
\end{tikzcd}$$

in where the vertical maps are the evident restrictions. We have already argued that the bottom horizontal map is an equivalence, so it remains to argue as much for each fiber. Fix a map $\Delta^{\{0\}} \xrightarrow{f} \Delta^q$. Denote by $0 \le i \le q$ the stratum containing the image of f. In the case $i=q$, the fiber of the lefthand vertical map over i is terminal, while the fiber of the righthand vertical map over f is also terminal, by inspection. So assume $i<q$. The fiber of the lefthand vertical map is

$$\mathbf{\Delta}\big(\{1<\cdots<p\},\{i<\cdots<q\}\big) ~\simeq~ \coprod_{0\le k\le p} \mathbf{\Delta}\big(\{k+1<\cdots<p\},\{i+1<\cdots<q\}\big)~.$$

Now, recognize $\Delta^j = \overline{\mathsf{C}}(\Delta^{j-1})$, so that Corollary 2.5.6 canonically identifies the fiber of the righthand vertical map as

$$\coprod_{D_0 \underset{\mathsf{cbl,opn}}{\subset} \Delta^{\{1<\cdots<p\}}} \mathcal{S}\mathsf{trat}\big(D_0,\Delta^{\{i+1<\cdots<q\}}\big)~.$$

Now recognize that each constructible open subspace $D_0 \subset \Delta^{\{1<\cdots<p\}}$ is of the form $(\Delta^{\{1<\cdots<p\}})_{\ge k+1}$ for some $0\le k\le p$. For $0\le k<p$, the collapse map $\{1<\cdots<p\}\to\{k+1<\cdots<p\}$ induces a stratified map $\Delta^{\{1<\cdots<p\}}\to\Delta^{\{k+1<\cdots<p\}}$ that restricts to a stratified map $(\Delta^{\{1<\cdots<p\}})_{\ge k+1}\to \Delta^{\{k+1<\cdots<p\}}$ that is isomorphic to the projection $[0,1)^k\times\Delta^{\{k+1<\cdots<p\}}\to\Delta^{\{k+1<\cdots<p\}}$; and for $k=p$ there is a unique such projection. Summarizing with Observation 2.4.4, we identify the map of fibers over $i\mapsto f$ as a map

$$\coprod_{0\le k\le p}\mathbf{\Delta}\big(\{k+1<\cdots<p\},\{i+1<\cdots<q\}\big)\longrightarrow \coprod_{0\le k\le p}\mathcal{S}\mathsf{trat}\big(\Delta^{\{k+1<\cdots<p\}},\Delta^{\{i+1<\cdots<q\}}\big)~.$$

By inspection, this map respects the coproduct structure, and restricts on each cofactor as that induced by the functor $\mathbf{\Delta}\to\mathcal{S}\mathsf{trat}$. That this map of fibers over $i\mapsto f$ is an equivalence of spaces follows from the inductive hypothesis.

□

3.3. **Exit-paths.** Here we use the functor $\mathbf{\Delta}\to\mathcal{S}\mathsf{trat}$ to define exit-path ∞-categories.

Definition 3.3.1 (Exit)**.** The *exit-path ∞-category* functor is the restricted Yoneda functor

$$\mathsf{Exit}\colon \mathcal{S}\mathsf{trat} \xrightarrow{j} \mathsf{PShv}(\mathcal{S}\mathsf{trat}) \xrightarrow{\mathsf{st}^*} \mathsf{PShv}(\mathbf{\Delta})~.$$

Explicitly, the value $\mathsf{Exit}(X)$ is the simplicial space $[p]\mapsto \mathcal{S}\mathsf{trat}(\Delta^p,X)$, the values of which are incarnated as a Kan complex for which the set of q-simplices is $\mathsf{Strat}(\Delta^p\times\Delta^q_e,X)$.

Remark 3.3.2. As Lemma 3.3.9 we explain that Definition 3.3.1 is consistent with the exit-path ∞-category defined in Appendix §A of [Lu2]. The difference is model specific. Namely, in [Lu2] the exit-path ∞-category is given as a quasi-category, whereas here we present it as a complete Segal space (Corollary 3.3.6).

Observation 3.3.3. For each pair of stratified spaces X and X', the canonical map

$$\mathsf{Exit}(X\times X') \xrightarrow{\simeq} \mathsf{Exit}(X)\times\mathsf{Exit}(X')$$

is an equivalence of simplicial spaces. This is direct from the equivalence $\mathsf{Exit}(-)\simeq\mathcal{S}\mathsf{trat}(\Delta^\bullet,-)$ and using that the stratified space $X\times X'$ is the product of X and X' in the ∞-category $\mathcal{S}\mathsf{trat}$.

Observation 3.3.4. For each smooth manifold M, there is a canonical identification

$$\mathsf{Exit}(M) \ \simeq \ M$$

as the constant simplicial space at the underlying space of M. Indeed, the space of p-simplices of $\mathsf{Exit}(M)$ is the Kan complex $\mathcal{S}\mathsf{trat}(\Delta^p \times \Delta^\bullet_e, M)$. The degeneracy map from the space of 0-simplices $\mathsf{Sing}(M) \simeq \mathcal{S}\mathsf{trat}(\Delta^\bullet_e, M) \xrightarrow{\simeq} \mathcal{S}\mathsf{trat}(\Delta^p \times \Delta^\bullet_e, M)$ is an equivalence of Kan complexes.

Next we identify the spaces of 0- and 1-simplices of $\mathsf{Exit}(X)$.

Lemma 3.3.5. *Let $X = (X \to P)$ be a stratified space. The space of 0-simplices of $\mathsf{Exit}(X)$ is canonically identified*

$$\mathsf{Exit}(X)_{|[0]} \ \simeq \ \coprod_{p\in P} X_p$$

as the coproduct of the underlying spaces of the strata of X. For each pair of strata $X_p, X_{p'} \subset X$, the space of 1-simplices from X_p to $X_{p'}$ is canonically identified

$$\big(X_p \times X_{p'}\big) \underset{\mathsf{Exit}(X)_{|\partial[1]}}{\times} \mathsf{Exit}(X)_{|[1]} \ \simeq \ \mathsf{Link}_{X_p}(X)_{p'}$$

as the underlying space of the p'-stratum of the link of the p-stratum.

Proof. The first statement is direct from the definition of $\mathcal{S}\mathsf{trat}$. Namely, $\mathcal{S}\mathsf{trat}(*, X)$ is the Kan complex $[q] \mapsto \mathsf{Strat}\big(\Delta^q_e, X\big)$. Because Δ^q_e is a connected smooth manifold for each $q \geq 0$, then any map from it to X factors through a stratum of X. More precisely, there is an identification of Kan complexes

$$\mathcal{S}\mathsf{trat}(*, X) \ \cong \ \mathsf{Sing}\Big(\coprod_{p\in P} X_p\Big) \ .$$

Let $x_0 \in X$ be a point, and choose a basic neighborhood $x_0 \in \mathbb{R}^i \times \mathsf{C}(L) \subset X$. Using that $\Delta^q = \overline{\mathsf{C}}(\Delta^{q-1})$, Corollary 2.5.6 canonically identifies the fiber of the evaluation map $\mathsf{Exit}(X)_{|[q]} \xrightarrow{\mathsf{ev}_{\{0\}}} \mathsf{Exit}(X)_{|[0]}$ over $x_0 \in X$ as the space

$$\coprod_{0\leq k\leq q} \mathsf{Exit}(L)_{|\{k+1<\cdots<q\}} \ .$$

As the case $q = 1$, the second statement then follows from the first.

□

Corollary 3.3.6. *The functor $\mathsf{Exit}\colon \mathcal{S}\mathsf{trat} \to \mathsf{PShv}(\boldsymbol{\Delta})$ takes values in complete Segal spaces.*

Proof. Corollary 3.2.3 gives that, for each $0 \leq k \leq p$, the diagram of spaces

$$\begin{tikzcd}
\mathcal{S}\mathsf{trat}(\Delta^{\{0<\cdots<p\}}, X) \arrow[r] \arrow[d] & \mathcal{S}\mathsf{trat}(\Delta^{\{k<\ldots<p\}}, X) \arrow[d] \\
\mathcal{S}\mathsf{trat}(\Delta^{\{0<\cdots<k\}}, X) \arrow[r] & \mathcal{S}\mathsf{trat}(\Delta^{\{k\}}, X)
\end{tikzcd}$$

is a pullback. This verifies the Segal condition for $\mathsf{Exit}(X)$.

Notice that a conically smooth map $\Delta^2 \xrightarrow{\sigma} X$ factors through a single stratum of X if and only if both $\sigma_{|\Delta^{\{0\}}}$ and $\sigma_{|\Delta^{\{2\}}}$ factor through the same stratum of X. It follows that the only retracts in the Segal space $\mathsf{Exit}(X)$ are equivalences. The completeness of $\mathsf{Exit}(X)$ follows.

□

Convention 3.3.7. By way of Corollary 3.3.6, we will henceforth regard Exit as a functor to ∞-categories, in their incarnation as complete Segal spaces (as established by [Re1]).

The colimits examined in §2.5 are preserved by the exit-path functor.

Proposition 3.3.8. *The exit-path functor $\mathsf{Exit} : \mathcal{S}\mathsf{trat} \to \mathsf{Cat}_\infty$ preserves the following colimits.*

(1) *For each covering sieve* $\mathcal{U}$ *of a stratified space* X*, the composite functor*

$$\mathcal{U}^{\triangleright} \to \mathcal{S}\mathsf{trat} \xrightarrow{\mathsf{Exit}} \mathsf{Cat}_{\infty}$$

is a colimit diagram.

(2) *The functor* Exit *carries each pushout diagram in* Strat

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X'' \arrow[d] \\ X' \arrow[r] & X, \end{tikzcd}$$

in which each map is a constructible bundle and the horizontal maps are embeddings, to a colimit diagram of ∞*-categories.*

(3) *For each compact stratified space* L*, the functor* Exit *carries each diagram*

$$\begin{tikzcd} \overline{\mathsf{C}}(\emptyset) \arrow[r, "\overline{\mathsf{C}}(\emptyset \hookrightarrow L)"] \arrow[d, "\{1\}"'] & \overline{\mathsf{C}}(L) \arrow[d, "\{1\}"] \\ \overline{\mathsf{C}}^2(\emptyset) \arrow[r, "\overline{\mathsf{C}}^2(\emptyset \hookrightarrow L)"] & \overline{\mathsf{C}}^2(L) \end{tikzcd}$$

to a colimit diagram of ∞*-categories.*

Proof. **(1)** We use the following sufficient condition for identifying certain colimits in $\mathsf{Cat}_\infty \simeq \mathsf{PShv}^{\mathsf{Segal,cplt}}(\boldsymbol{\Delta})$:

A functor $\mathcal{U}^{\triangleright} \to \mathsf{Cat}_\infty$ is a colimit diagram if the composite functor

$$\mathcal{U}^{\triangleright} \longrightarrow \mathsf{Cat}_\infty \longrightarrow \mathsf{PShv}(\boldsymbol{\Delta})$$

is a colimit diagram.

Let $\mathcal{U} \subset \mathsf{Strat}_{/X}$ be a covering sieve. It is enough to show that, for each $p \geq 0$, the functor

$$\mathcal{U}^{\triangleright} \to \mathsf{Strat} \xrightarrow{\mathsf{Exit}} \mathsf{Cat}_\infty \xrightarrow{\mathsf{ev}_{[p]}} \mathcal{S}\mathsf{paces}$$

is a colimit diagram. We do this by induction on p.

For each stratified space $Y = (Y \to Q)$, Lemma 3.3.5 offers the identification of spaces: $\mathsf{Exit}(Y)_{|[0]} \simeq \mathcal{S}\mathsf{trat}(*, Y) \simeq \coprod_{q \in Q} Y_q$. So the $p = 0$ assertion is that

$$\mathcal{U}^{\triangleright} \longrightarrow \mathsf{Strat} \xrightarrow{(Y \to Q) \mapsto \coprod_{q \in Q} Y_q} \mathsf{Man} \longrightarrow \mathcal{S}\mathsf{paces}$$

is a colimit diagram, where the last arrow is underlying space functor. After Lemma 2.5.1, it is enough to argue that the composite of the first two arrows $\mathcal{U} \to \mathsf{Man}$ generates a covering sieve. This is to say that, for each $q \in Q$, an open cover of $Y = (Y \to Q)$ restricts as an open cover of the stratum Y_q, which is directly the case. This establishes the base case of the induction.

Let $x_0 \in X$ be a point, and choose a basic neighborhood $x_0 \in \mathbb{R}^i \times \mathsf{C}(L) \subset X$. Using that $\Delta^p = \overline{\mathsf{C}}(\Delta^{p-1})$, Corollary 2.5.6 canonically identifies the fiber of the evaluation map $\mathsf{Exit}(X)_{|[p]} \xrightarrow{\mathsf{ev}_{\{0\}}} \mathsf{Exit}(X)_{|[0]}$ over $x_0 \in X$ as the space

$$\coprod_{0 \leq k \leq p} \mathsf{Exit}(L)_{|\{k+1 < \cdots < p\}} \ .$$

For each $q \in Q$, an open cover of $Y = (Y \to Q)$ restricts as an open cover of the link $\mathsf{Link}_{Y_q}(Y)$. This supplies the inductive step.

(2) After **(1)**, because each such X admits an open cover by basics $\mathbb{R}^i \times \mathsf{C}(L)$, using that $\mathcal{S}\mathsf{paces}$ is an ∞-topos, we can assume that $X = \mathbb{R}^i \times \mathsf{C}(L)$ and the square is the standard one witnessing a pushout $\mathbb{R}^i \coprod_{\mathbb{R}^i \times L} \mathbb{R}^i \times L \times [0,1) \cong \mathbb{R}^i \times \mathsf{C}(L)$. Because Exit factors through $\mathcal{S}\mathsf{trat}$, Observation 2.4.4

offers the reduction to the case $i = 0$. Through Lemma 3.3.5 one calculates $\mathsf{Exit}\big(\mathsf{C}(L)\big) \simeq \mathsf{Exit}(L)^{\triangleleft}$. Observation 3.3.3 is the equivalence $\mathsf{Exit}\big(L \times [0,1)\big) \simeq \mathsf{Exit}(L) \times \mathsf{Exit}\big([0,1)\big)$. The result follows, by identifying a pushout in ∞-categories.

(3) Point **(2)** gives an identification $\mathsf{Exit}\big(\overline{\mathsf{C}}(K)\big) \simeq \mathsf{Exit}(K)^{\triangleleft} := * \underset{\mathsf{Exit}(K)\times\{0\}}{\amalg} \mathsf{Exit}(K) \times [1]$. The result follows by calculating a double pushout in ∞-categories.

□

For the next result we reference, for $X = (X \to P)$ a stratified space, the exit-path quasi-category $\mathsf{Sing}^P(X)$ defined in Appendix A of [Lu2].

Lemma 3.3.9. *For each stratified space $X = (X \to P)$, there is an equivalence of ∞-categories*

$$\mathsf{Exit}(X) \ \simeq \ \mathsf{Sing}^P(X)\ .$$

Proof. By construction, the stratified space X is *conically stratified* in the sense defined in Appendix A of [Lu2]. It follows from those results that $\mathsf{Sing}^P(X)$ is a quasi-category. We identify the complete Segal space associated to this quasi-category as $\mathsf{Exit}(X)$. Explicitly, the complete Segal space associated to $\mathsf{Sing}^P(X)$ has as its space of p-simplices the space of functors $[p] \to \mathsf{Sing}^P(X)$, which we identify as the Kan complex $\Delta^p \times \Delta^\bullet_e \to X$, which is the space of p-simplices of $\mathsf{Exit}(X)$. These identifications respect the simplicial structure maps.

□

Remark 3.3.10. The reader can compare the preceding proof that $\mathsf{Exit}(K)$ is a complete Segal space with the subdivision-based proof from §A.6 [Lu2] that the exit-path simplicial set $\mathsf{Sing}^P(X)$ is a quasi-category. This indicates our point of view that sheaves on stratified spaces offer a tractable avenue for constructing ∞-categories by hand from geometry, where existence of regular neighborhoods make open covers often more manageable than subdivisions.

This provides the following connection to constructible sheaves.

Corollary 3.3.11. *For each stratified space X there is an equivalence*

$$\mathsf{Fun}(\mathsf{Exit}(X), \mathcal{S}\mathsf{paces}) \ \simeq \ \mathsf{Shv}^{\mathsf{cbl}}(X)$$

between copresheaves on the exit-path ∞-category of X and sheaves on X which are constructible with respect to the given stratification.

Proof. This is immediate after Lemma 3.3.9, using the results of §A.9 from [Lu2]. □

4. Striation sheaves

In this section we introduce *striation sheaves*. These are constructible sheaves on stratified spaces that satisfy an additional locality with respect to blow-ups along closed substratified spaces and iterated cones.

4.1. **Localities for stratified spaces.** We define striation sheaves.

Definition 4.1.1. The ∞-category of *striation sheaves* is the full ∞-subcategory of space-valued presheaves on Strat

$$\mathcal{S}\mathsf{tri} \ \subset \ \mathsf{PShv}(\mathsf{Strat})$$

consisting of those $\mathcal{F}$ that satisfy the following properties.

1. **Sheaf:** For each covering sieve $\mathcal{U} \subset \mathsf{Strat}_{/K}$, the restriction of $\mathcal{F}$ along the adjoint diagram $\mathcal{U}^{\triangleright} \to \mathsf{Strat}$ is a limit diagram of spaces.
2. **Constructible:** For each stratified space K, the value of $\mathcal{F}$ on the projection $K \times \mathbb{R} \to K$ is an equivalence of spaces.

(3) **Cone-local:** For each compact stratified space L, the value of $\mathcal{F}$ on the diagram of stratified spaces

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L \times \mathbb{R}_{\geq 0} \arrow[d] \\ * \arrow[r] & \mathsf{C}(L) \end{tikzcd}$$

is a pullback diagram of spaces.

(4) **Consecutive:** For each $p > 0$, the value of $\mathcal{F}$ on the diagram of stratified spaces

$$\begin{tikzcd} \Delta^{\{1\}} \arrow[r] \arrow[d] & \Delta^{\{1<\cdots<p\}} \arrow[d] \\ \Delta^{\{0<1\}} \arrow[r] & \Delta^{\{0<\cdots<p\}} \end{tikzcd}$$

is a pullback diagram of spaces.

(5) **Univalent:** The value of $\mathcal{F}$ on the diagram of stratified spaces

$$\begin{tikzcd} \Delta^{\{0<2\}} \amalg \Delta^{\{1<3\}} \arrow[r] \arrow[d] & \Delta^{\{0<1<2<3\}} \arrow[d] \\ \Delta^{\{0=2\}} \amalg \Delta^{\{1=3\}} \arrow[r] & * \end{tikzcd}$$

is a pullback diagram of spaces.

In §5, we describe a technique for constructing striation sheaves from point-set data.

Remark 4.1.2. The latter four of our conditions can be naturally strengthened.

(2) **Constructible (strong version):** The value of $\mathcal{F}$ on each stratified homotopy equivalence $X \to Y$ is an equivalence of spaces.

