ZERO-POINTED MANIFOLDS

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Abstract. We formulate a theory of pointed manifolds, accommodating both embeddings and Pontryagin-Thom collapse maps, so as to present a common generalization of Poincaré duality in topology and Koszul duality in $\mathcal{E}_n$-algebra.

Contents

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Introduction

In this work, we introduce zero-pointed manifolds as a tool to solve two apparently separate problems. The first problem, from manifold topology, is to generalize Poincaré duality to factorization homology; the second problem, from algebra, is to show the Koszul self-duality of n-disk, or $\mathcal{E}_n$, algebras. The category of zero-pointed manifolds can be thought of as a minimal home for manifolds generated by two kinds of maps, open embeddings and Pontryagin-Thom collapse maps of open embeddings. In this work, we show that this small formal modification of manifold topology gives rise to an inherent duality. Before describing zero-pointed manifolds, we recall these motivating problems in greater detail.

Factorization homology theory, after Lurie [Lu2], is a comparatively new area, growing out of ideas about configuration spaces from both conformal field theory and algebraic topology. Most directly, it is a topological analogue of Beilinson & Drinfeld's algebro-geometric factorization algebras of [BD]. In algebraic topology, factorization homology generalizes both usual homology and the labeled configuration spaces of Salvatore [Sa] and Segal [Se4]. See [AF1] for a more extended introduction. The last few years has seen great activity in this subject, well beyond the basic foundations laid in [Lu2], [AF1], and [AFT2], including Gaitsgory & Lurie's application of algebro-geometric factorization techniques to Tamagawa numbers in [Lu4], and Costello & Gwilliam's work on perturbative quantum field theory in [CG], where Costello's renormalization machine is made to output a factorization homology theory, an algebraic model for the observables in a quantum field theory.

One can ask if these generalized avatars of homology carry a form of Poincaré duality. An initial glitch in this question is that factorization homology is only covariantly natural with respect to open embeddings of manifolds, and one cannot formulate even usual Poincaré duality while only using push-forwards with respect to embeddings. One can then ask, en route to endowing factorization homology with a form of duality, as to the minimal home for manifold topology for which just usual Poincaré duality can be formulated. That is, a homology theory defines a covariant functor from $\mathcal{M}\mathsf{fld}_n$, n-manifolds with embeddings; a cohomology theory likewise defines a contravariant functor from $\mathcal{M}\mathsf{fld}_n$. For the formulation of duality results, what is the common geometric home for these two concepts?

As one answer to this question, in **Section 1** we define zero-pointed manifolds. Our category $\mathcal{Z}\mathcal{M}\mathsf{fld}_n$ consists of pointed topological spaces M_* for which the complement $M := M_* \smallsetminus *$ is an n-manifold; the essential example is a space $\overline{M}/\partial\overline{M}$, the quotient of an n-manifold by its boundary. The interesting feature of this category is the morphisms: a morphism of zero-pointed manifolds is a pointed map $f : M_* \to N_*$ such that the restriction away from the zero-point, $f^{-1}N \to N$, is an open embedding. This category $\mathcal{Z}\mathcal{M}\mathsf{fld}_n$ contains both $\mathcal{M}\mathsf{fld}_n$ and $\mathcal{M}\mathsf{fld}_n^{\mathsf{op}}$, the first by adding a disjoint basepoint and the second by 1-point compactifying. A functor from $\mathcal{Z}\mathcal{M}\mathsf{fld}_n$ thus has both push-forwards and extensions by zero, and both homology and cohomology can be thought of as covariant functor from $\mathcal{Z}\mathcal{M}\mathsf{fld}_n$. Lemma 1.6.7 implies an isomorphism $\neg : \mathcal{Z}\mathcal{M}\mathsf{fld}_n \cong \mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{op}}$ between the category of zero-pointed n-manifolds and its own opposite, which presages further duality.

In **Section 3**, we extend the notion of factorization homology to zero-pointed manifolds. This gives a geometric construction of additional functorialities for factorization homology with coefficients in an augmented n-disk algebra. Namely, there exists extension by zero maps. In particular, the factorization homology

$$\int_{(\mathbb{R}^n)^+} A$$

has the structure of an n-disk coalgebra via the pinch map, where $(\mathbb{R}^n)^+$ is the 1-point compactification of $\mathbb{R}^n$. By identifying the factorization homology of $(\mathbb{R}^n)^+$ with the n-fold iterated bar construction, we arrive at an n-disk coalgebra structure on the n-fold iterated bar construction, or topological André-Quillen homology, of an augmented n-disk algebra.