(3) **Cone-local (strong version):** For each stratified space K with a deepest stratum $K_d \subset K$, the value of $\mathcal{F}$ on the blow-up square

$$\begin{tikzcd} \mathsf{Link}_{K_d}(K) \arrow[r] \arrow[d] & \mathsf{Unzip}_{K_d}(K) \arrow[d] \\ K_d \arrow[r] & K \end{tikzcd}$$

is a pullback diagram of spaces.

(4) **Consecutive (strong version):** For each compact stratified space L, the value of $\mathcal{F}$ on the diagram

$$\begin{tikzcd} \overline{\mathsf{C}}(\emptyset) \arrow[r] \arrow[d] & \overline{\mathsf{C}}(L) \arrow[d] \\ \overline{\mathsf{C}}^2(\emptyset) \arrow[r] & \overline{\mathsf{C}}^2(L) \end{tikzcd}$$

is a pullback diagram of spaces.

(5) **Univalent (strong version):** Consider the simplicial set $E[p]$ that is the nerve of the connected groupoid whose underlying set is $\{0, \ldots, p\}$. By way of $\mathbf{\Delta} \xrightarrow{\Delta^\bullet} \mathsf{Strat}$ there results a simplicial stratified space E^p. We extend $\mathcal{F}$ to simplicial stratified spaces via right Kan extension $\mathcal{F}(Z_\bullet) := \lim_{\Delta^q \to Z_\bullet} \mathcal{F}(\Delta^q)$. For each stratified space K, and for each $p \geq 0$, the value of $\mathcal{F}$ on the projection $K \times E^p \to K$ is an equivalence.

Each of these conditions specializes to their weaker versions, and so a sheaf on Strat satisfying these strong conditions is in particular a striation sheaf. Conversely, every striation sheaf automatically satisfies these strengthened conditions.

Remark 4.1.3. Each of the defining properties of a striation sheaf $\mathcal{F}$ gives a conceptual reduction of information.

- The **sheaf** condition implies $\mathcal{F}$ is determined by its values on basic singularity types: $\mathbb{R}^i \times \mathsf{C}(L)$.
- The **constructible** condition implies $\mathcal{F}$ factors through $\mathcal{S}\mathsf{trat}$. Together these conditions imply $\mathcal{F}$ is determined by its values on cones.
- The **cone-local** condition implies the value of $\mathcal{F}$ near a singularity is determined by the value of $\mathcal{F}$ on the link of the singularity. Together these conditions imply $\mathcal{F}$ is determined by its values on standard simplices, $\Delta^p = \overline{\mathsf{C}}^{p+1}(\emptyset)$, which is to say $\mathcal{F}$ is equivalent data as a simplicial space.
- The **consecutive** condition implies equivalences among such $\mathcal{F}$ are detected by their values on $*$ and on Δ^1. Together these conditions imply the simplicial space characterizing $\mathcal{F}$ is a Segal space.
- The **univalent** condition implies equivalences among such $\mathcal{F}$ are detected by their values on Δ^1 together wtih surjectivity of components of their values on $*$. Together these conditions imply that the Segal space characterizing $\mathcal{F}$ is *complete*.

4.2. **Characterization.** In proving our characterization of striation sheaves, we will need the following result, which asserts that the value of a striation sheaf $\mathcal{F}$ on the cone of a stratified space L is determined by the values of $\mathcal{F}$ on the cones of a cover of L.

Lemma 4.2.1. *Let $\overline{\mathcal{U}}$ be a collection of conically smooth maps $\mathbb{D}^j \times \overline{\mathsf{C}}(W) \to L$ for which the collection $\mathcal{U}$ consisting of the precompositions with interiors $\mathbb{R}^j \times \mathsf{C}(W) \to L$ forms an open hypercover of L. For each cone-local constructible sheaf $\mathcal{F}$, and each $p \geq 0$, the canonical map of spaces*

$$\mathcal{F}\big(\overline{\mathsf{C}}^p(L)\big) \xrightarrow{\simeq} \lim_{\overline{U} \in \overline{\mathcal{U}}} \mathcal{F}\big(\overline{\mathsf{C}}^p(\overline{U})\big)$$

is an equivalence.

Proof. We proceed by induction on $p \geq 0$. The case $p = 0$ is immediate from Lemma 1.2.4. We assume the case p and deduce the case $p+1$ from the following sequence of canonical equivalences:

$$\begin{aligned}
\mathcal{F}\big(\mathsf{C}(\overline{\mathsf{C}}^p(L))\big) &\xrightarrow{\simeq} \mathcal{F}(*) \underset{\mathcal{F}\left(\overline{\mathsf{C}}^p(L)\right)}{\times} \mathcal{F}\big(\overline{\mathsf{C}}^p(L) \times \mathbb{R}_{\geq 0}\big) \\
&\xrightarrow{\simeq} \mathcal{F}(*) \underset{\lim_{\overline{U} \in \overline{\mathcal{U}}} \mathcal{F}\left(\overline{\mathsf{C}}^p(\overline{U})\right)}{\times} \lim_{\overline{U} \in \overline{\mathcal{U}}} \mathcal{F}\big(\overline{\mathsf{C}}^p(\overline{U}) \times \mathbb{R}_{\geq 0}\big) \\
&\simeq \lim_{\overline{U} \in \overline{\mathcal{U}}} \Big(\mathcal{F}(*) \underset{\mathcal{F}\left(\overline{\mathsf{C}}^p(\overline{U})\right)}{\times} \mathcal{F}\big(\overline{\mathsf{C}}^p(\overline{U}) \times \mathbb{R}_{\geq 0}\big)\Big) \\
&\xleftarrow{\simeq} \lim_{\overline{U} \in \overline{\mathcal{U}}} \mathcal{F}\big(\mathsf{C}(\overline{\mathsf{C}}^p(\overline{U}))\big)
\end{aligned}$$

The first map is an equivalence because $\mathcal{F}$ is cone-local. The second map is an equivalence by the inductive hypothesis for p, using that the collection $\big\{\overline{\mathsf{C}}^p(\overline{U})\big\}$ of subspaces of $\overline{\mathsf{C}}^p(L)$, as well as the collection $\big\{\overline{\mathsf{C}}^p(\overline{U}) \times \mathbb{R}_{\geq 0}\big\}$ of subspaces of $\overline{\mathsf{C}}^p(L) \times \mathbb{R}_{\geq 0}$, is of the form to which the statement of the lemma applies. The third map is an equivalence because, formally, limits commute. The fourth map is an equivalence because $\mathcal{F}$ is cone-local. Finally, Lemma 2.1.3 gives that the canonical map $\mathcal{F}\big(\mathsf{C}(\overline{\mathsf{C}}^p(Z))\big) \xrightarrow{\simeq} \mathcal{F}\big(\overline{\mathsf{C}}^{p+1}(Z)\big)$ is an equivalence for each stratified space Z.

□

We now prove the main result of this section: ∞-categories are striation sheaves. Recall the functor $\mathsf{st}\colon \mathbf{\Delta} \to \mathcal{S}\mathsf{trat}$ from §1. There results an adjunction

$$\mathsf{st}^*\colon \mathsf{PShv}(\mathcal{S}\mathsf{trat}) \rightleftarrows \mathsf{PShv}(\mathbf{\Delta})\colon \mathsf{st}_* \tag{17}$$

given by restriction and right Kan extension. Explicitly, this right Kan extension evaluates on $\mathcal{F}$ as

$$\mathsf{st}_*\mathcal{F}\colon X \mapsto \mathsf{Map}_{\mathsf{PShv}(\mathbf{\Delta})}\big(\mathsf{Exit}(X),\mathcal{F}\big)~. \tag{18}$$

Through Theorem 2.4.7, we identify the cone-local constructible sheaves $\mathsf{Shv}^{\mathsf{cone},\mathsf{cbl}}(\mathsf{Strat}) \subset \mathsf{Shv}(\mathcal{S}\mathsf{trat})$ as a full ∞-subcategory.

Lemma 4.2.2. *The adjunction (17) restricts as an equivalence of ∞-categories*

$$\mathsf{Shv}^{\mathsf{cone},\mathsf{cbl}}(\mathsf{Strat})~\simeq~\mathsf{PShv}(\mathbf{\Delta})~.$$

Proof. Lemma 3.2.4 grants that st_* is fully faithful. It remains to verify that the unit $\mathcal{F}\to \mathsf{st}_*\mathsf{st}^*\mathcal{F}$ is an equivalence if and only if $\mathcal{F}$ is a cone-local constructible sheaf.

We first point out that st_* takes values in cone-local constructible sheaves. Constructibility follows upon inspecting (18) because the projection $X\times\mathbb{R}\to X$ induces an equivalence $\mathsf{Exit}(X\times\mathbb{R})\xrightarrow{\simeq}\mathsf{Exit}(X)$, by definition of $\mathcal{S}\mathsf{trat}$. By inspecting (18), Proposition 3.3.8 gives that $\mathsf{st}_*\mathcal{F}$ is cone-local.

It remains to prove that this unit map $\mathcal{F}\to\mathsf{st}_*\mathsf{st}^*\mathcal{F}$ is an equivalence whenever $\mathcal{F}$ is a cone-local constructible sheaf. So let $\mathcal{F}$ be a cone-local constructible sheaf. By definition, each stratified space admits a hypercover by basic singularity types; so it is enough to show that, for each integer i and each compact stratified space Z, that the unit map $\mathcal{F}\big(\mathbb{R}^i\times\mathsf{C}(Z)\big)\to\mathsf{st}_*\mathsf{st}^*\mathcal{F}\big(\mathbb{R}^i\times\mathsf{C}(Z)\big)$ is an equivalence of spaces. Because $\mathcal{F}$ is constructible, then it is enough to show this for the case $i=0$.

So let Z be a compact stratified space. Consider the maximal $p\geq 0$ for which $Z\cong\overline{\mathsf{C}}^p(L)$ for some compact stratified space L. We proceed by downward induction on p. In the case $p>\mathsf{dim}(Z)$ then necessarily $L=\emptyset$ and $Z=\overline{\mathsf{C}}^{d+1}(\emptyset)=\Delta^d$, so the unit is an equivalence because st_* is fully faithful. We now give the inductive step.

From the proof of Lemma 1.2.4, there is a collection $\overline{\mathcal{U}}$ of conically smooth maps $\mathbb{D}^j\times\overline{\mathsf{C}}(W)\to L$ for which the collection $\mathcal{U}$ consisting of the precompositions with interiors $\mathbb{R}^j\times\mathsf{C}(W)\to L$ forms an open hypercover of L. So, through Lemma 4.2.1, we can assume $L=\mathbb{D}^j\times\overline{\mathsf{C}}(W)$ for some compact stratified space W and integer j. Because both $\mathcal{F}$ and $\mathsf{st}_*\mathsf{st}^*\mathcal{F}$ are constructible, we can assume $j=0$, so that $L=\overline{\mathsf{C}}(W)$. This case $Z=\overline{\mathsf{C}}^p(L)=\overline{\mathsf{C}}^{p+1}(W)$ follows by induction on p.

□

Theorem 4.2.3. *The adjunction (17) restricts to an equivalence of ∞-categories*

$$\mathcal{S}\mathsf{tri}~\simeq~\mathsf{Cat}_\infty~.$$

Proof. Through the equivalence $\mathsf{Shv}^{\mathsf{cone},\mathsf{cbl}}(\mathsf{Strat})\simeq\mathsf{PShv}(\mathbf{\Delta})$ of Lemma 4.2.2, the consecutive condition is identical to the Segal condition. Through the resulting equivalence $\mathsf{Shv}^{\mathsf{cons},\mathsf{cone},\mathsf{cbl}}(\mathsf{Strat})\simeq\mathsf{PShv}^{\mathsf{Segal}}(\mathbf{\Delta})$, the univalent condition is identical to the completeness condition. □

Remark 4.2.4. It was a choice to prove Theorem 4.2.3 by way of the *standard* cosimplicial stratified space $\mathsf{st}\colon\mathbf{\Delta}\to\mathsf{Strat}$ as opposed to the stratification of the topological simplices $\Delta^p\to[p]$ given by $t\mapsto\mathsf{Min}\big\{i\mid t_i\neq 0\big\}$. This choice resulted in our use of the exit-path ∞-category rather than its opposite, the enter-path ∞-category.

Together with the identifications of colimits in $\mathcal{S}\mathsf{trat}$ of §2.5, Theorem 2.4.7 has the following consequence.

Theorem 4.2.5. *The exit-path functor*

$$\mathsf{Exit}:\mathcal{S}\mathsf{trat}\longrightarrow\mathsf{Cat}_\infty$$

is fully faithful.

Proof. Consider the restricted Yoneda functor

$$\mathcal{S}\mathsf{trat}\longrightarrow\mathsf{PShv}(\mathcal{S}\mathsf{trat})\longrightarrow\mathsf{PShv}(\mathsf{Strat})~.$$

Theorem 2.4.7 grants that the second arrow is fully faithful. So, after Theorem 4.2.3, we need only verify that the essential image of this composite functor lies in $\mathcal{S}\mathsf{tri}$. Corollary 2.5.3, Lemma 2.5.1, and Corollary 3.2.3 grant that the image consists of constructible presheaves on Strat that satisfy the conditions **sheaf**, **cone-local**, and **consecutive**, respectively. Because each Segal space $\mathsf{Exit}(X)$ is complete, it follows lastly that objects in the image satisfy the **univalent** condition.

□

5. Transversality sheaves

Here we give a procedure for manufacturing examples of striation sheaves, hence ∞-categories, using even simpler stratified geometry mixed with ordinary category theory.

5.1. **Transversality sheaves.**

We now explicate checkable conditions that guarantee that the topologizing diagram of an isotopy sheaf is a striation sheaf.

Definition 5.1.1 (Transversality sheaves)**.** A transversality sheaf is a right fibration $\mathfrak{F} \to \mathsf{Strat}$ among ordinary categories for which its topologizing diagram $|\mathfrak{F}(- \times \Delta^\bullet_e)|$ is a striation sheaf. The category of transversality sheaves is the full subcategory

$$\mathsf{Trans} \ \subset\ \mathsf{Cat}_{/\,\mathsf{Strat}}$$

consisting of transversality sheaves.

We now give practicable point-set sufficient conditions for checking that a right fibration is a transversality sheaf.

Theorem 5.1.2. *If a right fibration $\mathfrak{F} \to \mathsf{Strat}$ satisfies the following conditions, it is a transversality sheaf, i.e., $|\mathfrak{F}(- \times \Delta^\bullet_e)|$ is a striation sheaf.*

(1) **Sheaf:** *For each covering sieve $\mathcal{U} \subset \mathsf{Strat}_{/K}$, the restriction of $\mathfrak{I}$ along the adjoint diagram $\mathcal{U}^{\triangleright} \to \mathsf{Strat}$ is a limit diagram of groupoids: for each covering sieve $\mathcal{U}$ of K, the canonical map of groupoids*

$$\mathfrak{I}(K) \xrightarrow{\simeq} \lim_{U \in \mathcal{U}} \mathfrak{I}(U)$$

is an equivalence.

(2) **Isotopy extension:** *For each weakly regular subspace $K_0 \subset K$, the canonical functor of groupoids*

$$\mathfrak{I}(K) \longrightarrow \mathfrak{I}(K_0)$$

is surjective on Hom-sets.

(3) **Isotopy equivalence:** *For each stratified space K, the projection $K \times \mathbb{R} \to K$ induces a bijection of isomorphism classes*

$$\pi_0\mathfrak{I}(K) \xrightarrow{\cong} \pi_0\mathfrak{I}(K \times \mathbb{R}) \ .$$

(4) **Cone-local:** *For each compact stratified space L, the value of $\mathfrak{F}$ on the diagram*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L \times \mathbb{R}_{\geq 0} \arrow[d] \\ \ast \arrow[r] & \mathsf{C}(L) \end{tikzcd}$$

is a limit diagram of groupoids.

(5) **Consecutive:** *For each $k \leq p$, the canonical map of groupoids*

$$\mathfrak{F}(\Delta^p) \longrightarrow \mathfrak{F}(\Delta^{\{0<\cdots<k\}}) \underset{\mathfrak{F}(\Delta^{\{k\}})}{\times} \mathfrak{F}(\Delta^{\{k<\cdots<p\}})$$

is essentially surjective and surjective on Hom-sets.

(6) **Univalent:** *The topologizing diagram* $|\mathfrak{F}(- \times \Delta^\bullet_e)|$ *carries the diagram in* Strat

$$\begin{tikzcd}
\Delta^{\{0<2\}} \amalg \Delta^{\{1<3\}} \arrow[r] \arrow[d] & \Delta^{\{0<1<2<3\}} \arrow[d] \\
\Delta^{\{0=2\}} \amalg \Delta^{\{1=3\}} \arrow[r] & *
\end{tikzcd}$$

to a pullback diagram of spaces.

Proof. Notice that the first three conditions coincide with those of an isotopy sheaf. So Theorem 2.3.7 grants that the topologizing diagram $|\mathfrak{F}(- \times \Delta^\bullet_e)|$ is a constructible sheaf. Clearly, the **univalent** condition on $\mathfrak{F}$ equals the univalent condition for $|\mathfrak{F}(- \times \Delta^\bullet_e)|$. It remains to show that **cone-locality** and **consecutivity** for $\mathfrak{F}$ imply the corresponding conditions for its topologizing diagram $|\mathfrak{F}(- \times \Delta^\bullet_e)|$. As in the proof of Theorem 2.3.7, we replace $\mathfrak{F}$, up to equivalence, by a presheaf on Strat valued in groupoids for which, for each weakly regular subspace $K_0 \subset K$, the restriction $\mathfrak{F}(K) \to \mathfrak{F}(K_0)$ is an isofibration.

Cone-local: From the cone-local condition, and since, for each $q \geq 0$, taking products with Δ^q_e commutes with pushouts, we have that the canonical diagram of groupoids

$$\begin{tikzcd}
\mathfrak{F}(\mathsf{C}(L) \times \Delta^p_e) \arrow[r] \arrow[d] & \mathfrak{F}(L \times \mathbb{R}_{\geq 0} \times \Delta^p_e) \arrow[d] \\
\mathfrak{F}(\Delta^p_e) \arrow[r] & \mathfrak{F}(L \times \Delta^p_e)
\end{tikzcd}$$

is a point-set pullback. Taking nerves and restricting to diagonals preserves limits, therefore the canonical diagram of Kan complexes

$$\begin{tikzcd}
\delta^*\mathfrak{F}(\mathsf{C}(L) \times \Delta^\bullet_e)_\star \arrow[r] \arrow[d] & \delta^*\mathfrak{F}(L \times \mathbb{R}_{\geq 0} \times \Delta^\bullet_e)_\star \arrow[d] \\
\delta^*\mathfrak{F}(\Delta^\bullet_e)_\star \arrow[r] & \delta^*\mathfrak{F}(L \times \Delta^\bullet_e)_\star
\end{tikzcd} \tag{19}$$

is a point-set pullback. From the proof of Theorem 2.3.7, the restriction map

$$\delta^*\mathfrak{F}(L \times \mathbb{R}_{\geq 0} \times \Delta^\bullet_e)_\star \longrightarrow \delta^*\mathfrak{F}(L \times \Delta^\bullet_e)_\star$$

is a Kan fibration. The diagram (19) is therefore a homotopy pullback among Kan complexes in Quillen's model structure on simplicial sets. In particular, the diagram (19) gives a pullback diagram in the ∞-category $\mathcal{S}\mathsf{paces}$.