This construction is closely bound to the Koszul self-duality of the $\mathcal{E}_n$ operad, first conceived by Getzler & Jones in [GJ] contemporaneously with Ginzburg & Kapranov's originating theory of [GK]. Namely, it has long been believed that the operadic bar construction $\mathsf{Bar}\,\mathcal{E}_n$ of the $\mathcal{E}_n$ operad is equivalent to an n-fold shift of the $\mathcal{E}_n$ co-operad. This is interesting because the bar construction extends to a functor $\mathsf{Alg}^{\mathsf{aug}}_{\mathcal{O}} \longrightarrow \mathsf{cAlg}^{\mathsf{aug}}_{\mathsf{Bar}\,\mathcal{O}}$ from augmented $\mathcal{O}$-algebras to augmented coalgebras. If one stably identifies $\mathsf{Bar}\,\mathcal{E}_n$ and a shift of $\mathcal{E}_n$, then one can construct a functor

$$\mathsf{Alg}^{\mathsf{aug}}_{\mathcal{E}_n} \longrightarrow \mathsf{cAlg}^{\mathsf{aug}}_{\mathcal{E}_n}$$

from augmented $\mathcal{E}_n$-algebras to augmented $\mathcal{E}_n$-coalgebras. We construct exactly such a functor using this zero-pointed variant of factorization homology, which is given by taking the factorization homology of the pointed n-sphere $(\mathbb{R}^n)^+$. In order to reduce from n-disk algebras to $\mathcal{E}_n$-algebras, we use the framed variant of the theory which is a special case of theory of structured zero-pointed manifolds developed in **Section 2**.

A construction of such a functor has been previously accomplished by other means. Fresse performed the chain-level calculation of the Koszul self-duality of $\mathsf{C}_*(\mathcal{E}_n, R)$ in [Fr]. A direct calculation of self-duality of the bar construction of the operad $\mathcal{E}_n$ in spectra has not yet been given. The construction of a functor as above was however accomplished in full generality by Lurie in [Lu3] using a formalism for duality given by twisted-arrow categories. We defer a comparison of our construction and theirs to a future work; we will not need to make use of any comparison in this work or its sequel [AF2].

A virtue of our construction is that it is easy, geometric, and for our purposes accomplishes more via the connection to factorization homology. That is, in **Section 4** we use this geometry to construct the Poincaré/Koszul duality map. Given a functor $\mathcal{F}$ taking values on zero-pointed n-manifolds M_*, we obtain maps

$$\int_{M_*} \mathcal{F}(\mathbb{R}^n_+) \longrightarrow \mathcal{F}(M_*) \longrightarrow \int^{M_*^\neg} \mathcal{F}\big((\mathbb{R}^n)^+\big) \ .$$

The lefthand map is a universal left approximation by a factorization homology theory; the righthand map is a universal right approximation by a factorization cohomology theory. The composite map is the Poincaré/Koszul duality map. While the operadic approach to constructing the functor from $\mathcal{E}_n$-algebras to $\mathcal{E}_n$-coalgebras requires one to work stably, such as in chain complexes or spectra, factorization homology applies unstably: in the case in which $\mathcal{F}$ is a functor to spaces, the Poincaré/Koszul duality map generalizes the scanning maps of McDuff [Mc] and Segal [Se1], as well as [Bö], [Ka], and [Sa], which arose in the theory of configuration space models of mapping spaces. Finally, we prove in Theorem 4.7.1 that the Poincaré/Koszul duality map is an equivalence for a Cartesian-presentable target. In particular, this gives a new proof of the non-abelian Poincaré duality of Lurie in [Lu2]. Our result further specializes to a version linear Poincaré duality, which assures that our duality map is an equivalence in the case of a stable ∞-category with direct sum; in this last case, our Poincaré/Koszul duality map becomes the Poincaré duality map of [DWW].

We note that the most appealing aspects of this work, such as the notions of zero-pointed manifolds and their basic properties, are not difficult. The comparatively technical stretch of this paper lies in **Section 3**, where we show that factorization homology of zero-pointed manifolds is well-defined and well-behaved. The mix of ∞-category theory and point-set topology around the zero-point introduces bad behavior in $\mathcal{D}\mathsf{isk}_{n,+/M_*}$, the slice ∞-category appearing in the definition of factorization homology. Consequently, we make recourse to a hand-crafted auxiliary version of this disk category, $\mathcal{D}\mathsf{isk}_+(M_*)$. This adaptation has two essential features. First, $\mathcal{D}\mathsf{isk}_+(M_*)$ is sifted, a property which is necessary to show that factorization homology exists as a symmetric monoidal functor. Second, $\mathcal{D}\mathsf{isk}_+(M_*)$ has a natural filatration by cardinality of embedded disks. This is an essential feature which $\mathcal{D}\mathsf{isk}_{n,+/M}$ lacks. This cardinality filtration gives rise to highly nontrivial filtrations on factorization homology. This generalizes the Goodwillie-Weiss embedding calculus of [We1] to functors on zero-pointed manifolds or, alternatively, to those functors on manifolds with boundary which are reduced on the boundary. In the case $n = 1$, in which case factorization homology of the circle is Hochschild homology, our cardinality filtration further specializes to the Hodge filtration on Hochschild homology developed by Burghelea-Vigu-Poirrier [BuVi], Feigin-Tsygan [FT], Gerstenhaber-Schack [GS], and Loday [Lo]; for general spaces, but still in the case of commutative algebras, our filtration specializes to the Hodge filtration of Pirashvili [Pi] and Glasman [Gl].

These small technical modifications involved in the construction of the auxiliary $\mathcal{D}\mathsf{isk}_+(M_*)$ play an essential role in the sequel [AF2]. An essential step therein shows that the Poincaré/Koszul duality map interchanges the cardinality filtration and the Goodwillie tower. That is, Goodwillie calculus and Goodwillie-Weiss calculus are Koszul dual in this context, a feature we ultimately use to present one solution as to when the Poincaré/Koszul duality map is an equivalence.

1. Zero-pointed spaces

For this section, we use the letters X, Y, and Z for locally compact Hausdorff topological spaces.

1.1. Heuristic motivation of definitions.

Definition 1.1.1 (Haus and Cov)**.** We denote the category of locally compact Hausdorff topological spaces, and open embeddings among them, as Haus, which we might regard as a simplicial set via the nerve construction. Consider the simplicial set $\mathsf{Ex}(\mathsf{Haus})$ for which a p-simplex is a map $\mathcal{P}_{\neq\emptyset}(\underline{p}) \to \mathsf{Haus}$ from the poset of non-empty subsets of $\underline{p} := \{0, \dots, p\}$. We define the sub-simplicial set

$$\mathsf{Cov} \ \subset \ \mathsf{Ex}(\mathsf{Haus})$$

for which a p-simplex $\mathcal{P}_{\neq\emptyset}(\underline{p}) \xrightarrow{X_-} \mathsf{Haus}$ belongs to Cov if for each $\emptyset \neq S \subset \underline{p}$ the collection of open embeddings $\bigl(X_T \hookrightarrow X_S\bigr)_{\{\emptyset\neq T\subsetneq S\}}$ is an open cover. More explicitly, a p-simplex of Cov is the data of an open cover $\bigl(X_i \hookrightarrow X\bigr)_{0\leq i\leq p}$ of a locally compact Hausdorff topological space, indexed by the set $\underline{p}$.

Disjoint union makes both Haus and Cov into commutative monoids in simplicial sets, which we think of as a symmetric monoidal simplicial set.

Remark 1.1.2. The simplicial set Cov is *not* a category; if it were, it would have been studied by now. Indeed, should Cov be a category then the canonical map of sets

$$\mathsf{Cov}[2] \longrightarrow \mathsf{Cov}\{0<1\} \underset{\mathsf{Cov}\{1\}}{\times} \mathsf{Cov}\{1<2\}$$

would be an isomorphism. Because of the open cover demand on the datum of a 2-simplex of Cov, this map is injective, but it is not surjective. For, given a pair of open covers of locally compact Hausdorff topological spaces $(O \hookrightarrow X \hookleftarrow O')$ and $(O' \hookrightarrow X' \hookleftarrow O'')$, while $(O \hookrightarrow X\coprod_{O'}X' \hookleftarrow O'')$ is an open cover of the pushout, this pushout need not be Hausdorff – an axiom that we maintain for many reasons.

The simplicial set Cov has the following properties.

1. The space $\emptyset$ is both initial and final, and is the symmetric monoidal unit.
2. There is a monomorphism $\mathsf{Haus} \longrightarrow \mathsf{Cov}$ given by regarding a finite sequence of open embeddings among locally compact Hausdorff topological spaces $\bigl(X_0 \hookrightarrow \dots \hookrightarrow X_p\bigr)$ as a cover of the final space X_p. And so we identify Haus with its image in Cov.
3. There is an isomorphism $\mathsf{Cov} \cong (\mathsf{Cov})^{\mathsf{op}}$ given by reversing the index-ordering of a sequence $\bigl(O_i \hookrightarrow X\bigr)_{0\leq i\leq p}$ of open embeddings. In particular, there is a monomorphism $\mathsf{Haus}^{\mathsf{op}} \to \mathsf{Cov}$.

(4) For each locally compact Hausdorff topological space X there is a preferred morphism $X \to X^{\mathsf{op}}$ in Cov – this morphism is natural in the sense that for each open embedding $X \xrightarrow{f} Y$ among such, the diagram in Cov

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & X^{\mathsf{op}} \\
Y \arrow[r] & Y^{\mathsf{op}} \arrow[u, "f^{\mathsf{op}}"']
\end{tikzcd}$$

commutes.

The cumulative effect of these four points is that any symmetric monoidal map out of Cov is augmented and coaugmented, and admits a compatible notion of 'extension by the unit' and 'restriction' for each open embedding. More precisely, the above points describe a restriction

$$\mathsf{Fun}^{\otimes}(\mathsf{Cov}, \mathcal{V}) \longrightarrow \mathsf{Fun}^{\otimes,\mathsf{aug}}\big(\mathsf{Haus}, \mathsf{TwAr}(\mathcal{V})\big)$$

to maps to the twisted arrow category of an arbitrary symmetric monoidal category $\mathcal{V}$. We may discuss this connection in a future work. Ultimately, we are interested only in the codomain of this restriction, which does not reference Cov (which will become "zero-pointed spaces" in just a moment) at all. An advantage of working with the domain of this restriction is that it is geometric, and it is intrinsic to Haus (i.e., without manipulation to $\mathcal{V}$).

We now follow-up on Remark 1.1.2, where we saw that the obstruction to Cov being a category was witnessed as the occurrence of non-Hausdorff pushouts along open embeddings. One might account for this by placing additional structure on vertices so that the target of an edge agrees with the source of another precisely if the corresponding pushout *is* Hausdorff. One articulation of such additional structure is that of a *pointed extension* in the sense of Definition 1.2.1, and we will explore others as well. In order to retain property (3), we further restrict our attention to *well-pointed* extensions. The resulting simplicial set is constructed just so it is a category, ZHaus, that we refer to as the (discrete) category of *zero-pointed spaces.*

It is conceivable to finish the story here, which would make for a more concise paper. However, we are compelled by other works and applications, and so we develop the story just outlined but for $\mathcal{H}\mathsf{aus}$ in place of Haus – the difference being that the Hom-sets are endowed with the compact-open topology, thereby resulting in an ∞-category. The most primitive reason we concern ourselves with this continuous version of this story is because *continuous* invariants are considerably more tractable than non-continuous ones (while those are quite interesting in their own right, they are less amenable to general classification results). We are referring here to the works [AFT1] and [AFT2] which, in particular, premise the work [AF2] thereby explaining a sense in which continuous sheaves are combinatorial entities ($\mathcal{E}_n$-algebras), and are the foundation of a large consideration of deformation problems (Koszul duality). And so, the bulk of this work amounts to establishing the above outline with topology present on sets of open embeddings.