Consecutive: We now show $|\mathfrak{F}(- \times \Delta^\bullet_e)| : \mathsf{Strat}^{\mathsf{op}} \to \mathcal{S}\mathsf{paces}$ is consecutive. That is, we show that the composite

$$\mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{st}} \mathsf{Strat}^{\mathsf{op}} \xrightarrow{|\mathfrak{F}(- \times \Delta^\bullet_e)|} \mathcal{S}\mathsf{paces}$$

is a Segal space. Fix $0 \leq k \leq p$, with associated maps $\Delta^k \to \Delta^p$ and $\Delta^{p-k} \to \Delta^p$ induced by the inclusions of $[k] = \{0 < \ldots k\}$ and $[p-k] = \{k < \ldots < p\}$ into $[p] = \{0 < \ldots < p\}$. We will show that the square among bisimplicial sets

$$\begin{tikzcd}
\mathfrak{F}(\Delta^p \times \Delta^\bullet_e)_\star \arrow[r] \arrow[d] & \mathfrak{F}(\Delta^{p-k} \times \Delta^\bullet_e)_\star \arrow[d] \\
\mathfrak{F}(\Delta^{\leq k} \times \Delta^\bullet_e)_\star \arrow[r] & \mathfrak{F}(\Delta^{\{k\}} \times \Delta^\bullet_e)_\star
\end{tikzcd}$$

is a homotopy pullback in the diagonal model structural. The limit of the diagram is

$$\mathfrak{F}\Big(\Delta^k \underset{\Delta^{\{k\}}}{\cup} \Delta^{p-k} \times \Delta^\bullet_e\Big)_\star$$

so we show that the natural map

$$\mathfrak{F}(\Delta^p \times \Delta^\bullet_e)_\star \longrightarrow \mathfrak{F}\Big(\Delta^k \underset{\Delta^{\{k\}}}{\cup} \Delta^{p-k} \times \Delta^\bullet_e\Big)_\star$$

is an acyclic fibration for the diagonal model structure on bisimplicial sets. By [Ja], acyclic fibrations for the diagonal model structure are detected by having the right lifting property with respect to the cofibrations

$$\partial\Delta[q] \boxtimes \Delta[r] \coprod_{\partial\Delta[q]\boxtimes\partial\Delta[r]} \Delta[q] \boxtimes \partial\Delta[r] \hookrightarrow \Delta[q] \boxtimes \Delta[r] \tag{20}$$

where $\boxtimes : \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set}) \times \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set}) \to \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}} \times \mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set})$ is the external product of simplicial sets. In the case at hand, for each stratified space Z and each $q \geq 0$, the simplicial set $\mathfrak{I}(Z \times \Delta^q_e)_\star$ is 1-coskeletal, since it is the nerve of a groupoid. Thus, it suffices to verify the right lifting property in the cases $r = 0, 1$.

First, note that the subspace $\Delta^k \bigcup_{\Delta^{\{k\}}} \Delta^{p-k} \subset \Delta^p$ is weakly regular and, moreover, a stratified homotopy equivalence as a consequence of Corollary 3.2.3.

Case $r = 0$**:** We solve the lifting problem

$$\begin{tikzcd}
\partial\Delta[q] \boxtimes \Delta[0] \arrow[r] \arrow[d] & \mathfrak{F}(\Delta^p \times \Delta^\bullet_e)_\star \arrow[d] \\
\Delta[q] \boxtimes \Delta[0] \arrow[r] \arrow[ur, dashed] & \mathfrak{F}\Big(\Delta^k \bigcup_{\Delta^{\{k\}}} \Delta^{p-k} \times \Delta^\bullet_e\Big)_\star .
\end{tikzcd}$$

That is, given an object of the groupoid

$$\mathfrak{F}(\Delta^p \times \partial\Delta^q_e) \qquad \text{and an object of} \qquad \mathfrak{F}\Big(\Delta^k \underset{\Delta^{\{k\}}}{\cup} \Delta^{p-k} \times \Delta^q_e\Big)$$

whose images in

$$\mathfrak{F}\Big(\Delta^k \underset{\Delta^{\{k\}}}{\cup} \Delta^{p-k} \times \partial\Delta^q_e\Big)$$

are identified, then we lift this to an object of $\mathfrak{F}(\Delta^p \times \Delta^q_e)$. From the sheaf property, this reduces to showing that the functor induced by $\mathfrak{F}$ on the inclusion of subspaces

$$\Delta^p \times \partial\Delta^q_e \bigcup_{\Delta^k \underset{\Delta^{\{k\}}}{\cup} \Delta^{p-k} \times \partial\Delta^q_e} \Delta^k \underset{\Delta^{\{k\}}}{\cup} \Delta^{p-k} \times \Delta^q_e \subset \Delta^p \times \Delta^q_e \tag{21}$$

is surjective on objects. Note this is weakly regular and a stratified homotopy equivalence. Since the restriction is an isofibration, it suffices to show that the map $\pi_0\mathfrak{F}$ is surjective when applied to the inclusion above. By isotopy equivalence, there is an equivalence of sets $\pi_0\mathfrak{F}(X) \simeq \pi_0|\mathfrak{F}(X \times \Delta^\bullet_e)|$ for all X. It therefores suffices to show that applying the topologizing diagram $|\mathfrak{F}(- \times \Delta^\bullet_e)|$ to (21) induces a bijection on π_0. Since $\mathfrak{F}$ is an isotopy sheaf, the topologizing diagram is a constructible sheaf. Since (21) is a stratified homotopy equivalence, our identification of the localization of Strat with respect to stratified homotopy equivalences implies that $|\mathfrak{F}(- \times \Delta^\bullet_e)|$ applied to (21) is an equivalence. In particular, it is a bijection on $\pi_0\mathfrak{F}$.

Case $r = 1$**:** We solve the lifting problem

$$\begin{tikzcd}
\partial\Delta[q] \boxtimes \Delta[1] \coprod_{\partial\Delta[q]\boxtimes\partial\Delta[1]} \Delta[q] \boxtimes \partial\Delta[1] \arrow[r] \arrow[d] & \mathfrak{F}(\Delta^p \times \Delta^\bullet_e)_\star \arrow[d] \\
\Delta[q] \boxtimes \Delta[1] \arrow[r] \arrow[ur, dashed] & \mathfrak{F}\Big(\Delta^k \bigcup_{\Delta^{\{k\}}} \Delta^{p-k} \times \Delta^\bullet_e\Big)_\star .
\end{tikzcd}$$

Equivalently, we are given a morphism in the groupoid

$$\mathfrak{F}\Big(\Delta^k \bigcup_{\Delta^{\{k\}}} \Delta^{p-k} \times \Delta^q_e\Big)$$

and a compatible morphism in the groupoid $\mathfrak{F}(\Delta^p \times \partial\Delta^q_e)$ together with lifts of the source and target to $\mathfrak{F}(\Delta^p \times \Delta^q_e)$. We are required to lift the morphism with prescribed source and target. Reformulating using the sheaf property of $\mathfrak{F}$, we have a morphism in the groupoid

$$\mathfrak{F}\Big(\Delta^p \times \partial\Delta^q_e \bigcup_{\Delta^k \underset{\Delta^{\{k\}}}{\cup} \Delta^{p-k}\times\partial\Delta^q_e} \Delta^k \underset{\Delta^{\{k\}}}{\cup} \Delta^{p-k} \times \Delta^q_e\Big)$$

and lifts of the source and target to $\mathfrak{F}(\Delta^p \times \Delta^q_e)$. The assertion now follows from the isotopy extension property assuring surjectivity on Hom-sets, again using that the inclusion (21) is weakly regular.

□

Remark 5.1.3. Restriction along $\mathbf{\Delta} \xrightarrow{\Delta^\bullet} \mathsf{Strat}$ gives the functor $\mathsf{PShv}(\mathsf{Strat}) \to \mathsf{PShv}(\mathbf{\Delta})$ to simplicial spaces. By inspection, the map of cosimplicial stratified spaces $\Delta^\bullet \to \Delta^\bullet_e$, which is the standard map on underlying topological spaces, factors as

$$\Delta^\bullet \longrightarrow E^\bullet \longrightarrow \Delta^\bullet_e$$

where E^p is the simplicial stratified space $[r] \mapsto \coprod_{[r]\to E^p} \Delta^r$. There results a natural transformation

$$|\mathcal{F}(- \times \Delta^\bullet_e)|_{|\mathbf{\Delta}^{\mathsf{op}}} \longrightarrow |\mathcal{F}(- \times E^\bullet)|$$

from the restriction of the topologizing diagram to the classifying diagram. After Rezk's reformulation of the completeness condition for Segal spaces (see §10 of [Re2]), our **univalence** condition gives that this natural transformation is an equivalence on *transversality sheaves.*

6. Constructible bundles

We will think of the category Strat as a category of parametrizing objects for stratified manifold structures, just as affine schemes parametrize algebro-geometric structures. In this section, we investigate a universal such example, $\mathcal{B}\mathsf{un}$, for which a K-point is a constructible bundle $X \to K$. This universal striation sheaf specializes to many interesting situations. For instance, from it one can recover the ∞-category of smooth n-manifolds and smooth embeddings among them, for any n, as well as the simplicial category $\mathbf{\Delta}$ and the category of based finite sets.

6.1. **Closure properties of constructible bundles.**

We verify that constructible bundles are closed under composition and pullback, among other formations. We first recall the definition.

Definition 6.1.1. A conically smooth map $X \to K$ is a *fiber bundle* if there is a basis for the topology of Y consisting of images $\phi(O) \subset Y$ which are indexed by pullback diagrams

$$\begin{tikzcd} F \times O \arrow[r] \arrow[d] & X \arrow[d] \\ O \arrow[r, "\phi"] & Y \end{tikzcd}$$

such that the horizontal maps are open embeddings. A conically smooth map $X \to K$ is a *constructible bundle* if the restriction $X_{|K_q} \to K_q$ is a fiber bundle for each stratum $K_q \subset K$.

We observe several common classes of constructible bundles.

Observation 6.1.2. A fiber bundle $X \to K$ is a constructible bundle. A constructible bundle $X \to S$ with a smooth base is a fiber bundle. For each stratified space $X = (X \to P)$, and each consecutive subposet $Q \subset P$, the inclusion of the preimage of Q

$$X_Q \hookrightarrow X$$

is a constructible bundle. For a constructible bundle $X \to K$ and a stratum $X_p \subset X$, the composite $X_p \to K$ is again a constructible bundle.

The condition of being a constructible bundle is local in the base.

Observation 6.1.3. Let $X \to K$ be a conically smooth map, and let $\mathcal{U}_0 \subset K$ be an open cover of K. Then $X \to K$ is a constructible bundle if and only if, for each $U \in \mathcal{U}_0$, the restriction $X_{|U} \to U$ is a constructible bundle.

The following is routine, and is proved in [AFT].

Lemma 6.1.4 ([AFT])**.** *For each fiber bundle $X \xrightarrow{f} K \times \mathbb{R}^i \times \mathsf{C}(L)$, there is an isomorphism*

$$X \;\cong\; X_{|K\times\{0\}} \times \mathbb{R}^i \times \mathsf{C}(L)$$

over $K \times \mathbb{R}^i \times \mathsf{C}(L)$ and under $K \times \{0\}$.

Corollary 6.1.5. *The composition of conically smooth maps $X \to Y \to Z$ is a fiber bundle if both $X \to Y$ and $Y \to Z$ are fiber bundles.*

Proof. It is sufficient to consider the case $Z = \mathbb{R}^i \times \mathsf{C}(L)$ is a basic. Lemma 6.1.4 grants that the map $Y \to Z$ is isomorphic to a projection $Y_0 \times \mathbb{R}^i \times \mathsf{C}(L) \to \mathbb{R}^i \times \mathsf{C}(L)$. Another application of Lemma 6.1.4 gives that $X \to Y$ is isomorphic to a product map $X_0 \times \mathbb{R}^i \times \mathsf{C}(L) \to Y_0 \times \mathbb{R}^i \times \mathsf{C}(L)$. It follows that the composition $X \to Z$ is isomorphic to a projection $X_0 \times \mathbb{R}^i \times \mathsf{C}(L) \to \mathbb{R}^i \times \mathsf{C}(L)$. □

The designed regularity within stratified spaces gives the following easy criterion that characterizes constructibility.

Lemma 6.1.6. *The following two conditions on a conically smooth map $f\colon X \to K$ are equivalent.*

(1) *The map $X \to K$ is a constructible bundle.*
(2) *The composite $X_p \to X \to K$ is a constructible bundle for each stratum $X_p \subset X$.*

Proof. Suppose (1) is true. Each composition $X_p \to X \to K$ factors through a stratum $K_q \subset K$. Because $\emptyset \to L$ is constructible for any L, it is enough to show that the factorized map $X_p \to K_q$ is a fiber bundle. By assumption, there is an open cover $\mathcal{U}_0$ of K_q together with isomorphisms $X_{|U} \cong F \times U$ over U for each $U \in \mathcal{U}_0$. In particular, there are isomorphisms among fiberwise strata $(X_p)_{|U} \cong F_p \times U$ over U for each $U \in \mathcal{U}_0$. This implies $X_p \to K_q$ is a fiber bundle.

Now suppose (2) is true. We must show, for each stratum $K_q \subset K$, that the restriction $X_{|K_q} \to K_q$ is a fiber bundle. So we can assume K is trivially stratified, in which case we are to show $X \to K$ is a fiber bundle.

Each stratified space Z admits an open cover $\{Z^{\leq n} \mid n \in \mathbb{Z}\}$ with each $Z^{\leq n} \subset Z$ the locus of those points $z \in Z$ whose local dimension $\mathsf{dim}_z(Z)$ is at most n. Evidently, each assignment $Z \mapsto Z^{\leq n}$ is functorial with respect to conically smooth open embeddings. Therefore, we can assume X has depth which is bounded above. We will induct on the depth of X.

Should this depth be zero, which is to say that each connected component of X is trivially stratified, the assertion is true because $X \to K$ is assumed to be a fiber bundle. Denote the union of deepest strata $X_d \subset X$, and its complement $X_{>d}$. By induction, each of the restrictions $X_d \to K$ and $X_{>d} \to K$ is a fiber bundle. Consider the link $\mathsf{Link}_{X_d}(X) \xrightarrow{\pi} X_d$ and the fiberwise open cone $\mathsf{C}(\pi)$ – it is equipped with the cone-locus section $X_d \to \mathsf{C}(\pi)$. Choose a conically smooth open embedding $\mathsf{C}(\pi) \hookrightarrow X$ under X_d – such a choice is possible after the results of [AFT]. This witnesses a pushout

$$\mathsf{C}(\pi) \coprod_{\mathsf{Link}_{X_d}(X)\times\mathbb{R}_{>0}} X_{>d} \;\cong\; X\ .$$

Both of the projections $\mathsf{Link}_{X_d}(X) \xrightarrow{\pi} X_d$ and $\mathsf{C}(\pi) \to X_d$ are fiber bundles; after Corollary 6.1.5 we see that both of these projections further project to K as fiber bundles. Thus, to verify that $X \to K$ is a fiber bundle, it suffices to show that the conically smooth open embedding $\mathsf{C}(\pi) \hookrightarrow X$ can be re-chosen as one over K. For this, we can work locally in K and assume $K = \mathbb{R}^i$ for some i.

Fix isomorphisms $X_d \cong F_d \times \mathbb{R}^i$ and $X_{>d} \cong F_{>d} \times \mathbb{R}^i$, each over $\mathbb{R}^i$ – such a choice is possible because $\mathbb{R}^i$ is contractible. Likewise, choose an isomorphism $\mathsf{Link}_{X_d}(X) \cong L_d \times \mathbb{R}^i$ over the first of these isomorphisms. Denote the projection $\pi_d\colon L_d \to F_d$. There results an isomorphism $\mathsf{C}(\pi) \cong \mathsf{C}(\pi_d) \times \mathbb{R}^i$. We have thus established a conically smooth open embedding

$$e\colon \mathsf{C}(\pi_d) \times \mathbb{R}^i \cong \mathsf{C}(\pi) \hookrightarrow X$$

under $F_d \times \mathbb{R}^i \cong X_d$. Optimistically, we seek a conically smooth open embedding $\phi\colon \mathsf{C}(\pi_d) \times \mathbb{R}^i \hookrightarrow \mathsf{C}(\pi_d) \times \mathbb{R}^i$ under $F_d \times \mathbb{R}^i$ so that the diagram

$$\begin{tikzcd}
\mathsf{C}(\pi_d) \times \mathbb{R}^i \arrow[r, "\phi"] \arrow[dr, "\mathsf{pr}"'] & \mathsf{C}(\pi_d) \times \mathbb{R}^i \arrow[r, "e"] & X \arrow[dl, "f"] \\
& \mathbb{R}^i &
\end{tikzcd}$$

commutes, though we will only approximate as much.

Consider the composition $fe\colon \mathsf{C}(\pi_d) \times \mathbb{R}^i \to \mathbb{R}^i$. By construction, for each point $x_d \in F_d$ of the cone-locus, the restriction $fe_{|\{x_d\}}\colon \mathbb{R}^i \to \mathbb{R}^i$ is the identity map. The inverse function theorem of [AFT] grants that, for each $R > 0$, there is an open neighborhood of this cone-locus $F_d \subset O \subset \mathsf{C}(\pi_d)$ for which, for each $x \in O$, the restriction $fe_{|\{x\}}\colon \mathbb{R}^i \to \mathbb{R}^i$ is a smooth open embedding whose image contains the ball $\mathsf{B}_R(0)$ of radius R about the origin. Because conically smooth open self-embeddings $\mathsf{C}(L_d) \hookrightarrow \mathsf{C}(L_d)$ form a basis for the topology about the origin $* \in \mathsf{C}(L_d)$, there is a conically smooth open self-embedding $\underline{\phi}\colon \mathsf{C}(\pi_d) \hookrightarrow \mathsf{C}(\pi_d)$ over F_d whose image lies in O. Consider $\phi_R\colon \mathsf{C}(\pi_d) \times \mathsf{B}_R(0) \hookrightarrow \mathsf{C}(\pi_d) \times \mathbb{R}^i$ whose projection to $\mathsf{C}(\pi_d)$ is $\underline{\phi}$ and whose projection to $\mathbb{R}^i$ is $(fe_{|\{-\}})^{-1}$. This map is a conically smooth open embedding. By construction, the diagram

$$\begin{tikzcd}
\mathsf{C}(\pi_d) \times \mathsf{B}_R(0) \arrow[r, "\phi_R"] \arrow[d, "\mathsf{pr}"'] & \mathsf{C}(\pi_d) \times \mathbb{R}^i \arrow[r, "e"] & X \arrow[d, "f"] \\
\mathsf{B}_R(0) \arrow[rr] & & \mathbb{R}^i
\end{tikzcd}$$

commutes, where the bottom arrow is the standard inclusion. This is our approximate solution indicated earlier. This implies that the restriction $X_{|\mathsf{B}_R(0)} \to \mathsf{B}_R(0)$ is a fiber bundle. Because the collection $\{\mathsf{B}_R(0) \mid R > 0\}$ is an open cover of $\mathbb{R}^i$, we conclude that $X \to \mathbb{R}^i$ is a fiber bundle. This proves (1).

□

Proposition 6.1.7. *The composition of conically smooth maps $X \to Y \to Z$ is a constructible bundle if both $X \to Y$ and $Y \to Z$ are constructible bundles.*

Proof. Using Lemma 6.1.6, we need only verify that, for each stratum $X_p \subset X$, the restriction of the composite $X_p \to Z$ is constructible. From the definition of conically smooth maps, this composition factors as a composition through strata: $X_p \to Y_q \to Z_r$. By assumption, each of these factored maps is a fiber bundle among smooth manifolds. The result follows because the composition of smooth fiber bundles is again a smooth fiber bundle.

□

Lemma 6.1.8. *For each constructible bundle $X \xrightarrow{f} K$, and each conically smooth map $K' \xrightarrow{g} K$, the pullback*

$$\begin{tikzcd}
X \underset{K}{\times} K' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
K' \arrow[r, "g"] & K
\end{tikzcd}$$

exists, and the left vertical map is a constructible bundle. Furthermore, the left vertical map is proper whenever f is proper.

Proof. The properness assertion is immediate. Suppose we are given the existence of such pullbacks. To verify that the left vertical map in the square of the statement of the lemma is a constructible bundle, we can assume K' is trivially stratified. In this case, the map $K' \to K$ factors through a single stratum, $K_q \subset K$. The problem reduces showing that K too is trivially stratified and so that $X \to K$ is a fiber bundle. This problem is local in K', so we can assume that both K' and K are Euclidean spaces. In this case, there is an isomorphism $X \cong F \times K$ over K, and the problem is immediately solved.

We now prove the existence of such pullbacks. Open covers give colimit diagrams in Strat, and open covers pull back along conically smooth maps. Given these considerations, because K admits an open cover by stratified spaces of bounded depth, we can assume K has bounded depth. We proceed by induction on the depth of K. The statement is standard provided $X \to K$ is a fiber bundle, which proves the case that K has depth zero.

We now prove the inductive step. Consider a deepest stratum $K_0 \subset K$. Consider the preimages $K_0' := g^{-1}K_0$ and $X_0 := f^{-1}K_0$. Provided the existence of the codomain, there is a canonical conically smooth map

$$(22) \quad \big(X_0 \underset{K_0}{\times} K_0'\big) \coprod_{\big(\mathsf{Link}_{X_0}(X) \underset{\mathsf{Link}_{K_0}(K)}{\times} \mathsf{Link}_{K_0'}(K')\big)} \big(\mathsf{Unzip}_{X_0}(X) \underset{\mathsf{Unzip}_{K_0}(K)}{\times} \mathsf{Unzip}_{K_0'}(K')\big) \longrightarrow X \underset{K}{\times} K' .$$

The lefthand pushout exists, for it is a constructible cover, provided the existence of the fiber products in the lefthand expression. By inspection, should each of the fiber products in the lefthand expression exist, this pushout exhibits the desired pullback, and the above arrow will be an isomorphism. The base case verifies the existence of the left term of the pushout. The inductive step verifies the existence of the intermediate term of the pushout. After the existence of regular neighborhoods, there is a conically smooth map $\mathsf{Unzip}_{K_0}(K) \xrightarrow{\alpha} \mathbb{R}_{\geq 0}$ with the following properties:

- The preimage
$$\mathsf{Link}_{K_0}(K) = \alpha^{-1}0$$
is the link. Thereafter, we identify $\mathsf{Link}_{K_0'}(K') = (\alpha g)^{-1}0$ and $\mathsf{Link}_{X_0}(X) = (\alpha f)^{-1}0$.
- There is an open conically smooth map
$$\mathsf{Link}_{K_0}(K) \times [0,1) \hookrightarrow \mathsf{Unzip}_{K_0}(K)$$
under $\mathsf{Link}_{K_0}(K)$ and over the standard inclusion $[0,1) \to \mathbb{R}_{\geq 0}$. Likewise, there are open conically smooth maps $\mathsf{Link}_{K_0'}(K') \times [0,1) \hookrightarrow \mathsf{Unzip}_{K_0'}(K')$ and $\mathsf{Link}_{X_0}(X) \times [0,1) \hookrightarrow \mathsf{Unzip}_{X_0}(X)$, respectively under $\mathsf{Link}_{K_0'}(K')$ and $\mathsf{Link}_{X_0}(X)$, each over the standard inclusion $[0,1) \to \mathbb{R}_{\geq 0}$.

So, after Observation 6.1.3, to prove the existence of the right term in the pushout we are reduced to proving the pullback $X \underset{K}{\times} K'$ exists where

- $L \times [0,1) = K$;

- the map $X \xrightarrow{f} K = L \times [0,1)$ fits into a diagram among stratified spaces

$$\begin{tikzcd} E \times [0,1) \arrow[r, "r"] \arrow[d, "f_0 \times \mathsf{id}"'] & X \arrow[d, "f"] \\ L \times [0,1) \arrow[r] & K \end{tikzcd}$$

with r a refinement and f_0 constructible;
- the map $K' \xrightarrow{g} K = L \times [0,1)$ fits into a diagram among stratified spaces

$$\begin{tikzcd} L' \times [0,1) \arrow[r, "s"] \arrow[d, "g_0 \times \mathsf{id}"'] & K' \arrow[d, "g"] \\ L \times [0,1) \arrow[r] & K \end{tikzcd}$$

with s a refinement.

Using that limits commute, we have the identification

$$X \underset{Y \times Z}{\times} K' \;\cong\; (X \underset{Y}{\times} K') \underset{Y \times Z}{\times} (X \underset{Z}{\times} K')$$

provided existence of each pullback on the right. Applying this to our case at hand for which $K = L \times [0,1)$, by induction on depth, we are reduced to the case that $L = *$. The identity

$$\big(E \times [0,1)\big) \underset{[0,1)}{\times} \big(L' \times [0,1)\big) \cong \big(E \times L'\big) \times [0,1)$$

verifies the existence of the pullback term in this expression. Finally, we recognize the desired pullback

$$\big(E \times [0,1)\big) \underset{[0,1)}{\times} \big(L' \times [0,1)\big) \longrightarrow X \underset{[0,1)}{\times} K'$$

as the domain of a refinement, thereby demonstrating its existence.

□

From this we conclude in that constructible covers pull back along constructible bundles.

Corollary 6.1.9. *For each constructible bundle $X \to K$ and each constructible cover*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \widetilde{K} \arrow[d] \\ K_0 \arrow[r] & K, \end{tikzcd} \qquad \textit{the pullback diagram} \qquad \begin{tikzcd} X \underset{K}{\times} L \arrow[r] \arrow[d] & X \underset{K}{\times} \widetilde{K} \arrow[d] \\ X \underset{K}{\times} K_0 \arrow[r] & X \end{tikzcd}$$

is a constructible cover.

Proof. Lemma 6.1.8 grants that the pullback square exists. Because the first displayed diagram in the statement is a pullback, then so is the second displayed diagram. Inspection of expression (22) verifies that the second displayed diagram is a pushout.

□

Lemma 6.1.10. *For each constructible bundle $X \to Z \times \mathbb{R}$, there is an isomorphism*

$$X \;\cong\; X_{|Z \times \{0\}} \times \mathbb{R}$$

under $X_{|Z \times \{0\}} \times \{0\}$ and over $Z \times \mathbb{R}$.

Proof. Fix a constructible bundle $X \xrightarrow{f} Z \times \mathbb{R}$, and denote the stratification $X = (X \to P)$. There is the map of poset $P \xrightarrow{\mathsf{depth}} \mathbb{Z}_{\geq 0}$ that reports the codimension of $X_p \subset X$. For $S \subset \mathbb{Z}_{\geq 0}$, we notate $X_S := X_{\mathsf{depth}^{-1}S} \subset X$ for the union of strata whose depth is an element of S.

Consider the parallel vector field ∂_t on $Z \times \mathbb{R}$. Its flow is a conically smooth map $(Z \times \mathbb{R}) \times \mathbb{R} \to Z \times \mathbb{R}$ given by $(z, s, t) \mapsto (z, s + t)$. This flow is defined for all time. In a moment, we will argue that there exist a parallel vector field V on X lifting ∂_t. Provided this, the flow of V is defined for all time. In particular, we have an isomorphism $\gamma \colon X_{|Z\times\{0\}} \times \mathbb{R} \to X$ over $Z \times \mathbb{R}$, as desired.

Lemma 6.1.6 implies that, for each $k \geq 0$, the restriction $X_k \to Z \times \mathbb{R}$ factors through some stratum $Z_q \times \mathbb{R} \subset Z \times \mathbb{R}$ as a fiber bundle. Through Lemma 6.1.4, choose an isomorphism $X_k \cong (X_k)_{|Z_q\times\{0\}} \times \mathbb{R}$ under $(X_k)_{|Z_q\times\{0\}} \times \{0\}$ and over $Z_k \times \mathbb{R}$. By way of the above isomorphism, there results a parallel vector field V_k on X_k lifting ∂_t on $Z_q \times \mathbb{R}$. For each $k \geq 0$, choose a tubular neighborhood $X_k \subset \nu_k \subset X$. (Such exists through the results of [AFT] discussed in §1 of this article). This open neighborhood $\nu_k \subset X$ is equipped with a conically smooth retraction $\nu_k \to X_k$ which is a fiber bundle, equipped with a section. Another application of Lemma 6.1.4 gives the isomorphism $\nu_k \cong (\nu_k)_{|Z_q\times\{0\}} \times \mathbb{R}$ over and under $X_k \cong (X_k)_{|Z_q\times\{0\}} \times \mathbb{R}$. In particular, there is a parallel vector field $\widetilde{V}_k$ on ν_k extending V_k on X_k. The chosen collection of tubular neighborhoods $\{\nu_k\}_{k\geq 0}$ is an open cover of X. Choose a partition of unity $\{\psi_k\}_{k\geq 0}$ subordinate to this open cover. Consider the parallel vector field on X

$$V' \ := \ \sum_{k\geq 0} \psi_k \widetilde{V}_k \ .$$

We now modify V' to ensure that it lies over ∂_t on $Z \times \mathbb{R}$.

We define a sequence $\{V'_k\}_{k\geq 0}$ of parallel vector fields on X with the following properties:

- For each $0 \leq i \leq k \leq l$, the restrictions $(V'_k)_{|X_i} = (V'_l)_{|X_i}$ agree.
- For each $0 \leq i \leq k$, the restriction $(V'_k)_{|X_i}$ lies over ∂_t on $Z \times \mathbb{R}$.

We define this sequence by induction on $k \geq 0$. For $k = 0$, set $V'_k = V'$. It is immediate to verify that the restriction $V'_{|X_0}$ lies over ∂_t, because $V'_{|X_0} = V_0$ (for $(\psi_k)_{|X_0} = 0$ unless $k = 0$). Now suppose V'_k has been constructed for $k < d > 0$. The projection $X_d \to Z\times\mathbb{R}$ factors through some stratum $Z_q\times\mathbb{R}$ as a fiber bundle. In particular, the map of sheaves of parallel vector fields $Df_{|d} : \chi_{X_d} \to f^*\chi_{Z_q\times\mathbb{R}}$ has locally constant rank, and it admits a splitting $\chi_{X_d} \simeq \mathsf{Ker}(Df_{|d}) \oplus \chi_{Z_q\times\mathbb{R}}$. By way of this splitting, choose a vector field W_d on X_d for which $(V'_{d-1})_{|X_d} - W_d$ lies over ∂_t on $Z_q \times \mathbb{R}$. Because the restriction $(V'_{d-1})_{|X_{<d}}$ lies over ∂_t, we can take W_d to vanish conically smoothly as it approaches $X_{<d}$. In particular $(V'_{d-1})_{|X_d} - W_d$ gives an extension of $(V'_{d-1})_{|X_{d-1}}$ to $X_{\leq d}$. Call this extension W'_d. It is a parallel vector field on $X_{\leq d}$ that agrees with (V'_{d-1}) on $X_{<d}$. As performed previously, choose a regular neighborhood $X_{\leq d} \subset O \subset X$ as in §1, and extend W'_d to a parallel vector field $\widetilde{W'_d}$ on O. Choose a partition of unity $\{\phi_O, \phi_{>d}\}$ subordinate to the open cover $O \cup (X \smallsetminus X_{\leq d})$ of X. Take

$$V_d \ := \ \phi_O \widetilde{W'_d} + \phi_{>d} V'_{d-1} \ .$$

This is a parallel vector field on X, and it satisfies the two required properties by construction.

The desired parallel vector field on X is given by the expression

$$V \ := \ \lim_{k\geq 0} V'_k$$

which is defined because X is locally compact, so in particular the map $\mathsf{depth} \colon X \to \mathbb{Z}_{\geq 0}$ is locally bounded. By construction, this parallel vector field lifts ∂_t on $Z \times \mathbb{R}$.

□

6.2. **Decomposing constructible bundles.** We study how to break up total spaces of constructible bundles. The main result of this section is Corollary 6.2.11.

Observation 6.2.1. Each conically smooth map $X \to \overline{\mathsf{C}}(Z)$ to a closed cone determines a map between inclusions

$$\Big(\mathsf{Link}_{X_{|*}}(X) \to \mathsf{Unzip}_{X_{|*}}(X)\Big) \longrightarrow \Big(Z \xrightarrow{\{0\}} Z \times [0,1]\Big) ,$$

over the inclusion $* \to \overline{\mathsf{C}}(Z)$. Furthermore, should $X \to \overline{\mathsf{C}}(Z)$ be a constructible bundle, then each of the maps above are also.

Lemma 6.2.2. *For each constructible bundle $X \to \overline{\mathsf{C}}(Z)$ over a closed cone, there is an open conically smooth map under $\mathsf{Link}_{X_{|*}}(X)$*

$$\mathsf{Link}_{X_{|*}}(X) \times [0,1] \longrightarrow \mathsf{Unzip}_{X_{|*}}(X)$$

over $Z \times [0,1]$. Furthermore, for each open conically smooth map $e : X \to Y$ of constructible bundles over $\overline{\mathsf{C}}(Z)$, the diagram

$$\begin{tikzcd}
\mathsf{Link}_{X_{|*}}(X) \arrow[r, "\{0\}"] \arrow[dr] & \mathsf{Link}_{X_{|*}}(X) \times [0,1] \arrow[r, "\mathsf{Link}(e)\times\mathsf{id}"] \arrow[d, dashed] & \mathsf{Link}_{Y_{|*}}(Y) \times [0,1] \arrow[d, dashed] & \mathsf{Link}_{Y_{|*}}(Y) \arrow[l, "\{0\}"'] \arrow[dl] \\
& \mathsf{Unzip}_{X_{|*}}(X) \arrow[r, "\mathsf{Unzip}(e)"] & \mathsf{Unzip}_{Y_{|*}}(Y) &
\end{tikzcd}$$

over $Z \times [0,1]$ can be filled with the dashed arrows are open conically smooth maps.

Proof. Consider the vector field ∂_t on $Z\times[0,1]$. Its flow is the conically smooth map $\chi\colon (Z\times[0,1])\times \mathbb{R}_{\geq 0} \dashrightarrow Z \times [0,1]$ given by $(z,s,t) \mapsto (z, s+t)$. This is defined on $Z \times \{(s,t) \in [0,1]\times\mathbb{R}_{\geq 0} \mid 0 \leq s+t \leq 1\}$. We now now argue that there exists a vector field V on $\mathsf{Unzip}_{X_{|*}}(X)$ which lifts ∂_t.

From its construction, and after Lemma 6.1.10, the constructible bundle $\mathsf{Unzip}_{X_{|*}}(X) \to Z\times[0,1]$ admits an atlas consisting of maps among basics of the form $f_\alpha \times \mathsf{pr} : \mathbb{R}^{\mathsf{i}_\alpha} \times \mathsf{C}(\mathsf{L}_\alpha) \times \mathsf{I}_\alpha \to \mathsf{U}_\alpha \times \mathsf{I}_\alpha$ with $I_\alpha \subset [0,1]$ an open subspace which is an interval and $U_\alpha \hookrightarrow Z$ a member of the atlas of Z. In particular, there is a standard lift V_α of the restriction of ∂_t to $U_\alpha \times I_\alpha$. Choose a conically smooth partition of unity $\{\phi_\alpha\}$ subordinate to this atlas (such exists, as established in [AFT]). The vector field $V := \sum_\alpha \phi_\alpha V_\alpha$ defines a vector field on $\mathsf{Unzip}_{X_{|*}}(X)$. By design, V_α lifts ∂_t on $Z \times [0,1]$.

The flow of V is a conically smooth map $\gamma\colon \mathsf{Unzip}_{X_{|*}}(X) \times \mathbb{R}_{\geq 0} \dashrightarrow \mathsf{Unzip}_{X_{|*}}(X)$ over χ, and is defined on the preimage of the domain of χ. Furthermore, for each $t \in \mathbb{R}_{\geq 0}$, the conically smooth map $\gamma_t\colon \mathsf{Unzip}_{X_{|*}}(X) \dashrightarrow \mathsf{Unzip}_{X_{|*}}(X)$ is open, where it is defined. In particular, the restriction

$$\mathsf{Link}_{X_{|*}}(X) \times [0,1] \longrightarrow \mathsf{Unzip}_{X_{|*}}(X)_{|Z}$$

is an open conically smooth map over the isomorphism $\chi_1\colon (Z \times \{0\}) \times [0,1] \xrightarrow{\cong} Z \times [0,1]$.

Finally, the relative assertion is manifest from the above construction. □

Lemma 6.2.3. *For each constructible bundle $X \to \overline{\mathsf{C}}(Z)$ over a closed cone, there is a pushout diagram in* Strat

$$\begin{tikzcd}
\mathsf{Link}_{X_{|*}}(X) \times (0,1] \arrow[r] \arrow[d] & X_{|Z} \times (0,1] \arrow[d] \\
\mathsf{Link}_{X_{|*}}(X) \times [0,1] \arrow[r] & \mathsf{Unzip}_{X_{|*}}(X)
\end{tikzcd}$$

over the trivial pushout diagram witnessing $Z \times [0,1] \coprod_{Z\times(0,1]} Z \times (0,1] \cong Z \times [0,1]$.