1.2. **Pointed extensions and negation.** We define *pointed extensions* and *negations* thereof.

Definition 1.2.1. A pointed extension X_* of X is a compactly generated Hausdorff topology on the underlying set of $* \amalg X$ extending the given topology on X. The category of *pointed extensions (of X)* is the full subcategory

$$\mathsf{Point}_X \ \subset\ \mathsf{Top}^{X_+/}$$

of the undercategory consisting of the pointed extensions of X.

Proposition 1.2.2. *The following assertions concerning the category* Point_X *are true.*

- *It is a poset.*
- *It admits small colimits and, in particular, has an initial object.*
- *It admits small limits and, in particular, has a final object.*
- *Products distribute over small colimits.*

Proof. First, Point_X is a poset because a morphism $X_* \to X'_*$ in Point_X is necessarily the identity map on the underlying set. The category $\mathsf{Top}^{X_+/}$ admits small colimits and limits, with finite products distributing over small colimits. By inspection, $\mathsf{Point}_X \subset \mathsf{Top}_*^{X_+/}$ is closed under limits and colimits, so the result follows. $\square$

The data of a pointed extension X_* of X determines which sequences in X that leave compact subsets converge to infinity. As such, one can informally contemplate the *negation* $X_*^\neg$ of X_* by making the complementary declaration: such a sequence belongs to $X_*^\neg$ if and only if it does not belong to X_*. In other words, $X_*^\neg$ is endowed with the complementary topology about $*$ from that of X_*. The following makes this heuristic precise.

Corollary 1.2.3 (Negation)**.** *There is functor*

$$\neg\colon \mathsf{Point}_X^{\mathsf{op}} \longrightarrow \mathsf{Point}_X$$

characterized by assigning to each X_* the final object $X_*^\neg$ for which the natural map

$$X_+ \longrightarrow X_* \underset{X^+}{\times} X'_*$$

is a homeomorphism. This functor preserves limits, and negating twice receives a canonical natural transformation $\mathsf{id} \to \neg\neg$.

In other words, there exists a map of pointed extension $X'_* \to X_*^\neg$ if and only if the natural map $X_+ \to X_* \times_{X^+} X'_*$ is a homeomorphism. We will be largely interested in the following special class of pointed extensions.

Definition 1.2.4. A pointed extension X_* of X is *well-pointed* if the canonical map $X_* \xrightarrow{\cong} X_*^{\neg\neg}$ is a homeomorphism.

Consider the full subcategories $\mathsf{Point}_X^{\mathsf{well}} \subset \mathsf{Point}_X$ consisting of the well-pointed extensions. Twice-negation is a left adjoint to this inclusion

$$\neg\neg\colon \mathsf{Point}_X \leftrightarrows \mathsf{Point}_X^{\mathsf{well}} \tag{1}$$

implementing a localization. We will sometimes refer to this functor as *well-pointed replacement.*

Remark 1.2.5. While we are chiefly interested only in well-pointed extensions, they are not closed under typical constructions among pointed extensions. So we open our consideration to (not necessarily well-)pointed extensions so that various constructions are defined, and we recognize that well-pointed replacement can always be applied thereafter.

Remark 1.2.6. By construction, this category $\mathsf{Point}_X^{\mathsf{well}}$ is a Boolean algebra.

Example 1.2.7. The initial object of Point_X is X_+, the space X with a disjoint based point. The final object of Point_X is X^+, the one-point compactification of X. Both of these objects are well-pointed, and they are equal to each other's negations: $X_+^\neg = X^+$ and $X_+ = (X^+)^\neg$.

Example 1.2.8. The poset $\mathsf{Point}_\mathbb{R}$ has exactly four elements, and is a square. These are enumerated as $\mathbb{R}_+$ and $\overrightarrow{\mathbb{R}} := \big([-\infty,\infty),\{-\infty\}\big)$ and $\overleftarrow{\mathbb{R}} = \big((-\infty,\infty],\{\infty\}\big)$ and $\mathbb{R}^+$. These are abstractly related as $-1\colon \overrightarrow{\mathbb{R}} \cong \overleftarrow{\mathbb{R}} : -1$ and $(\overrightarrow{\mathbb{R}})^\neg = \overleftarrow{\mathbb{R}}$ and $\overrightarrow{\mathbb{R}} \underset{\mathbb{R}_+}{\amalg} \overleftarrow{\mathbb{R}} \cong \mathbb{R}^+$ and $\mathbb{R}_+ \cong \overrightarrow{\mathbb{R}} \underset{\mathbb{R}^+}{\times} \overleftarrow{\mathbb{R}}$.