Proof. For each open neighborhood $\mathsf{Link}_{X_{|*}}(X) \subset \nu \subset \mathsf{Unzip}_{X_{|*}}(X)$ there is an open cover in Strat

$$\begin{tikzcd}
\nu \smallsetminus \mathsf{Link}_{X_{|*}}(X) \arrow[r] \arrow[d] & \mathsf{Unzip}_{X_{|*}}(X) \smallsetminus \mathsf{Link}_{X_{|*}}(X) \arrow[d] \\
\nu \arrow[r] & \mathsf{Unzip}_{X_{|*}}(X),
\end{tikzcd}$$

thereby demonstrating a pushout diagram. Now, take ν to be the image of a conically smooth map

$$\mathsf{Link}_{X_{|*}}(X) \times [0,1] \longrightarrow \mathsf{Unzip}_{X_{|*}}(X)$$

over $Z \times [0,1]$, as produced by Lemma 6.2.2. The result follows because this map is, in particular, a monomorphism of underlying sets.

□

Corollary 6.2.4. *For each constructible bundle $X \to \overline{\mathsf{C}}(Z)$ over a closed cone, there is a diagram among stratified spaces*

$$\begin{tikzcd}
 & \mathsf{Link}_{X_{|*}}(X) \times (0,1] \arrow[r] \arrow[d] & X_{|Z} \times (0,1] \arrow[dd] \\
\mathsf{Link}_{X_{|*}}(X) \arrow[r, "\{0\}"] \arrow[d] & \mathsf{Link}_{X_{|*}}(X) \times [0,1] \arrow[dr] & \\
X_{|*} \arrow[rr] & & X
\end{tikzcd}$$

witnessing a colimit.

Remark 6.2.5. The above span $X_{|*} \xleftarrow{\pi} \mathsf{Link}_{X_{|*}}(X) \xrightarrow{\gamma} X_{|Z}$ has the property that π is proper and constructible, and γ is open; furthermore, any refinement $\mathsf{Link}_{X_{|*}}(X) \to L'$ factoring π and γ is in fact an isomorphism.

We notice a sort of converse to Corollary 6.2.4.

Observation 6.2.6. For each span among stratified spaces $\big(X_0 \xleftarrow{\pi} L \xrightarrow{\gamma} X_1\big)$ in which π is proper and constructible and γ is open, the colimit of the diagram

$$X_0 \longleftarrow L \xrightarrow{\{0\}} L \times [0,1] \longleftarrow L \times (0,1] \longrightarrow X_1 \times (0,1]$$

exists; call it X. Furthermore, for each constructible bundle $X_1 \xrightarrow{f} Z$ for which the composite $L \xrightarrow{f\gamma} Z$ is also constructible, the resulting canonical map $X \to \overline{\mathsf{C}}(Z)$ too is constructible, manifestly.

The next result is phrased in terms of the following categories. Fix a stratified space Z. There is the full subcategory

$$\mathsf{Cbl}(Z) \ \subset \ \mathsf{Strat}_{/Z}$$

consisting of the constructible bundles over Z. There is the category

$$\mathsf{Burn}_1(Z)$$

for which an object is a diagram in Strat

$$\begin{tikzcd}
 & L \arrow[dl, "\pi"'] \arrow[d] \arrow[dr, "\gamma"] & \\
X_0 & Z & X_Z \arrow[l]
\end{tikzcd}$$

in which π is proper and constructible, γ is open, and the maps to Z are constructible. A morphism is a natural transformation of such diagrams that restricts as the identity map on Z; composition is given by composing natural transformations. Supposing Z is compact, Observation 6.2.6 offers the functor

$$\mathsf{Burn}_1(Z) \longrightarrow \mathsf{Cbl}\big(\overline{\mathsf{C}}(Z)\big) \tag{23}$$

given by assigning to $(X_0 \xleftarrow{\pi} L \xrightarrow{\gamma} X_Z)$ the colimit term of the diagram

$$\begin{tikzcd}
 & L \times (0,1] \arrow[rr, "\gamma\times\mathsf{id}"] \arrow[d] & & X_Z \times (0,1] \arrow[dd] \\
L \arrow[r, "\{0\}"] \arrow[d, "\pi"'] & L \times [0,1] \arrow[drr] & & \\
X_0 \arrow[rrr] & & & X_0 \coprod_L L \times [0,1] \coprod_{L\times[0,1)} X_Z \times [0,1),
\end{tikzcd}$$

which is a constructible bundle over $\overline{\mathsf{C}}(Z)$.

With Lemma 6.2.3, Corollary 6.2.4 quickly implies the following result. To state this result we denote by $\mathsf{Ar}^{\mathsf{open}}(-)$ the full subcategory of those arrows that are by *open* conically smooth maps.

Corollary 6.2.7. *For each compact stratified space Z, each of the functors*

$$\mathsf{Burn}_1(Z) \xrightarrow{(23)} \mathsf{Cbl}\big(\overline{\mathsf{C}}(Z)\big) \qquad \text{and} \qquad \mathsf{Ar}^{\mathsf{open}}\big(\mathsf{Burn}_1(Z)\big) \xrightarrow{\mathsf{Ar}^{\mathsf{open}}(23)} \mathsf{Ar}^{\mathsf{open}}\Big(\mathsf{Cbl}\big(\overline{\mathsf{C}}(Z)\big)\Big)$$

is essentially surjective.

Remark 6.2.8. The functor (23) is far from conservative. Namely, for each span $(X_0 \xleftarrow{\pi} L \xrightarrow{\gamma} X_1)$ with π proper and constructible and γ open, any refinement $L \to L'$ factoring both π and γ induces the same colimit of Observation 6.2.6. Nevertheless, there is a terminal such L' under L.

The effect of Corollary 6.2.7 can be iterated, but doing so requires more set-up. Fix a stratified space Z, and an integer $p \geq 0$. Consider the poset $\mathcal{P}_p$ of non-empty convex subsets $S \subset \{0 < \cdots < p < +\}$ for which $p \in S$ implies $+ \in S$. Order $\mathcal{P}_p$ by reverse inclusion. Consider the full subcategory

$$\mathsf{Burn}_p(Z) \ \subset\ \mathsf{Fun}\big(\mathcal{P}_p, \mathsf{Strat}\big)$$

consisting of those functors $\mathcal{P}_p \xrightarrow{X_-} \mathsf{Strat}$ that satisfy the following conditions.

- The value $X_{\{+\}} = Z$, and for each $+ \in S \in \mathcal{P}_p$ the map of stratified spaces
$$X_S \longrightarrow Z$$
is constructible.
- For each relation $S \leq T$ in $\mathcal{P}_p$ for which $\mathsf{Min}(S) = \mathsf{Min}(T)$, the map of stratified spaces
$$X_S \longrightarrow X_T$$
is *proper* and *constructible*.
- For each relation $S \leq T$ in $\mathcal{P}_p$ for which $\mathsf{Min}(S) = \mathsf{Min}(T)$, the map of stratified spaces
$$X_S \longrightarrow X_T$$
is *open*.
- For each pair $S, T \in \mathcal{P}_p$ for which $S \cap T \neq \emptyset$, the square among stratified spaces
$$\begin{tikzcd}
X_{S\cup T} \arrow[r] \arrow[d] & X_T \arrow[d] \\
X_S \arrow[r] & X_{S\cap T}
\end{tikzcd}$$
is a pullback.

For $p > 0$, this category fits into an evident square

$$\begin{tikzcd}
\mathsf{Burn}_p(Z) \arrow[r] \arrow[d] & \mathsf{Burn}_{p-1}(Z) \arrow[d] \\
\mathsf{Burn}_1(*) \arrow[r] & \mathsf{Burn}_0(*)
\end{tikzcd}$$

in where the horizontal arrows are given by restriction along the standard inclusion $\{1 < \cdots < p < +\} \hookrightarrow \{0 < \cdots < p < +\}$, and the vertical arrows are induced from the standard inclusion $\{0 < 1\} \hookrightarrow \{0 < \cdots < p < +\}$ with terminal value on $+$. Directly so, all of the functors in this square are isofibrations. Because constructible bundles admit base-change (Lemma 6.1.8), this square of categories is in fact a pullback:

$$\mathsf{Burn}_p(Z) \xrightarrow{\simeq} \mathsf{Burn}_1(*) \underset{\mathsf{Burn}_0(*)}{\times} \mathsf{Burn}_{p-1}(Z)~. \tag{24}$$

We now construct a functor

$$\mathsf{Burn}_p(Z) \longrightarrow \mathsf{Cbl}\big(\overline{\mathsf{C}}^p(Z)\big) \tag{25}$$

by induction on p. For $p = 0$ the assertion is trivial. For $p = 1$ this is the functor (23). Now suppose $p > 1$. There is the diagram among categories

$$\begin{tikzcd}
\mathsf{Burn}_p(Z) \arrow[rrr, "\mathsf{ev}_{\{0,1\}}"] \arrow[d] & & & \mathsf{Ar}^{\mathsf{cbl}}(\mathsf{Strat}) \arrow[d, "\mathsf{ev}_1"] \\
\mathsf{Ar}^{\mathsf{open}}\big(\mathsf{Burn}_{p-1}(Z)\big) \arrow[r, "\mathsf{ev}_0"] & \mathsf{Burn}_{p-1}(Z) \arrow[r, "(Z\to *)"] & \mathsf{Burn}_{p-1}(*) \arrow[r, "\mathsf{ev}_{\{0,1\}}"] & \mathsf{Strat}
\end{tikzcd}$$

with the left vertical functor given by restriction along the inclusion of posets

$$\cup\colon \mathcal{P}_{p-1} \times [1] \cong \mathcal{P}_{\{1<\cdots<p\}} \times \big\{\emptyset \subset \{0\}\big\} ~\cong~ \big\{S \in \mathcal{P}_p \mid S \neq \{0\}\big\} ~\subset~ \mathcal{P}_p~.$$

This square is a pullback

$$\mathsf{Burn}_p(Z) \xrightarrow{\simeq} \mathsf{Ar}^{\mathsf{open}}\big(\mathsf{Burn}_{p-1}(Z)\big) \underset{\mathsf{Strat}}{\times} \mathsf{Ar}^{\mathsf{cbl}}(\mathsf{Strat}) \tag{26}$$

since there is a pushout expression of posets

$$\big\{0 \subset \{0,1\}\big\} \coprod_{\{0,1\}} \big\{S \in \mathcal{P}_p \mid S \neq \{0\}\big\} \xrightarrow{\cong} \mathcal{P}_p$$

and because constructible bundles admit base-change (Lemma 6.1.8). We now define the functor (25) as composite of the sequence of functors

$$\begin{tikzcd}
\mathsf{Burn}_p(Z) \arrow[d, "\simeq"', "(26)"] & & \mathsf{Burn}_1\big(\overline{\mathsf{C}}^{p-1}(Z)\big) \arrow[r, "(23)"] \arrow[d, "\simeq"', "(26)"] & \mathsf{Cbl}\big(\overline{\mathsf{C}}^p(Z)\big) \\
\mathsf{Ar}^{\mathsf{open}}\big(\mathsf{Burn}_{p-1}(Z)\big) \underset{\mathsf{Strat}}{\times} \mathsf{Ar}^{\mathsf{cbl}}(\mathsf{Strat}) \arrow[rr, "\text{induction}"] & & \mathsf{Ar}^{\mathsf{open}}\big(\mathsf{Cbl}\big(\overline{\mathsf{C}}^{p-1}(Z)\big)\big) \underset{\mathsf{Strat}}{\times} \mathsf{Ar}^{\mathsf{cbl}}(\mathsf{Strat}) &
\end{tikzcd} \tag{27}$$

This inductive definition of the functor (25) makes the next result immediate.

Corollary 6.2.9. *For each compact stratified space Z, and for each $p \geq 0$, each of the functors*

$$\mathsf{Burn}_p(Z) \xrightarrow{(25)} \mathsf{Cbl}\big(\overline{\mathsf{C}}^p(Z)\big) \qquad \textit{and} \qquad \mathsf{Ar}^{\mathsf{open}}\big(\mathsf{Burn}_p(Z)\big) \xrightarrow{\mathsf{Ar}^{\mathsf{open}}(25)} \mathsf{Ar}^{\mathsf{open}}\big(\mathsf{Cbl}\big(\overline{\mathsf{C}}^p(Z)\big)\big)$$

is essentially surjective.

Proof. We proceed by induction on p. The case $p = 0$ is immediate. The case $p = 1$ is Corollary 6.2.7. Now suppose $p > 1$. From the definition of (25) in terms of the diagram (27), it is enough to verify thta each of the arrows in (27) is essentially surjective. For this it is remains to verify that the arrow labeled by "induction" is essentially surjective. This arrow is between fiber products, each of which is among isofibrations. Essential surjectivity of this arrow therefore follows by induction.

□

We insert the following observation, which is implied by Corollary 6.2.9.

Corollary 6.2.10. *Let $X \to \overline{\mathsf{C}}^2(Z)$ be a constructible bundle over a double closed cone. There are open conically smooth maps γ_{1Z} and γ_{0Z} and γ_{010Z}, and γ_{0Z} over Z, as well as γ_{01}, fitting into a commutative diagram*

$$\begin{tikzcd}
\mathsf{Link}_{X_{|*}}(X_{|\overline{\mathsf{C}}^{\{0\}}(Z)}) \arrow[drrr, bend left, "\gamma_{0Z}"] \arrow[dddr, bend right, "\pi_{0Z}"'] & & & \\
& \mathsf{Link}_{\mathsf{Link}_{X_{|*}}(X_{|\overline{\mathsf{C}}^2(\emptyset)})}\big(\mathsf{Link}_{X_{|*}}(X)\big) \arrow[ul, "\gamma_{010Z}"'] \arrow[r, "\gamma_{01Z}"'] \arrow[d, "\pi_{01Z}"] & \mathsf{Link}_{X_{|\overline{\mathsf{C}}^{\{1\}}(\emptyset)}}\big(X_{|\overline{\mathsf{C}}^{\{1\}}(Z)}\big) \arrow[r, "\gamma_{1Z}"'] \arrow[d, "\pi_{1Z}"] & X_{|Z} \\
& \mathsf{Link}_{X_{|*}}(X_{|\overline{\mathsf{C}}^2(\emptyset)}) \arrow[r, "\gamma_{01}"'] \arrow[d, "\pi_{01}"] & X_{|\overline{\mathsf{C}}^{\{1\}}(\emptyset)} & \\
& X_{|*} & &
\end{tikzcd}$$

in which the square is a pullback, and the map γ_{010Z} is terminal among refinements from the pullback.

Corollary 6.2.11. *For each compact stratified space Z, and each $0 \leq k < p$, the canonical functor*

$$\mathsf{Cbl}\big(\overline{\mathsf{C}}^p(Z)\big) \longrightarrow \mathsf{Cbl}\big(\overline{\mathsf{C}}^k(*)\big) \underset{\mathsf{Cbl}(*)}{\times} \mathsf{Cbl}\big(\overline{\mathsf{C}}^{p-k}(Z)\big)$$

is essentially surjective.

Proof. Consider a weakly regular subspace $K_0 \subset K$, and choose a refinement $\widetilde{K}$ of K with respect to which the subspace $K_0 \subset \widetilde{K}$ is constructible and closed. Consider a constructible bundle $X \to K$. Consider the pullback $\widetilde{X} := X \underset{\widetilde{K}}{\times} K$, and regard it as a constructible bundle over $\widetilde{K}$. There is the pushout over $\widetilde{K}$:

$$\widetilde{X}_{|K_0} \coprod_{\mathsf{Link}_{\widetilde{X}_{|K_0}}(\widetilde{X})} \mathsf{Unzip}_{\widetilde{X}_{|K_0}}(\widetilde{X}) \ \cong\ X\ .$$

Through these considerations, we see that both of the restriction functors $\mathsf{Cbl}\big(\overline{\mathsf{C}}^k(*)\big) \to \mathsf{Cbl}(*)\big) \leftarrow \mathsf{Cbl}\big(\overline{\mathsf{C}}^{p-k}(Z)\big)$ are isofibrations.

There is a diagram of categories

$$\begin{tikzcd}
\mathsf{Burn}_p(Z) \arrow[r] \arrow[d] & \mathsf{Burn}_k(*) \underset{\mathsf{Burn}_0(*)}{\times} \mathsf{Burn}_{p-k}(Z) \arrow[d] \\
\mathsf{Cbl}\big(\overline{\mathsf{C}}^p(Z)\big) \arrow[r] & \mathsf{Cbl}\big(\overline{\mathsf{C}}^k(*)\big) \underset{\mathsf{Cbl}(*)}{\times} \mathsf{Cbl}\big(\overline{\mathsf{C}}^{p-k}(Z)\big).
\end{tikzcd}$$

The top horizontal map is an equivalence of categories, as implied by the equivalence (24). Corollary 6.2.9 gives that the downward functors are essentially surjective. We conclude that the bottom horizontal functor is essentially surjective as well.

$\square$

6.3. **Classifying constructible bundles.** We examine a category of constructible bundles, through which we define the ∞-category $\mathcal{B}\mathsf{un}$. First, recall from the end of §1.1 the notion of *finitary*.

Definition 6.3.1 (Bun)**.** An object of the category Bun is a constructible bundle $f : X \to K$ whose fibers are finitary. A morphism from $(X \xrightarrow{f} K)$ to $(X' \xrightarrow{f'} K')$ is a pullback diagram among stratified spaces

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ K \arrow[r, "g"'] & K' \end{tikzcd} .$$

Composition is given by concatenating such squares horizontally, then composing horizontal maps.

Lemma 6.3.2. *The forgetful functor*

$$\mathsf{Bun} \longrightarrow \mathsf{Strat}~, \qquad (X \to K) \mapsto K$$

is a right fibration, and as so it is a cone-local sheaf.

Proof. Lemma 6.1.8 immediately implies that this projection is a Cartesian fibration. By design, the fibers are groupoids. It follows that it is a right fibration. Observation 6.1.3, together with the fact that open covers pull back, implies $\mathsf{Bun} \to \mathsf{Strat}$ is a sheaf. Corollary 6.1.9 implies $\mathsf{Bun} \to \mathsf{Strat}$ is cone-local.

$\square$

Recall Definition 5.1.1 of a transversality sheaf.

Theorem 6.3.3. *The functor* $\mathsf{Bun} \to \mathsf{Strat}$ *is a transversality sheaf.*

Proof. We will employ Theorem 5.1.2. Lemma 6.3.2 states that $\mathsf{Bun} \to \mathsf{Strat}$ is a right fibration and a sheaf. Once and for all, choose an weakly regular subspace $K_0 \subset K$, and choose a refinement $\widetilde{K}$ of K with respect to which the subspace $K_0 \subset \widetilde{K}$ is constructible and closed. Also, fix an arbitrary constructible bundle $X \to K$. Consider the pullback $\widetilde{X} := X \underset{\widetilde{K}}{\times} K$, and regard it as a constructible bundle over $\widetilde{K}$. There is a pushout over K:

$$\widetilde{X}_{|K_0} \coprod_{\mathsf{Link}_{\widetilde{X}_{|K_0}}(\widetilde{X})} \mathsf{Unzip}_{\widetilde{X}_{|K_0}}(\widetilde{X}) ~\cong~ X~. \tag{28}$$

Isofibration: An isomorphism $X_0' \cong X_{|K_0}$ over K_0 pulls back to an isomorphism $\widetilde{X}_0' \cong \widetilde{X}_{|K_0}$ over $\widetilde{K}_0$. Through the aforementioned colimit (28), this isomorphism extends to an isomorphism $\widetilde{X}' \cong \widetilde{X}$ over $\widetilde{K}$. This isomorphism determines an isomorphism $X' \cong X$ over K, extending the given one over K_0.
Isotopy extension: This follows a nearly identical argument as the verification of the **Isofibration** condition.
Isotopy equivalence: For each constructible bundle $X \to K \times \mathbb{R}$, there is an isomorphism $X \cong X_0 \times \mathbb{R}$ over $K \times \mathbb{R}$ to a product map. The desired essential surjectivity follows.
Consecutive: We must show, for each $0 \leq k \leq p$, that the restriction functor

$$\mathsf{Bun}(\Delta^p) \longrightarrow \mathsf{Bun}(\Delta^{\{0<\cdots<k\}}) \underset{\mathsf{Bun}(\Delta^{\{k\}})}{\times} \mathsf{Bun}(\Delta^{\{k<\cdots<p\}})$$

is essentially surjective, and surjective on Hom-sets. Essential surjectivity follows immediately from Corollary 6.2.11 upon recognizing these two points.

- For each $p \geq 0$, there is an isomorphism $\overline{\mathsf{C}}^p(\ast) \cong \Delta^p$.
- For each stratified space Z, $\mathsf{Bun}(Z)$ as the underlying groupoid of $\mathsf{Cbl}(Z)$.

We now argue the surjectivity on Hom-sets. Let $X \to \Delta^p$ be a constructible bundle. Recall that the inclusion $\Delta^{\{0<\cdots<k\}} \bigcup_{\Delta^{\{k\}}} \Delta^{\{k<\cdots<p\}} \subset \Delta^p$ is a weakly regular subspace. Inspecting the expression (28) applied to this weakly regular subspace, one notices that any automorphism of the restriction $X_{|\Delta^{\{0<\cdots<k\}} \underset{\Delta^{\{k\}}}{\bigcup} \Delta^{\{k<\cdots<p\}}}$ over $\Delta^{\{0<\cdots<k\}} \underset{\Delta^{\{k\}}}{\bigcup} \Delta^{\{k<\cdots<p\}}$ extends to an automorphism of X over Δ^p. This implies the desired surjectivity on Hom-sets, because $\mathsf{Bun}(-)$ takes values in groupoids.

Univalent: As in Remark 5.1.3, to prove univalence for Bun amounts to proving that the map of spaces

$$\mathcal{B}\mathsf{un}(\Delta^0) \longrightarrow \mathcal{B}\mathsf{un}(E^1)$$

is an equivalence, where E^1 is the simplicial stratified space corresponding, via $\Delta^\bullet$, to the nerve of the free isomorphism. Noncanonically, we can identify these spaces as disjoint unions indexed by the collection of isomorphisms classes $[X] \in \pi_0 \mathcal{B}\mathsf{un}(\Delta^0)$, as

$$\mathcal{B}\mathsf{un}(\Delta^0) \simeq \coprod_{[X]} \mathsf{Aut}(X) \qquad \text{and} \qquad \mathcal{B}\mathsf{un}(E^1) \simeq \coprod_{[X]} \mathsf{Emb}(X)~.$$

Here $\mathsf{Aut}(X)$ is the space of (conically smooth) automorphisms of X, and $\mathsf{Emb}(X) := \mathsf{Emb}^{\sim}(X,X)$ is the space of (conically smooth) self-embeddings of X which are isotopic to an automorphism (via a conically smooth isotopy consisting of embeddings). To show the desired equivalence, we therefore reduce to showing that for any finitary stratified space X, the natural map

$$\mathsf{Aut}(X) \longrightarrow \mathsf{Emb}(X)$$

is a homotopy equivalence of Kan complexes. We now prove this equivalence.

Since X is finitary, it is the interior of a compact stratified space with boundary $\overline{X}$. We choose an open collar $C \cong \partial\overline{X} \times (0,1]$ of the boundary $\partial\overline{X}$, and then set $Z := X \smallsetminus C$ to be the complement of the open collar. By sucking in along the parametrization of the collar, we can define a 1-parameter family of automorphisms

$$\phi \colon (0,1] \longrightarrow \mathsf{Aut}(X)$$

such that:

- $\phi_1 = \mathsf{id}_X$;
- $\phi_{t|_Z} = \mathsf{id}_Z$ for each $t \in (0,1]$;
- for each $x \in C$ and each open $U \subset X$ containing Z, there exists t such that $\phi_s(x) \in U$ for all values $0 < s < t$.

Fix such a Z and such a ϕ. Denote by $\mathsf{Emb}_Z(X) \subset \mathsf{Emb}(X)$ the submonoid of those self-embeddings of X that are the identity map on Z, and set $\mathsf{Aut}_Z(X) = \mathsf{Aut}(X) \cap \mathsf{Emb}_Z(X)$. Since Z is compact, the isotopy extension theorem grants that the natural restrictions $\mathsf{Aut}(X) \to \mathsf{Emb}(Z,X)$ and $\mathsf{Emb}(X) \to \mathsf{Emb}(Z,X)$ are surjective Kan fibrations. We thus have a natural map of homotopy fiber sequences

$$\begin{tikzcd}
\mathsf{Aut}_Z(X) \arrow[r] \arrow[d] & \mathsf{Aut}(X) \arrow[r] \arrow[d] & \mathsf{Emb}(Z,X) \arrow[d, equal] \\
\mathsf{Emb}_Z(X) \arrow[r] & \mathsf{Emb}(X) \arrow[r] & \mathsf{Emb}(Z,X)
\end{tikzcd}$$

which is the identity on the base $\mathsf{Emb}(Z,X)$. Consequently, to prove the equivalence of $\mathsf{Aut}(X) \to \mathsf{Emb}(X)$, we can reduce to proving that the map of fibers $\mathsf{Aut}_Z(X) \to \mathsf{Emb}_Z(X)$ is an equivalence. To do so, we construct an explicit homotopy inverse using our previously chosen ϕ.

Consider the map

$$\gamma\colon \mathsf{Emb}_Z(X) \times (0,1] \longrightarrow \mathsf{Emb}_Z(X)~, \qquad (f,t) \mapsto \phi_t^{-1} \circ f \circ \phi_t$$

given by conjugating with ϕ. Since $\mathsf{Emb}_Z(X)$ consists of conically smooth maps, there is an extension of γ to a map $\overline{\gamma}\colon \mathsf{Emb}_Z(X) \times [0,1] \to \mathcal{S}\mathsf{trat}_Z(X,X)$. We denote the restriction of this extension along $\{0\}$ as

$$D_Z\colon \mathsf{Emb}_Z(X) \longrightarrow \mathcal{S}\mathsf{trat}_Z(X,X)~.$$

To conclude the argument, we show that the values of D_Z are automorphism of X. Let $f\colon X \to X$ be an embedding that is the identity on Z. Then there is an open neighborhood $Z \subset O \subset X$ on which there is a conically smooth inverse $f^{-1}\colon X \dashrightarrow X$. Denote the pre-image $O' := f^{-1}(O) \subset X$. The chain rule of [AFT] gives the identities $\mathsf{id}_O = D_Z(f \circ f^{-1}) = D_Z f \circ D_Z f^{-1}$ and $\mathsf{id}_{O'} = D_Z(f^{-1} \circ f) = D_Z f^{-1} \circ D_Z f$. This demonstrates an inverse to $D_Z f$. We thus conclude that the map factors as $D_Z\colon \mathsf{Emb}_Z(X) \to \mathsf{Aut}_Z(X)$, and therefore that the isotopy $\overline{\gamma}$ witnesses D_Z as a homotopy inverse to the inclusion $\mathsf{Aut}_Z(X) \to \mathsf{Emb}_Z(X)$.

□

The efforts of this article culminate as the following example of an ∞-category $\mathcal{B}\mathsf{un}$ that classifies constructible bundles. Recall the Definition 6.3.1 of Bun, which Theorem 6.3.3 verifies is a transversality sheaf.

Definition 6.3.4 ($\mathcal{B}\mathsf{un}$)**.** $\mathcal{B}\mathsf{un}$ is the ∞-category associated to the transversality sheaf Bun by way of the functor $\mathsf{Trans} \to \mathfrak{S}\mathsf{tri}$ and the equivalence $\mathfrak{S}\mathsf{tri} \simeq \mathsf{Cat}_\infty$ of Theorem 4.2.3.

Remark 6.3.5. For each stratified space K, there is an identification of the space of functors

$$\mathsf{Map}\big(\mathsf{Exit}(K), \mathcal{B}\mathsf{un}\big) \ \simeq\ \Big([q] \mapsto \big\{X \underset{\mathrm{cbl}}{\longrightarrow} K \times \Delta^q_e\big\}\Big)$$

as the space associated to the simplicial set for which a q-simplex is a constructible bundle over $K \times \Delta^q_e$. Each object of the ∞-category $\mathcal{B}\mathsf{un}$ is represented by a stratified space; the space of morphisms in $\mathcal{B}\mathsf{un}$ from X_0 to X_1 is represented by the simplicial set for which a q-simplex is a constructible bundle $X \to \Delta^1 \times \Delta^q_e \cong [0,1] \times \mathbb{R}^q$ with identifications $X_0 \times \mathbb{R}^q \cong X_{|\{0\}\times\mathbb{R}^q}$ and $X_1 \times \mathbb{R}^q \cong X_{|\{1\}\times\mathbb{R}^q}$ over $\mathbb{R}^q$.

Remark 6.3.6. For each stratified space K, the set of components of the space of functors

$$\pi_0 \,\mathsf{Map}\big(\mathsf{Exit}(K), \mathcal{B}\mathsf{un}\big) \ \cong\ \Big\{X \underset{\mathrm{cbl}}{\longrightarrow} K\Big\}_{/\mathrm{conc}}$$

is identified as the set of smooth concordance classes of constructible bundles over K. In this sense, the ∞-category $\mathcal{B}\mathsf{un}$ can be interpreted as one that classifies constructible bundles.

Remark 6.3.7. Corollary 6.2.7 implies a surjection

$$\big\{X_0 \xleftarrow{\mathsf{p.cbl}} L \xrightarrow{\mathsf{open}} X_1\big\}_{/\mathsf{iso}} \longrightarrow \pi_0\, \mathcal{B}\mathsf{un}(\Delta^1)$$

to the set of path components of the space of morphisms of $\mathcal{B}\mathsf{un}$, from the set of isomorphism classes of spans among stratified spaces for which the leftward map is proper and constructible and the rightward map is open. Corollary 6.2.9 generalizes this as a surjection

$$\big\{X_0 \xleftarrow{\mathsf{p.cbl}} L_{0,1} \xrightarrow{\mathsf{open}} X_1 \xleftarrow{\mathsf{p.cbl}} L_{1,2} \xrightarrow{\mathsf{open}} \dots \xleftarrow{\mathsf{p.cbl}} L_{p-1,p} \xrightarrow{\mathsf{open}} X_p\big\}_{/\mathsf{iso}} \longrightarrow \pi_0\, \mathcal{B}\mathsf{un}(\Delta^p)$$

for each $p \geq 0$. Entertain the existence of a pre-Burnside category of Strat (with leftward factor *proper and constructible* and with rightward factor *open*); composition in this pre-Burnside category is defined precisely because constructible bundles admit base-change. The surjections above can be improved as a functor from this pre-Burnside category to the ∞-category $\mathcal{B}\mathsf{un}$ which, on the level of homotopy categories, is essentially surjective and full. Because it is not important for our purposes, we forgo justifying this assertion.

There are subcategories

$$\mathsf{cBun} \ \subset\ \mathsf{Bun}^{\mathsf{cpt}} \ \subset\ \mathsf{Bun}$$

consisting of those constructible bundles $X \to K$ that are *proper*, and those that are have compact fibers.

Observation 6.3.8. Following the proof that Bun is a transversality sheaf (Theorem 6.3.3), the subcategories cBun and $\mathsf{Bun}_{\leq n}$ and $\mathsf{Bun}^{\mathsf{cpt}}$ too are transversality sheaves.

After Observation 6.3.8, there results ∞-subcategories

$$\mathsf{c}\mathcal{B}\mathsf{un} \ \subset\ \mathcal{B}\mathsf{un}^{\mathsf{cpt}} \ \subset\ \mathcal{B}\mathsf{un} \ \supset\ \mathcal{B}\mathsf{un}_{\leq n}$$

as well as their intersections $\mathsf{c}\mathcal{B}\mathsf{un}_{\leq n} \subset \mathcal{B}\mathsf{un}^{\mathsf{cpt}} \subset \mathcal{B}\mathsf{un}$, the second inclusion being full.

Lemma 6.3.9. *There are equivalences of ∞-categories*

$$\mathsf{c}\mathcal{B}\mathsf{un}_{\leq 0} \xrightarrow{\simeq} \mathsf{Fin}^{\mathsf{op}} \qquad \text{and} \qquad \mathcal{B}\mathsf{un}^{\mathsf{cpt}}_{\leq 0} \xrightarrow{\simeq} \mathsf{Fin}^{\mathsf{op}}_{*} \ .$$

Proof. We will prove the second equivalence. Consider a K-point $\mathsf{Exit}(K) \xrightarrow{X\to K} \mathcal{B}\mathsf{un}_{\leq 0^{\mathsf{cpt}}}$ classifying the indicated constructible bundle. Consider the topological space X^* over K, equipped with an embedding $X \hookrightarrow X^*$ over K, as well as a section $K \to X^*$ for which $X^* \to K$ is proper and the map $K \amalg X \to X^*$ is a bijection. There exists a unique such X^*, and it inherits the structure of a stratified space with respect to which the maps $K \amalg X \to X^* \to K$ are conically smooth. The map $X^* \to K$ has the property that each diagram among stratified spaces

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\{1\}"'] & X^* \arrow[d] \\ Z \times \Delta^1 \arrow[r] \arrow[ur, dashed] & K, \end{tikzcd}$$

in which Z is compact, has a unique filler.

The functor $\mathcal{B}\mathsf{un}^{\mathsf{cpt}}_{\leq 0} \to \mathsf{Fin}^{\mathsf{op}}_{*}$ is given by assigning to such a K-point the functor

$$\mathsf{Exit}(X^*) \longrightarrow \mathsf{Exit}(K)$$

together with its section. The path lifting property implies this functor is a right fibration, and therefore is classified by a functor $(\mathcal{B}\mathsf{un}^{\mathsf{cpt}}_{\leq 0})^{\mathsf{op}} \to \mathsf{Fin}_*$. This functor is evidently essentially surjective.

To argue that this functor is fully faithful we construct an inverse on mapping spaces. Notice that, for each map $I_+ \xrightarrow{f} J_+$ among based finite sets, the reversed mapping cylinder (Definition 6.6.12)

$$\mathsf{Cylr}(f) \longrightarrow \Delta^1$$

is equipped with a standard map to the topological 1-simplex, which is in fact a constructible bundle of stratified spaces, and the base-point determines a section. The map $\mathsf{Cylr}(f) \smallsetminus \Delta^1 \to \Delta^1$ is a constructible bundle with compact zero-dimensional fibers, and thus describes a morphism of $\mathcal{B}\mathsf{un}^{\mathsf{cpt}}_{\leq 0}$. It is routine to verify that this assignment is in fact an inverse on the level of mapping spaces.

By direct inspection, the first equivalence is a restriction of the second.

$\square$

6.4. **The absolute exit-path ∞-category.** We introduce another interesting striation sheaf, the absolute exit-path ∞-category. This example ties together the previously considered $\mathcal{B}\mathsf{un}$ and exit-paths.

Definition 6.4.1. The category of *absolute exit-paths* $\mathfrak{Exit}$ has as an object a constructible bundle $X \xrightarrow{f} K$ together with a section $K \xrightarrow{s} X$. A morphism from $(X \underset{f}{\overset{s}{\leftrightarrows}} K)$ to $(X' \underset{f'}{\overset{s'}{\leftrightarrows}} K')$ is a pullback diagram in Strat

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ K \arrow[r, "g"] & K' \end{tikzcd} \qquad \text{such that the diagram of sections} \qquad \begin{tikzcd} X \arrow[r, "g'"] & X' \\ K \arrow[r, "g"] \arrow[u, "s"] & K' \arrow[u, "s'"'] \end{tikzcd}$$

commutes. Composition is given by concatenating such squares horizontally and composing horizontal arrows.

Notice the projection $\mathfrak{Exit} \to \mathsf{Bun}$, given by $(X \underset{f}{\overset{s}{\leftrightarrows}} K) \mapsto (X \xrightarrow{f} K)$, and the composite functor $\mathfrak{Exit} \to \mathsf{Bun} \to \mathsf{Strat}$.

Theorem 6.4.2. *The functor* $\mathfrak{Exit} \to \mathsf{Strat}$ *is a transversality sheaf.*

Proof. We show the six conditions, in sequence. Most of these arguments follow those proving that Bun is a transversality sheaf (Theorem 6.3.3), so we will be brief.

Right fibration: Since sections restrict contravariantly, the projection $\mathfrak{Exit} \to \mathsf{Bun}$ is a category fibered in sets. In particular, this projection is a right fibration. It follows that the composition $\mathfrak{Exit} \to \mathsf{Bun} \to \mathsf{Strat}$ is a right fibration, because $\mathsf{Bun} \to \mathsf{Strat}$ is. Note also that, like Bun, for each weakly regular subspace $K_0 \subset K$, the restriction $\mathfrak{Exit}(K) \to \mathfrak{Exit}(K_0)$ is an isofibration, since sections can be extended along regular neighborhoods.

Isotopy extension: This follows a nearly identical argument as the verification of the isofibration condition for Bun.

Isotopy equivalence: For each constructible bundle with a section $X \leftrightarrows K \times \mathbb{R}$, there is an isomorphism $X \cong X_0 \times \mathbb{R}$ over $K \times \mathbb{R}$ to a product map with the zero-section. The desired essential surjectivity follows.

Cone-local sheaf: After the corresponding properties of $\mathsf{Bun} \to \mathsf{Strat}$, the sheaf condition, and also the cone-local condition, follows for $\mathfrak{Exit}$ because covering sieves, as well as blow-up squares, are colimits in Strat.

Consecutive: Because $\mathfrak{Exit} \to \mathsf{Bun}$ is fibered in sets, and because Bun satisfies the consecutive condition, then we are reduced to the following problem.

For each constructible bundle $X \to \Delta^p$, each section

$$\Delta^{\{0<\cdots<k\}} \bigcup_{\Delta^{\{k\}}} \Delta^{\{k<\cdots<p\}} \xrightarrow{s_0} X_{|\Delta^{\{0<\cdots<k\}} \underset{\Delta^{\{k\}}}{\cup} \Delta^{\{k<\cdots<p\}}}$$

can be extended to a section $\Delta^p \xrightarrow{s} X$.