Observation 1.2.9. The datum of a pointed extension of a locally compact Hausdorff topological space X can be rephrased in a number of ways: Namely, there are natural isomorphisms among the following posets.

- The poset of topologies $X_* = (X \amalg *, \tau)$ on the underlying set of $X \amalg *$ that satisfy the following conditions.
  - The inclusion of sets $X \hookrightarrow X \amalg *$ is an open embedding of topological spaces.
  - The topological space X_* is Hausdorff.

  The partial order $X_* \leq X'_*$ means the identity map $X_* \to X'_*$ is continuous.

- The poset of collections $\mathcal{P}$ of closed subsets of X that satisfy the following conditions.
  - Let $C \subset X$ be a closed subset, and let $P \in \mathcal{P}$. Then the intersection $P \cap C$ is a member of $\mathcal{P}$.
  - Let P and P' be members of $\mathcal{P}$. Then the union $P \cup P'$ is a member of $\mathcal{P}$.
  - Let $K \subset X$ be a compact subspace. Then there is an element $P \in \mathcal{P}$ for which $K \subset \mathsf{Int}(P)$ is contained in the interior.

  The partial order $\mathcal{P} \leq \mathcal{P}'$ means $\mathcal{P}' \subset \mathcal{P}$.
- The poset of subsets of the (point-set) boundary $\partial\overline{X} \subset \overline{X}$ of the Stone-Čech compactification of X.

  The partial order $S \leq S'$ means $S \subset S'$.

Indeed, such a Hausdorff extension X_* determines such a collection $\mathcal{P}_X := \{P \subset X \mid P \subset X_* \text{ is closed}\}$. Such a collection of closed subsets $\mathcal{P}$ determines such a subset $S_{\mathcal{P}} := \bigcap\limits_{P \in \mathcal{P}} (\partial\overline{X} \setminus \mathsf{Clsr}(P \subset \overline{X})) \subset \partial\overline{X}$. Such a subset $S \subset \partial\overline{X}$ determines such a Hausdorff extension $X_{*_S} := *\coprod\limits_{S}(S \cup X)$. Straightforward is that each of these assignments sends one partial order relation to another, and that the composition of any consecutive pair of these assignments is inverse to the third.

1.3. **Pointed embeddings.** We define *zero-pointed embeddings* among pointed extensions.

Definition 1.3.1 (Zero-pointed embeddings)**.** For X_* and Y_* be pointed extensions of X and Y, respectively, the space of *zero-pointed embeddings* from X_* to Y_* is the subspace

$$\mathsf{ZEmb}(X_*, Y_*) \ \subset \ \mathsf{Map}(X_*, Y_*)$$

of the compact-open topology consisting of those based maps $f\colon X_* \to Y_*$ for which the restriction $f_|\colon f^{-1}Y \to Y$ is an open embedding.

Note that $f \circ g$ is a zero-pointed embedding if both f and g are.

Lemma 1.3.2. *Let X_* and Y_* be pointed extensions of X and Y. Negation implements a continuous map*

$$\neg\colon \mathsf{ZEmb}(X_*, Y_*) \longrightarrow \mathsf{ZEmb}(Y_*^{\neg}, X_*^{\neg}) \ .$$

This map is a homeomorphism provided both X_* and Y_* are well-pointed.

Proof. The map of underlying based sets $\mathsf{ZEmb}(X_*, Y_*) \wedge Y_*^{\neg} \to X_*^{\neg}$ is determined by declaring the preimage of $x \in X$ to be the subset $\{(f, y) \mid f(x) = y\}$. Fix a continuous based map $q\colon Z_* \to Y^+$ for which the projection $Y_* \underset{Y^+}{\times} Z_* \to Y_*$ factors through Y_+. Consider the composite map of based sets

$$\alpha\colon \mathsf{ZEmb}(X_*, Y_*) \times Z_* \to \mathsf{ZEmb}(X_*, Y_*) \wedge Y_*^{\neg} \to X_*^{\neg} \to X^+ \ .$$

Let $K \subset X$ be a compact subspace. The preimage $\alpha^{-1}K$ consists of those pairs (f, z) for which $q(z) \in f(K)$, which is a closed subspace of the domain of α. It follows that α is continuous.

There is a commutative diagram of continuous maps among topological spaces

$$\begin{tikzcd}
X_+ \arrow[dd] & \Big(\mathsf{ZEmb}(X_*, Y_*) \times Z_*\Big) \underset{X^+}{\times} X_* \arrow[l, dashed, "(3)"'] \arrow[r, dashed, "(1)"] \arrow[d] & \mathsf{ZEmb}(X_*, Y_*) \times \big(Z_* \underset{Y^+}{\times} Y_*\big) \arrow[r, dashed, "(2)"] \arrow[d] & Y_+ \arrow[dd] \\
 & \Big(\mathsf{ZEmb}(X_*, Y_*) \times X_*\Big) \times Z_* \arrow[r, "(1{,}\mathsf{ev}) \times \mathsf{id}_Z"] \arrow[d] & \Big(\mathsf{ZEmb}(X_*, Y_*) \times Y_*\Big) \times Z_* \arrow[d] & \\
X_* & \mathsf{ZEmb}(X_*, Y_*) \times X_* \arrow[l] \arrow[r, "(1{,}\mathsf{ev})"] & \mathsf{ZEmb}(X_*, Y_*) \times Y_* \arrow[r] & Y_* \ .
\end{tikzcd}$$

The factorization depicted as (1) exists by inspection. By construction of q, the factorization depicted as (2) exists. The factorization depicted as (3) follows, which verifies that α factors continuously through $X_*^\neg$.

If X_* and Y_* are well-pointed, then $\neg\colon \mathsf{ZEmb}(X_*, Y_*) \leftrightarrows \mathsf{ZEmb}(Y_*^\neg, X_*^\neg)\colon \neg$ are mutual inverses, by inspection.

□

1.4. **Constructions.** We explain a few constructions among pointed extensions.
Throughout what follows, we consider only compactly generated Hausdorff topological spaces; in particular colimits are taken in that category.

Construction 1.4.1. Fix pointed extensions X_*, Y_*, and Z_* of X, Y, and Z, respectively.

Wedge: The wedge sum $X_* \vee Y_*$ is a pointed extension of the coproduct $X \amalg Y$.

Smash: The smash product $X_* \wedge Y_* := (X_* \times Y_*)/(X_* \vee Y_*)$ is a pointed extension of the product $X \times Y$. Smash distributes over wedge:

$$X_* \wedge (Y_* \vee Z_*) = (X_* \wedge Y_*) \vee (X_* \wedge Z_*)\ .$$

Coinv: Let G be a finite group acting on the based space X_*. The coinvariants $(X_*)_G$ is a pointed extension of the coinvariants X_G.

Sub: Let $W \subset X$ be a subspace. The union $W_{X_*} := * \cup W \subset X_*$ is a pointed extension of W. This pointed extension has the following universal property:

Let $f\colon Z_* \to X_*$ be a zero-pointed embedding. Suppose $f(Z_*) \smallsetminus * \subset W \subset X$. Then f factors through $W_{X_*} \to X_*$.

Quot: Let $W \subset X$ be a subspace. The quotient

$$W^{X_*} := X_*/(X_* \smallsetminus W)$$

is a pointed extension of W. This pointed extension has the following universal property:

Let $f\colon X_* \to Z_*$ be a zero-pointed embedding. Suppose $f^{-1}(Z_* \smallsetminus *) \subset W \subset X$. Then f factors through W^{X_*}.

1.5. **Negating.** We present methods for recognizing negations.

Proposition 1.5.1. *Let*

$$\begin{tikzcd}
X_+ \arrow[r] \arrow[d] & X''_* \arrow[d] \\
X'_* \arrow[r] & X^+
\end{tikzcd}$$

be a commutatve diagram of pointed extensions of X. If this diagram is simultaneously pullback and pushout in the category of compactly generated Hausdorff topological spaces, then the canonical relations

$$X'_* \to (X''_*)^\neg \qquad \textit{and} \qquad X''_* \to (X'_*)^\neg \tag{2}$$

are the universal relations to well-pointed replacements.

Proof. The canonical relations (2) happen through the characterizing property of negation, using that the diagram is a pullback. Because negation $\neg$ sends colimit diagrams to limit diagrams, then the negation of the diagram is again pullback. There results canonical relations $(X_*^-)^\neg \to (X_*^+)^{\neg\neg}$ and $(X_*^+)^\neg \to (X_*^-)^{\neg\neg}$. These relations are inverse to the negations of those in (2).

□

The following result gives intrinsic characerizations of negation and well-pointedness.

Lemma 1.5.2. *Let X be a locally compact Hausdorff topological space, and let X_* be a pointed extension of X. The following statements are true.*

(1) *The collection of subsets* $\{* \cup (X \smallsetminus K)\}$*, indexed by compact neighborhoods* $* \in \mathsf{Int}(K) \subset K \subset X_*$*, is a local base about* $* \in X_*^{\neg}$ *(setting this collection to consist solely of* $\{*\}$ *if the indexing set is empty).*

(2) *The collection of subsets* $\{* \subset V \subset X_* \mid \overline{V} \text{ compact}\}$*, indexed by open neighborhoods contained in compact subspaces, is a local base about* $* \in X_*^{\neg\neg}$.

(3) *The following statements are equivalent:*
   (a) X_* *is well-pointed.*
   (b) *There is a compact neighborhood* $* \in \mathsf{Int}(K) \subset K \subset X_*$.
   (c) *The topological space* X_* *is locally compact and Hausdorff.*
   (d) *There is a local base about* $* \in X_*$ *consisting of compact neighborhoods, meaning each open neighborhood* $* \in U \subset X_*$ *contains a compact neighborhood* $* \in \mathsf{Int}(K) \subset K \subset U$.

(4) *If* X_* *is well-pointed then its negation*

$$X_*^{\neg} = (X_*)^+ \smallsetminus *$$

is the one-point compactification of X_* *minus the point* $*$.

(5) *If* X *is paracompact and* X_* *is well-pointed, then* X_* *is paracompact Hausdorff.*

Proof. Consider the topological space X'_* which is the compactly generated topology on $X \sqcup *$ generated by that of X and the named collection – it is Hausdorff because X is locally compact Hausdorff. Manifestly, there exist a pair of neighborhoods $* \in U \subset X_*$ and $* \in V \subset X'_*$ for which $\emptyset = U \cap X \cap V$. So the universal relation $X_+ \xrightarrow{=} X_* \underset{X^+}{\times} X'_*$ is an equality. Conversely, let X''_* be another pointed extension of X and suppose the universal relation $X_+ \xrightarrow{=} X_* \underset{X^+}{\times} X''_*$ is an equality.