Denote $X_0 := X_{|\Delta^{\{0<\cdots<k\}} \underset{\Delta^{\{k\}}}{\cup} \Delta^{\{k<\cdots<p\}}}$. Necessarily, the section s_0 is a constructible embedding. So there is a refinement $\widetilde{X}_0 \to X_0$, over the identity refinement of $\Delta^{\{0<\cdots<k\}} \bigcup_{\Delta^{\{k\}}} \Delta^{\{k<\cdots<p\}}$, with respect to which s_0 is constructible. This data $\widetilde{X}_0 \to \Delta^{\{0<\cdots<k\}} \bigcup_{\Delta^{\{k\}}} \Delta^{\{k<\cdots<p\}}$ is an object of $\mathsf{Bun}\big(\Delta^{\{0<\cdots<k\}} \bigcup_{\Delta^{\{k\}}} \Delta^{\{k<\cdots<p\}}\big)$. Choose an extension of $(\widetilde{X}_0 \to \Delta^{\{0<\cdots<k\}} \bigcup_{\Delta^{\{k\}}} \Delta^{\{k<\cdots<p\}})$ to $(\widetilde{X} \to \Delta^p)$ – such a choice is possible since Bun is consecutive. Consider the intersection $S \subset \widetilde{X}$ of constructible subspaces that contain the image of the constructible section s_0 – this is a constructible subspace. From Lemma 6.1.6, the projection $S \to \Delta^p$ is a constructible bundle; by construction, it is a bijection. It follows through standard reasoning that the projection $S \to \Delta^p$ is in fact an isomorphism. The inverse of this isomorphism is a constructible section $\widetilde{s}\colon \Delta^p \to \widetilde{X}$ extending s_0. Finally, Corollary 6.2.9 implies there is a refinement $\widetilde{X} \to X$ over the identity refinement of Δ^p. The composition $\Delta^p \xrightarrow{\widetilde{s}} \widetilde{X} \to X$ is the desired section.

Univalent: For Bun, the proof of univalence is implied by the equivalence $\mathsf{Aut}(X) \simeq \mathsf{Emb}^{\sim}(X, X)$. The statement for $\mathfrak{Exit}$ follows by the identical argument given the equivalence

$$\mathsf{Aut}_*(X) \simeq \mathsf{Emb}_*^{\sim}(X, X)$$

between pointed automorphisms and pointed self-embedding of a pointed stratified space X which isotopic to pointed automorphisms. The pointed assertion follows from the unpointed one, since these spaces are fibers of evaluation maps to X. ☐

The following result connects the three main striation sheaves considered in this paper, asserting that the fibers of $\mathcal{E}\mathsf{xit}$ over $\mathcal{B}\mathsf{un}$ consist of exit-path ∞-categories. This explains our calling $\mathcal{E}\mathsf{xit}$ the absolute exit-path ∞-category.

Proposition 6.4.3. *For each constructible bundle $X \xrightarrow{f} K$, classified by a functor* $\mathsf{Exit}(K) \xrightarrow{(X \xrightarrow{f} K)} \mathcal{B}\mathsf{un}$*, the diagram among ∞-categories*

$$\begin{tikzcd}
\mathsf{Exit}(X) \arrow[r, "(X \underset{K}{\times} X \rightleftarrows X)"] \arrow[d, "\mathsf{Exit}(f)"'] & \mathcal{E}\mathsf{xit} \arrow[d] \\
\mathsf{Exit}(K) \arrow[r, "(X \xrightarrow{f} K)"] & \mathcal{B}\mathsf{un}
\end{tikzcd}$$

is a pullback.

Proof. Let $Z \to K$ be a conically smooth map between stratified spaces. Denote the pullback $X_{|Z} := Z \times_K X$, which exists by Lemma 6.1.8. The space of Z-points of $\mathsf{Exit}(X)$ over $Z \to K$ is the space of sections $\Gamma\big(X_{|Z} \to Z\big)$. □

6.5. **Classes of constructible bundles.** We isolate several useful classes of constructible bundles, and show that some of them form a factorization system on the ∞-category $\mathcal{B}\mathsf{un}$.

Consider a constructible bundle $X \xrightarrow{f} K$, as well as a closed constructible subspace $K_0 \subset K$. Denote the preimage $X_0 := f^{-1}K_0 \subset X$. There are blow-up diagrams

$$\begin{tikzcd}
\mathsf{Link}_{X_0}(X) \arrow[r] \arrow[d, "\pi_{X_0}"'] & \mathsf{Unzip}_{X_0}(X) \arrow[d] \\
X_0 \arrow[r] & X
\end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}
\mathsf{Link}_{K_0}(K) \arrow[r] \arrow[d] & \mathsf{Unzip}_{K_0}(K) \arrow[d] \\
K_0 \arrow[r] & K
\end{tikzcd}$$

and a constructible map from the left diagram to the right diagram. Also, there exists an open conically smooth map under $\mathsf{Link}_{X_0}(X)$

$$\gamma_{X_0}\colon \mathsf{Link}_{X_0}(X) \times [0,1) \longrightarrow \mathsf{Unzip}_{X_0}(X)$$

over an open conically smooth map under $\mathsf{Link}_{K_0}(K)$

$$\mathsf{Link}_{K_0}(K) \times [0,1) \longrightarrow \mathsf{Unzip}_{K_0}(K)~.$$

Definition 6.5.1 (Class ψ)**.** For the situation above:

(1) The constructible bundle f is *closed* at K_0 if
- the constructible proper map $\pi_{X_0}\colon \mathsf{Link}_{X_0}(X) \to X_0$ is an embedding;
- the open conically smooth $\gamma_{X_0}\colon \mathsf{Link}_{X_0}(X) \times [0,1) \cong \mathsf{Unzip}_{X_0}(X)_{|\mathsf{Link}_{K_0}(K)\times[0,1)}$ can be chosen to be an isomorphism.

(2) The constructible bundle f is *creating* at K_0 if
- the constructible proper map $\pi_{X_0}\colon \mathsf{Link}_{X_0}(X) \to X_0$ is surjective;
- the open conically smooth $\gamma_{X_0}\colon \mathsf{Link}_{X_0}(X) \times [0,1) \cong \mathsf{Unzip}_{X_0}(X)_{|\mathsf{Link}_{K_0}(K)\times[0,1)}$ can be chosen to be an isomorphism.

(3) The constructible bundle f is *refining* at K_0 if
- the constructible proper map $\pi_{X_0}\colon \mathsf{Link}_{X_0}(X) \xrightarrow{\cong} X_0$ is an isomorphism;
- the open conically smooth $\gamma_{X_0}\colon \mathsf{Link}_{X_0}(X) \times [0,1) \to \mathsf{Unzip}_{X_0}(X)_{|\mathsf{Link}_{K_0}(K)\times[0,1)}$ can be chosen to be a refinement.

(4) The constructible bundle f is *embedding* at K_0 if
- the constructible proper map $\pi_{X_0}\colon \mathsf{Link}_{X_0}(X) \xrightarrow{\cong} X_0$ is an isomorphism;
- the open conically smooth $\gamma_{X_0}\colon \mathsf{Link}_{X_0}(X) \times [0,1) \hookrightarrow \mathsf{Unzip}_{X_0}(X)_{|\mathsf{Link}_{K_0}(K)\times[0,1)}$ can be chosen to be an embedding.

There are some umbrella classes.

(1) The constructible bundle f is *active* at K_0 if
- the constructible proper map $\pi_{X_0}\colon \mathsf{Link}_{X_0}(X) \to X_0$ is surjective.

(2) The constructible bundle f is *proper constructible* at K_0 if

- the open conically smooth map $\gamma_{X_0}\colon \mathsf{Link}_{X_0}(X)\times[0,1) \cong \mathsf{Unzip}_{X_0}(X)_{|\mathsf{Link}_{K_0}(K)\times[0,1)}$ can be chosen to be an isomorphism.

(3) The constructible bundle f is *open* at K_0 if
- the constructible proper map $\pi_{X_0}\colon \mathsf{Link}_{X_0}(X)\to X_0$ is an isomorphism.

For economy of language, we will use the placeholder "ψ" for any of the above classes of constructible bundles. We say f is of class ψ if it is at each closed constructible subspace $K_0\subset K$.

For the next observation, we consider the full subcategory $\mathsf{Cbl}^{\psi}(Y)\subset \mathsf{Cbl}(Y)$ of those constructible bundles over Y of class ψ.

Observation 6.5.2. For each class ψ of constructible bundle, the condition of a constructible bundle being of class ψ is independent of isomorphism-type over the base. Furthermore, for each compact stratified space Z, the preimage in $\mathsf{Burn}_1(Z)\to \mathsf{Cbl}\big(\mathsf{C}(Z)\big)$ of $\mathsf{Cbl}^{\psi}\big(\mathsf{C}(Z)\big)$ is the full subcategory consisting of those $(X_0 \xleftarrow{\pi} L \underset{\text{over } Z}{\xrightarrow{\gamma}} X_1)$ for which the constructible bundles $L\to Z\leftarrow X_1$ are of class ψ, and

- should ψ be *closed*, then π is an embedding and γ is isotopic to an isomorphism;
- should ψ be *creating*, then π is surjective and γ is isotopic to an isomorphism;
- should ψ be *refining*, then π is an isomorphism and γ is isotopic to a refinement;
- should ψ be *embedding*, then π is an isomorphism and γ is isotopic to an embedding;
- should ψ be *active*, then π is surjective;
- should ψ be *proper constructible*, then γ is isotopic to an isomorphism;
- should ψ be *open*, then π is an isomorphism.

Lemma 6.5.3. *For each class ψ of constructible bundle, the projection $\mathsf{Bun}^{\psi}\to\mathsf{Strat}$ is a transversality sheaf.*

Proof. Being of class ψ is preserved under base change, by inspection. It follows that the projection $\mathsf{Bun}^{\psi}\to\mathsf{Strat}$ is a right fibration. The proof that Bun satisfies the conditions of Theorem 5.1.2 specializes to a proof that Bun^{ψ} satisfies the conditions of Theorem 5.1.2, with the essential surjectivity part of the **consecutive** condition the only aspect that is not direct from inspection. To verify this essential surjectivity follows using Observation 6.5.2.

□

Definition 6.5.4. For each class ψ of constructible bundle, $\mathcal{B}\mathsf{un}^{\psi}$ is the ∞-category associated to the transversality sheaf Bun^{ψ} by way of the functor $\mathsf{Trans}\to\mathcal{S}\mathsf{tri}$ and the equivalence $\mathcal{S}\mathsf{tri}\simeq\mathsf{Cat}_\infty$ of Theorem 4.2.3.

For each class ψ of constructible bundle, the inclusion $\mathsf{Bun}^{\psi}\to\mathsf{Bun}$ of right fibrations over Strat determines a functor among ∞-categories $\mathcal{B}\mathsf{un}^{\psi}\to\mathcal{B}\mathsf{un}$.

Lemma 6.5.5. *For each class ψ of constructible bundle, the functor*

$$\mathcal{B}\mathsf{un}^{\psi}\to\mathcal{B}\mathsf{un}$$

is a monomorphism.

Proof. We must show that the map of spaces $\mathcal{B}\mathsf{un}^{\psi}(\Delta^p)\to\mathcal{B}\mathsf{un}(\Delta^p)$ is an inclusion of components for $p=0,1$. By construction, this map of spaces is represented by the inclusion of simplicial groupoids

$$\mathsf{Bun}^{\psi}(\Delta^p\times\Delta^{\bullet}_e)\ \hookrightarrow\ \mathsf{Bun}(\Delta^p\times\Delta^{\bullet}_e)\ .$$

As so, we can solve our problem by verifying the following assertion.

> Let $X_0\to\Delta^p$ and $X_1\to\Delta^p$ be two constructible bundles of class ψ. Suppose they are concordant as constructible bundles over Δ^p, which is to say there is a constructible bundle $X\to\Delta^p\times\Delta^1_e$ restricting over $\Delta^{\{i\}}_e$ as X_i. Then $X\to\Delta^p\times\Delta^1_e$ too is of class ψ.

Lemma 6.1.10 grants an isomorphism $X \cong X_{\frac{1}{2}} \times \Delta^1_e$ over $\Delta^p \times \Delta^1_e$. The assertion follows using the first part of Observation 6.5.2.

□

We phrase the next result in terms of *factorization systems*, which we first define. For $\mathcal{C}$ an ∞-category, we say a pair $(\mathcal{L}, \mathcal{R})$ of ∞-subcategories of $\mathcal{C}$ forms a *factorization system* on $\mathcal{C}$ if, for each morphism $[1] \xrightarrow{f} \mathcal{C}$ the ∞-category of $(\mathcal{L}, \mathcal{R})$-factorizations $\mathsf{Fact}_{\mathcal{L},\mathcal{R}}(f)$ is terminal. Here $\mathsf{Fact}_{\mathcal{L},\mathcal{R}}(f)$ is defined as the pullback in the diagram among functor ∞-categories

$$\begin{tikzcd}
\mathsf{Fact}_{\mathcal{L},\mathcal{R}}(f) \arrow[rrr] \arrow[d] & & & \mathcal{C}^{\{0<1<2\}} \arrow[d] \\
\mathcal{L}^{\{0<1\}} \times \mathcal{R}^{\{1<2\}} \arrow[r, "\{f\}"] & \mathcal{L}^{\{0<1\}} \times \mathcal{C}^{\{1<2\}} \times \mathcal{R}^{\{1<2\}} \arrow[rr] & & \mathcal{C}^{\{0<1\}} \times \mathcal{C}^{\{0<2\}} \times \mathcal{C}^{\{1<2\}}.
\end{tikzcd}$$

(This notion of factorization system is weaker than that defined in [Lu2].)

Theorem 6.5.6. *The pair of ∞-subcategories $(\mathcal{B}\mathsf{un}^{\mathsf{cls}}, \mathcal{B}\mathsf{un}^{\mathsf{act}})$ forms a factorization system on $\mathcal{B}\mathsf{un}$.*

Proof. Fix a constructible bundle $E \xrightarrow{f} \Delta^1$. Choose an open conically smooth map γ as in the span among stratified spaces

$$E_{|\Delta^{\{0\}}} \xleftarrow{\pi} \mathsf{Link}_{E_{|\Delta^{\{0\}}}}(E) \xrightarrow{\gamma} E_{|\Delta^{\{1\}}} \tag{29}$$

in where π is the standard projection from the link. By way of the equivalence (24), the concatenation of spans

$$E_{|\Delta^{\{0\}}} \xleftarrow{\pi} \mathsf{Link}_{E_{|\Delta^{\{0\}}}}(E) \xrightarrow{=} \mathsf{Link}_{E_{|\Delta^{\{0\}}}}(E) \xleftarrow{=} \mathsf{Link}_{E_{|\Delta^{\{0\}}}}(E) \xrightarrow{\gamma} E_{|\Delta^{\{1\}}} \tag{30}$$

determines object of $\mathsf{Burn}_2(*)$. The functor (25) thus assigns a constructible bundle $\overline{E} \to \overline{\mathsf{C}}^2(*) \cong \Delta^2$ to this concatenation of span.

Fix a functor $\mathsf{Exit}(Z) \xrightarrow{f_Z} \mathsf{Fact}_{\mathcal{B}\mathsf{un}^{\mathsf{cls}},\mathcal{B}\mathsf{un}^{\mathsf{act}}}(f)$, with Z a compact stratified space. This functor f_Z classifies a constructible bundle $X \xrightarrow{f_Z} Z \times \Delta^2$ with the following properties:

- f_Z is *closed* at $Z \times \Delta^{\{0<1\}}$,
- f_Z is *active* at $Z \times \Delta^{\{1<2\}}$,
- there is an isomorphism $X_{|Z\times\Delta^{\{0<2\}}} \cong Z \times E$ over $Z \times \Delta^{\{0<2\}}$.

Choose an open conically smooth maps γ_{01} over $Z \times \Delta^{\{0<1\}}$ and γ_{12} over $Z \times \Delta^{\{1<2\}}$ as in the concatenation of spans that map constructibly to Z:

$$X_{|Z\times\Delta^{\{0\}}} \xleftarrow{\pi_{01}} \mathsf{Link}_{X_{|Z\times\Delta^{\{0\}}}}(X_{|Z\times\Delta^{\{0<1\}}}) \xrightarrow{\gamma_{01}} X_{|Z\times\Delta^{\{1\}}} \xleftarrow{\pi_{12}} \mathsf{Link}_{X_{|Z\times\Delta^{\{1\}}}}(X_{|Z\times\Delta^{\{1<2\}}}) \xrightarrow{\gamma_{12}} X_{|Z\times\Delta^{\{2\}}}. \tag{31}$$

The equivalence (24) recognizes this concatenation of spans as an object of $\mathsf{Burn}_2(*)$ constructibly over Z. An ultimate application of Corollary 6.2.9 gives that, up to isomorphism over $Z \times \Delta^2$, constructible bundle f_Z is determined from the diagram (31).

The first condition on f_Z in particular implies the map γ_{01} can be taken to be an isomorphism, and so the composing the spans of (31) gives a span

$$X_{|Z\times\Delta^{\{0\}}} \xleftarrow{\pi_{12}\circ\gamma_{01}^{-1}\circ\pi_{01}} \mathsf{Link}_{X_{|Z\times\Delta^{\{1\}}}}(X_{|Z\times\Delta^{\{1<2\}}}) \xrightarrow{\gamma_{12}} X_{|Z\times\Delta^{\{2\}}}. \tag{32}$$

Through an ultimate application of Corollary 6.2.10, there results a map from the diagram (32) to the diagram

$$X_{|Z\times\Delta^{\{0\}}} \xleftarrow{\pi_{02}} \mathsf{Link}_{X_{|Z\times\Delta^{\{0\}}}}(X_{|Z\times\Delta^{\{0<2\}}}) \xrightarrow{\gamma_{02}} X_{|Z\times\Delta^{\{2\}}} \tag{33}$$

under $X_{|Z\times\Delta^{\{0\}}}$ and $X_{|Z\times\Delta^{\{2\}}}$, and constructibly over Z. The first condition on f_Z further implies π_{01} is an embedding, and the second condition on f_Z implies π_{12} is surjective. Because there are no refinements from $\mathsf{Link}_{X_{|Z\times\Delta^{\{1\}}}}(X_{|Z\times\Delta^{\{1<2\}}})$ factoring π_{12} and γ_{12}, then we conclude that there are no refinements from $\mathsf{Link}_{X_{|Z\times\Delta^{\{1\}}}}(X_{|Z\times\Delta^{\{1<2\}}})$ factoring $\pi_{12}\circ\gamma_{01}^{-1}\circ\pi_{01}$ and γ_{12}. Another ultimate application of Corollary 6.2.10 gives that this map of diagrams (32) $\to$ (33) is in fact an equivalence. Through the same ongoing reasoning, (33) determines, up to isomorphism, the restricted constructible bundle $X_{|Z\times\Delta^{\{0<2\}}} \to Z\times\Delta^{\{0<2\}}$. The third condition on f_Z thus implies (33) is isomorphic to (29). In this way, we conclude an isomorphism $X \cong Z\times\overline{E}$ over $Z\times\Delta^2$. This isomorphism is classified by an equivalence between the functor $\mathsf{Exit}(Z)\xrightarrow{f_Z}\mathsf{Fact}_{\mathcal{B}\mathsf{un}^{\mathsf{cls}},\mathcal{B}\mathsf{un}^{\mathsf{act}}}(f)$ and the constant functor at $\{f\}$.