Let $* \in K_* \subset X_*$ be a compact neighborhood, and denote $K := K_* \smallsetminus *$. Let X''_* be another pointed extension of X, and consider the subspace $(* \cup K) \subset X''_*$. By compactness, the universal continuous map $X_* \to X^+$ restricts to K_* as a homeomorphism onto its image, which we then identify as K^+. It follows that the pullback $K_* \underset{K^+}{\times} (* \cup K) \xrightarrow{\cong} (* \cup K) \subset X''_*$ projects as a homeomorphism onto its image. Suppose the universal relation $X_+ \xrightarrow{=} X_* \underset{X^+}{\times} X''_*$ is an equality. The identity of sets $K_+ \cong (* \cup K)$ is a homeomorphism. In particular $K \subset X''_*$ is closed. We conclude that $X''_* \to X'_*$ is continuous. This proves Statement (1).

Statement (2) follows by applying statement (1) to $X_*^{\neg}$, using that $(* \cup C) \subset X_*^{\neg}$ is compact if and only if $C \subset X \subset X_*$ is closed, because X_* is compactly generated. Statement (4) follows by inspecting local bases about base points, using that well-pointed implies locally compact. Statement (5) is standard after Statement (3).

For Statement (3), the equivalences of (b), (c), and (d) are nearly definitional, because X is locally compact Hausdorff. The equivalence of (a) and (c) follows from the second statement.

□

Corollary 1.5.3. *Let* $\overline{X}$ *be a compact Hausdorff topological space. Let* $\partial_L, \partial_R \subset \overline{X}$ *be a pair of disjoint closed subspaces. Write* $\partial\overline{X} := \partial_L \cup \partial_R \subset \overline{X}$ *and* $X := \overline{X} \smallsetminus \partial\overline{X}$*. There results the two pointed extensions of* X

$$X_* := * \underset{\partial_L}{\amalg} \left(\overline{X} \smallsetminus \partial_R\right) \qquad \text{and} \qquad X_*^{\neg} := * \underset{\partial_R}{\amalg} \left(\overline{X} \smallsetminus \partial_L\right) .$$

Both of these extension of X *are well-pointed, and they are each others negation.*

Proof. Both X_* and $X_*^\neg$ are locally compact and thus well-pointed. The result follows by applying Proposition 1.5.1 to the diagram of pointed extensions of X

$$\begin{tikzcd}
X_+ = * \underset{\partial_L \cap \partial_R}{\amalg} \big(X \cup (\partial_L \cap \partial_R)\big) \arrow[r] \arrow[d] & * \underset{\partial_R}{\amalg} (X \cup \partial_R) \arrow[d] \\
* \underset{\partial_L}{\amalg} (X \cup \partial_L) \arrow[r] & X^+ = * \underset{\partial \overline{X}}{\amalg} \overline{X} \ .
\end{tikzcd}$$

□

Corollary 1.5.4. *Let X_* and Y_* be pointed extensions of X and Y, respectively. The following statements concerning negations are true.*

Wedge: *There is an equality of pointed extensions of $X \amalg Y$*

$$X_*^\neg \vee Y_*^\neg = (X_* \vee Y_*)^\neg \ .$$

If X_* and Y_* are well-pointed, then so is $X_* \vee Y_*$.

Smash: *There is a canonical relation among pointed extensions of $X \times Y$*

$$X_*^\neg \wedge Y_*^\neg \to (X_* \wedge Y_*)^\neg \ .$$

This relation is the canonical one to the well-pointed replacement.

Coinv: *Let G be a finite group acting on the based space X_*. There is a preferred action of G on the based space $X_*^\neg$, and an equality of pointed extensions of the coinvariants X_G*

$$(X_*^\neg)_G \ = \ (X_{*G})^\neg \ .$$

If X_* is well-pointed, then so is X_{*G}.

Sub/Quot: *Let $W \subset X$ be an open subspace. There are canonical relations among pointed extensions of W*

$$W^{X_*^\neg} \longrightarrow (W_{X_*})^\neg \quad \textit{and} \quad (W^{X_*})^\neg \xrightarrow{(=)} W_{X_*^\neg} \ .$$

If X_* is well-pointed then so is W_{X_*}, the second relation is an equality as indicated, and the first relation is the canonical one to the well-pointed replacement.

Proof. This is an exercise in universal properties.

W: The statement concerning wedge sum is immediate by inspecting local bases about base points.

S/Q: Negating the canonical map $X_* \to W^{X_*}$ gives a map $(W^{X_*})^\neg \to X_*^\neg$. From the universal property of Construction **Sub**, there results a relation $(W^{X_*})^\neg \to W_{X_*^\neg}$, which is the right relation. Negating the canonical map $W_{X_*} \to X_*$ gives a map $X_*^\neg \to (W_{X_*})^\neg$. From the universal property of Construction **Quot**, there results the left relation $W^{X_*^\neg} \to (W_{X_*})^\neg$.