□

6.6. **Subcategories of $\mathcal{B}\mathsf{un}$.** We now realize certain ∞-subcategories of $\mathcal{B}\mathsf{un}$ in terms of subcategories of Strat.

Definition 6.6.1. The subcategory

$$\mathsf{Strat}^{\mathsf{open}}\subset\mathsf{Strat}$$

has as morphisms those maps of stratified spaces which are open embeddings of underlying topological spaces. The categories

$$\mathsf{Strat}^{\mathsf{ref}}\ \text{ and }\ \mathsf{Strat}^{\mathsf{emb}}$$

are the further subcategories whose morphisms are: homeomorphisms of underlying topological spaces (for $\mathsf{Strat}^{\mathsf{ref}}$); open embeddings of stratified spaces (for $\mathsf{Strat}^{\mathsf{emb}}$).

Remark 6.6.2. In [AFT], $\mathcal{S}\mathsf{trat}^{\mathsf{emb}}$ is denoted $\mathcal{S}\mathsf{nglr}$.

Definition 6.6.3. The category

$$\mathsf{Strat}^{\mathsf{p.cbl}}$$

has as objects stratified spaces and as morphisms those maps which are both proper and constructible. The categories

$$\mathsf{Strat}^{\mathsf{p.cbl,inj}}\qquad\text{and}\qquad\mathsf{Strat}^{\mathsf{p.cbl,surj}}$$

are the further subcategories whose morphisms are additionally injective or surjective, respectively.

Definition 6.6.4. Given a subcategory $\mathsf{Strat}^{\psi}\subset\mathsf{Strat}$ and two constructible bundles $X\to K$ and $Y\to K$, the set

$$\mathsf{Strat}^{\psi}_{K}(X,Y)\subset\mathsf{Strat}_{K}(X,Y)$$

consists of those maps which belong to Strat^{ψ} fiberwise: for every $t\in K$, the map of fibers $X_t\to Y_t$ belongs to Strat^{ψ}. A subcategory $\mathsf{Strat}^{\psi}\subset\mathsf{Strat}$ is parametrizable if for every pair of stratified spaces X and Y the simplicial set

$$\mathsf{Strat}^{\psi}_{\Delta^{\bullet}_{e}}(X\times\Delta^{\bullet}_{e},Y\times\Delta^{\bullet}_{e})$$

is a Kan complex. For Strat^{ψ} parametrizable, the associated Kan-enriched category is $\mathcal{S}\mathsf{trat}^{\psi}$.

Lemma 6.6.5. *For Strat^{ψ} one of the subcategories $\mathsf{Strat}^{\mathsf{open}}$, $\mathsf{Strat}^{\mathsf{ref}}$, $\mathsf{Strat}^{\mathsf{emb}}$, $\mathsf{Strat}^{\mathsf{p.cbl}}$, $\mathsf{Strat}^{\mathsf{p.cbl,surj}}$, or $\mathsf{Strat}^{\mathsf{p.cbl,surj}}$, then for any stratified spaces X and Y, the simplicial set*

$$\mathsf{Strat}^{\psi}_{\Delta^{\bullet}_{e}}(X\times\Delta^{\bullet}_{e},Y\times\Delta^{\bullet}_{e})$$

is a Kan complex.

Proof. The proof from [AFT] that $\mathsf{Strat}_{\Delta^{\bullet}_{e}}(X\times\Delta^{\bullet}_{e},Y\times\Delta^{\bullet}_{e})$ is itself a Kan complex immediately extends to these cases. □

We give a homotopy equivalent description of these Kan-enriched categories $\mathcal{S}\mathsf{trat}^{\psi}$ using the model of simplicial objects in simplicial groupoids.

Definition 6.6.6. The value of the bisimplicial groupoid $\mathsf{Strat}^{\psi}_{\star,\bullet}$ on $([p],[q])$ is the subcategory

$$\mathsf{Strat}^{\psi}_{p,q} \subset \mathsf{Fun}([p], \mathsf{Strat}_{/\Delta^q_e})^{\sim}$$

consisting of those sequences

$$\begin{tikzcd} X_0 \arrow[r] \arrow[drr] & X_1 \arrow[r] \arrow[dr] & \ldots \arrow[r] & X_p \arrow[dl] \\ & & \Delta^q_e & \end{tikzcd}$$

for which each map $X_i \to \Delta^q_e$ is a fiber bundle and each map $X_i \to X_{i+1}$ belongs to $\mathsf{Strat}^{\psi}_{\Delta^q_e}(X_i, X_{i+1})$.

Note that the existence of pullbacks along constructible bundles ensures that $\mathsf{Strat}^{\psi}_{\star,\bullet}$ is indeed a bisimplicial object.

Definition 6.6.7. The simplicial space $|\,\mathsf{Strat}^{\psi}_{\star,\bullet}\,|$ is the vertical geometric realization of $\mathsf{Strat}^{\psi}_{\star,\bullet}$ (i.e., in $\bullet$ direction).

Lemma 6.6.8. *The simplicial spaces* $|\,\mathsf{Strat}^{\mathsf{p.cbl}}_{\star,\bullet}\,|$ *and* $|\,\mathsf{Strat}^{\mathsf{open}}_{\star,\bullet}\,|$ *are complete Segal spaces.*

Proof. The Segal condition is immediate. Completeness follows by observing that the spaces of objects are a disjoint union over all isomorphism classes $[X]$ of stratified spaces of the spaces of automorphisms $\mathsf{Aut}(X)$. This is given by identifying $\Delta^{\bullet}_e$ and the interior of $\Delta^{\bullet}$. For $\mathcal{S}\mathsf{trat}^{\mathsf{open}}$, the underlying space of objects is a disjoint of union of spaces $\mathsf{Emb}^{\sim}(X, X)$, stratified embeddings which are isotopic to automorphisms, so we additionally use the equivalence $\mathsf{Aut}(X) \simeq \mathsf{Emb}^{\sim}(X, X)$ established in proving the univalence of Bun.

□

Lemma 6.6.9. *There is a natural equivalence*

$$\mathcal{S}\mathsf{trat}^{\psi} \simeq |\,\mathsf{Strat}^{\psi}_{\star,\bullet}\,|$$

for $\mathcal{S}\mathsf{trat}^{\psi}$ *one of the* ∞*-categories* $\mathcal{S}\mathsf{trat}^{\mathsf{open}}$, $\mathcal{S}\mathsf{trat}^{\mathsf{ref}}$, $\mathcal{S}\mathsf{trat}^{\mathsf{emb}}$, $\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}}$, $\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl,surj}}$, *or* $\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl,surj}}$.

Proof. There is a canonical map from $\mathcal{S}\mathsf{trat}^{\psi}$, as it is a localization. The equivalence on spaces of objects follows immediately, since in both sides are disjoint unions of automorphism spaces. The equivalence on morphism spaces follows from the fact that every constructible bundle $X \to \Delta^q_e$ splits as a product; once observed, the two sides have identical morphism spaces. □

In the following definition of the open cylinder of a composable sequence of p open stratified maps $X_0 \to \ldots \to X_p$, we make use of the open complements of the standard filtration of the p-simplex,

$$\Delta^p_{>p-1} \subset \Delta^p_{>p-2} \subset \ldots \Delta^p_{>0} \subset \Delta^p \ ,$$

where $\Delta^p_{>i} \cong \Delta^p \smallsetminus \Delta^p_{\leq i}$.

Definition 6.6.10 (Open cylinder)**.** Let $X_0 \to X_1$ be an open conically smooth map. The *open cylinder* $\mathsf{Cylo}(X_0 \to X_1)$ is the pushout in stratified spaces

$$\begin{tikzcd} X_0 \times \Delta^1 \smallsetminus \{0\} \arrow[r] \arrow[d] & X_1 \times \Delta^1 \smallsetminus \{0\} \arrow[d] \\ X_0 \times \Delta^1 \arrow[r] & \mathsf{Cylo}\big(X_0 \to X_1\big) \ . \end{tikzcd}$$

Given a sequence of open conically smooth maps $X_0 \to \ldots \to X_p$, the *open cylinder* $\mathsf{Cylo}(X_0 \to \ldots \to X_p)$ is the iterated pushout in stratified spaces

$$X_0 \times \Delta^p \coprod_{X_0 \times \Delta^p_{>0}} X_1 \times \Delta^p_{>0} \cdots \coprod_{X_{p-2} \times \Delta^p_{>p-2}} X_{p-1} \times \Delta^p_{>p-2} \coprod_{X_{p-1} \times \Delta^p_{>p-1}} X_p \times \Delta^p_{>p-1} \ .$$

Observation 6.6.11. Let $X_i \to K$ be a collection of constructible bundle, for $0 \leq i \leq p$, and let $X_0 \to \dots X_p$ be a sequence of open conicallly smooth maps. The natural projection

$$\mathsf{Cylo}\big(X_0 \to \dots \to X_p\big) \longrightarrow \Delta^p \times K$$

is a constructible bundle, where Δ^p carries the standard stratification.

In the following definition of the cylinder of a composable sequence of p stratified maps $X_p \to \dots \to X_0$, we again make use of the standard filtration of the p-simplex, $\Delta^p_{\leq 0} \subset \Delta^p_{\leq 1} \subset \dots \Delta^p_{\leq p} = \Delta^p$. For notational economy, we identify $\Delta^p_{\leq i} \cong \Delta^i$.

Definition 6.6.12 (Reversed cylinder)**.** Given a proper constructible bundle of stratified spaces $X_1 \to X_0$, the reversed cylinder is the pushout in stratified spaces

$$\mathsf{Cylr}\big(X_1 \to X_0\big) := X_0 \coprod_{X_1 \times \{0\}} X_1 \times \Delta^1 .$$

Given a composable sequence of stratified maps $X_p \to \dots \to X_0$, the reversed cylinder is

$$\mathsf{Cylr}\big(X_p \to \dots \to X_0\big) := X_0 \coprod_{X_1 \times \{0\}} X_1 \times \Delta^1 \coprod_{X_2 \times \Delta^1} X_2 \times \Delta^2 \coprod_{X_3 \times \Delta^2} \dots \coprod_{X_p \times \Delta^{p-1}} X_p \times \Delta^p .$$

Remark 6.6.13. The reversed cylinder admits an inductive description. There is a natural equivalence

$$\mathsf{Cylr}\big(X_p \to \dots \to X_0\big) \cong \mathsf{Cylr}\big(X_{p-1} \to \dots \to X_0\big) \coprod_{X_{p-1} \times \Delta^{p-1}} X_p \times \Delta^p .$$

Observation 6.6.14. Let $X_i \to K$ be a collection of constructible bundle, for $0 \leq i \leq p$, and let $X_p \to \dots \to X_0$ be a sequence of proper constructible bundles commuting with the maps to K. The natural projection

$$\mathsf{Cylr}\big(X_p \to \dots \to X_0\big) \longrightarrow \Delta^p \times K$$

is a constructible bundle, where Δ^p carries the standard stratification.

Theorem 6.6.15. *Taking reversed cylinders and open cylinders define functors to* $\mathcal{B}\mathsf{un}$

$$\mathcal{S}\mathsf{trat}^{\mathsf{open}} \xrightarrow{\ \mathsf{Cylo}\ } \mathcal{B}\mathsf{un} \xleftarrow{\ \mathsf{Cylr}\ } \big(\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl}}\big)^{\mathsf{op}}$$

which realize both sources as ∞-subcategories of $\mathcal{B}\mathsf{un}$. *These functors restrict to subsidiary equivalences:*

$$\mathcal{S}\mathsf{trat}^{\mathsf{emb}} \simeq \mathcal{B}\mathsf{un}^{\mathsf{emb}} \qquad \textit{and} \qquad \mathcal{S}\mathsf{trat}^{\mathsf{ref}} \simeq \mathcal{B}\mathsf{un}^{\mathsf{ref}} ,$$
$$\big(\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl,inj}}\big)^{\mathsf{op}} \simeq \mathcal{B}\mathsf{un}^{\mathsf{cls}} \qquad \textit{and} \qquad \big(\mathcal{S}\mathsf{trat}^{\mathsf{p.cbl,surj}}\big)^{\mathsf{op}} \simeq \mathcal{B}\mathsf{un}^{\mathsf{creat}} .$$

Proof. We first apply Observation 6.6.11 to show the open cylinder defines a map of bisimplicial groupoids

$$\mathsf{Cylr} : \mathsf{Strat}^{\mathsf{open}}_{\star,\bullet} \longrightarrow \mathsf{Bun}(\Delta^\star \times \Delta^\bullet_e) .$$

This sends an object of $\mathsf{Strat}^{\mathsf{open}}_{p,q}$ given by

$$\begin{tikzcd} X_0 \arrow[r] \arrow[drr] & X_1 \arrow[r] \arrow[dr] & \dots \arrow[r] & X_{p-1} \arrow[r] \arrow[dl] & X_p \arrow[dll] \\ & & \Delta^q_e & & \end{tikzcd}$$

to the constructible bundle of stratified spaces

$$\mathsf{Cylo}\big(X_0 \to \dots \to X_p\big) \longrightarrow \Delta^p \times \Delta^q_e$$

which is an object of the groupoid $\mathsf{Bun}(\Delta^p \times \Delta^q_e)$. This is exactly Observation 6.6.11 applied in the case $K = \Delta^q_e$. We show that the map of spaces

$$\big|\mathsf{Strat}^{\mathsf{open}}_{p,\bullet}\big| \longrightarrow \big|\mathsf{Bun}(\Delta^p \times \Delta^\bullet_e)\big|$$

is an inclusion of components. First, it is immediate that this is an equivalence for $p = 0$, since even more is true: the map

$$\mathsf{Strat}^{\mathsf{open}}_{0,\bullet} \longrightarrow \mathsf{Bun}(\Delta^\bullet_e)$$

is an isomorphism of simplicial groupoids. It now suffices, since both sides satisfy the Segal condition, to show the case of $p = 1$.

To do so, we introduce an auxiliary space $\mathcal{B}\mathsf{un}^{\mathsf{triv}}$ of constructible bundles with choices of trivializations along strata, as given by a parallel vector field. That is, there is an fibration $\mathcal{B}\mathsf{un}^{\mathsf{triv}} \to \left|\mathsf{Bun}(\Delta^1 \times \Delta^\bullet_e)\right|$ whose fiber over a point $X \to \Delta^1$ is the space of isomorphisms

$$X_{|\Delta^1_{>0}} \cong X_1 \times \Delta^1_{>0}$$

over $\Delta^1_{>0}$, where X_1 is the fiber over $\{1\}$. Note that the open cylinder functor naturally lifts to a map

$$\left|\mathsf{Strat}^{\mathsf{open}}_{1,\bullet}\right| \longrightarrow \mathcal{B}\mathsf{un}^{\mathsf{triv}}(\Delta^1)$$

since such trivializations are present in the construction. Let $\mathcal{B}\mathsf{un}^{\mathsf{triv}}(\Delta^1)_o$ be subspace of components in the image of the open cylinder. Then we have a diagram

$$\begin{tikzcd}
\mathcal{B}\mathsf{un}^{\mathsf{triv}}(\Delta^1)_o \arrow[r, hook] \arrow[d, dashed, bend right] & \mathcal{B}\mathsf{un}^{\mathsf{triv}}(\Delta^1) \arrow[d, "\simeq"] \\
\left|\mathsf{Strat}^{\mathsf{open}}_{1,\bullet}\right| \arrow[r] \arrow[u] & \left|\mathsf{Bun}(\Delta^1 \times \Delta^\bullet_e)\right|
\end{tikzcd}$$

where the map above has an immediate retraction $\mathcal{B}\mathsf{un}^{\mathsf{triv}}(\Delta^1)_o \to \left|\mathsf{Strat}^{\mathsf{open}}_{1,\bullet}\right|$. This is immediately seen to be a homotopy equivalence, so the assertion follows. Further, this exchanges embedding/refining maps $X_0 \to X_1$ with maps which are embedding/refining, direct from the definition of the latter.

We next apply Observation 6.6.14 to show the reversed cylinder defines a map of bisimplicial groupoids

$$\mathsf{Cylr} : \left(\mathsf{Strat}^{\mathsf{p.cbl}}_{\star,\bullet}\right)^{\mathsf{op}} \longrightarrow \mathsf{Bun}(\Delta^\star \times \Delta^\bullet_e)\ .$$

This sends an object of $\mathsf{Strat}^{\mathsf{p.cbl}}_{p,q}$ given by

$$\begin{tikzcd}
X_p \arrow[r] \arrow[drr] & X_{p-1} \arrow[r] \arrow[dr] & \dots \arrow[r] & X_1 \arrow[r] \arrow[dl] & X_0 \arrow[dll] \\
& & \Delta^q_e & &
\end{tikzcd}$$

to the constructible bundle of stratified spaces

$$\mathsf{Cylr}\big(X_p \to \dots \to X_0\big) \longrightarrow \Delta^p \times \Delta^q_e$$

which is an object of the groupoid $\mathsf{Bun}(\Delta^p \times \Delta^q_e)$. This is exactly Observation 6.6.14 applied in the case $K = \Delta^q_e$. We show that the map of spaces

$$\left|(\mathsf{Strat}^{\mathsf{p.cbl}}_{p,\bullet})^{\mathsf{op}}\right| \longrightarrow \left|\mathsf{Bun}(\Delta^p \times \Delta^\bullet_e)\right|$$

is an inclusion of components. First, it is immediate that this is an equivalence for $p = 0$, since even more is true: the map

$$(\mathsf{Strat}^{\mathsf{p.cbl}}_{0,\bullet})^{\mathsf{op}} \longrightarrow \mathsf{Bun}(\Delta^\bullet_e)$$

is an isomorphism of simplicial groupoids. It now suffices, since both sides satisfy the Segal condition, to show the case of $p = 1$.

To do so, we again use the auxiliary space $\mathcal{B}\mathsf{un}^{\mathsf{triv}}$ of constructible bundles with choices of trivializations along strata. Let $\mathcal{B}\mathsf{un}^{\mathsf{triv}}(\Delta^1)_r$ be subspace of components which are in the image of the reversed cylinder. Then we have a diagram

$$\begin{tikzcd}
\mathcal{B}\mathsf{un}^{\mathsf{triv}}(\Delta^1)_r \arrow[r, hook] \arrow[d, dashed, bend right] & \mathcal{B}\mathsf{un}^{\mathsf{triv}}(\Delta^1) \arrow[d, "\simeq"] \\
\left|(\mathsf{Strat}^{\mathsf{p.cbl}}_{1,\bullet})^{\mathsf{op}}\right| \arrow[r] \arrow[u] & \left|\mathsf{Bun}(\Delta^1 \times \Delta^\bullet_e)\right|
\end{tikzcd}$$

where the map above has an immediate retraction $\mathcal{B}\mathsf{un}^{\mathsf{triv}}(\Delta^1)_r \to \left|(\mathsf{Strat}^{\mathsf{p.cbl}}_{1,\bullet})^{\mathsf{op}}\right|$. This is immediately seen to be a homotopy equivalence, so the assertion follows. Further, this exchanges injective/surjective maps $X_1 \to X_0$ with maps which are closed/creation, direct from the definition of the latter.

□

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