Now suppose X_* is well-pointed. Negating this last relation gives the composite relation $W_{X_*} \to (W_{X_*})^{\neg\neg} \to (W^{X_*^\neg})^\neg$. Replacing X_* with $X_*^\neg$ gives the relation $W_{X_*^\neg} \to (W^{X_*^{\neg\neg}})^\neg = (W^{X_*})^\neg$ which is inverse to the right relation. From the universal property of Construction **Sub**, there is the universal relation $(W_{X_*})^{\neg\neg} \to W_{X_*^{\neg\neg}} = W_{X_*}$. This relation is inverse to the universal relation $W_{X_*} \to (W_{X_*})^{\neg\neg}$, and so W_{X_*} is well-pointed. Also, replacing X_* by $X_*^\neg$ in the negation of the left relation gives the right, and it follows that $(W^{X_*^\neg})^{\neg\neg} = (W_{X_*})^\neg$.

Sm: Consider pointed extensions X_*' and Y_*' of X and Y, respectively, and suppose $X_+ \xrightarrow{=} X_* \underset{X^+}{\times} X_*'$ and $Y_+ \xrightarrow{=} Y_* \underset{Y^+}{\times} Y_*'$. Then $(X \times Y)_+ = X_+ \wedge Y_+ \xrightarrow{=} (X_* \wedge Y_*) \underset{(X \times Y)^+}{\times} (X_*' \wedge Y_*')$. There results the relation $X_*^\neg \wedge Y_*^\neg \to (X_* \wedge Y_*)^\neg$.

Now suppose X_* and Y_* are well-pointed. Notice that the based spaces $X_* \times Y_*$ and $X_*^\neg \times Y_*^\neg$ are well-pointed. We are about to show that the relation just obtained is the canonical one to the well-pointed replacement. Through previously established identities, the righthand side can be rewritten as

$$(X_* \wedge Y_*)^{\neg} = \big((X \times Y)^{X_* \times Y_*}\big)^{\neg} = (X \times Y)_{(X_* \times Y_*)^{\neg}} \ .$$

This last pointed extension is well-pointed, and it is a subspace of a the pointed extension $(X_* \times Y_*)^{\neg}$ in where a local base about $*$ is given as $\{* \cup (X_* \times Y_*) \smallsetminus (K \times K')\}$, indexed by pairs of compact neighborhoods $* \in \mathsf{Int}(K) \subset K \subset X_*$ and $* \in \mathsf{Int}(K') \subset K' \subset Y_*$.

Let $* \in \mathsf{Int}(C) \subset C \subset X_*^{\neg} \wedge Y_*^{\neg}$ be a compact neighborhood. Consider the collection $\{\mathsf{Int}(K) \times \mathsf{Int}(K')\} \cup \{\mathsf{Int}(C)\}$ of open subsets of $X_*^{\neg} \wedge Y_*^{\neg}$, indexed by pairs of compact neighborhoods $* \in \mathsf{Int}(K) \subset K \subset X_*$ and $* \in \mathsf{Int}(K') \subset K' \subset Y_*$. Compactness of C implies there exists finitely many such pairs covering $C \smallsetminus \mathsf{Int}(C)$. Taking the union of these pairs concludes that there is a single such pair (K, K') for which $\mathsf{Int}(K) \times \mathsf{Int}(K') \supset C \smallsetminus \mathsf{Int}(C)$. Rewriting, we see that there is such a pair (K, K') for which $(X_* \times Y_*) \smallsetminus (K \times K') \subset C$. This shows that the relation at hand is an open map, and thus an equality.

Cv: In a standard way, the group G acts on the full subposet of Point_X consisting of those X'_* for which the universal relation $X_+ \xrightarrow{=} X_* \underset{X^+}{\times} X'_*$ is an equality. The action of G on $X_*^{\neg}$ follows. For both sides of the proported relation, a local base about $*$ is given by the colleciton $\big\{* \cup \big(\bigcap_{g \in G} g \cdot (X \smallsetminus K)\big)_G\big\}$, indexed by compact neighborhoods $* \in \mathsf{Int}(K) \subset K \subset X_*$.

□

1.6. **Zero-pointed spaces.** We define a category of *zero-pointed spaces* and *zero-pointed embeddings* among them.

Recall Definition 1.2.4 of well-pointed extensions, and Definition 1.3.1 of zero-pointed embeddings.

Definition 1.6.1 ($\mathcal{Z}\mathcal{H}\mathsf{aus}$)**.** We define the symmetric monoidal topology category

$$\mathcal{Z}\mathcal{H}\mathsf{aus}$$

of *zero-pointed spaces* as having:

Ob: An object is a pointed locally compact Hausdorff topological space X_*. We will use the notation $X := X_* \smallsetminus *$ and refer to it as the *underlying (topological) space* of X_*.

Mor: The space of morphism $X_* \xrightarrow{f} Y_*$ is the space of zero-pointed embeddings; in particular, an element $f \in \mathsf{ZEmb}(X_*, Y_*)$ is a continuous map of based spaces such that the restriction

$$f_| \colon f^{-1}X \to Y$$

is an open embedding.

Wedge sum of based spaces makes $\mathcal{Z}\mathcal{H}\mathsf{aus}$ into a symmetric monoidal topological category.

Remark 1.6.2 (Zero object = unit)**.** Notice that the zero-pointed space $*$, with underlying space $\emptyset$, is a zero-object in $\mathcal{Z}\mathcal{H}\mathsf{aus}$. In other words, for each zero-pointed space X_* there are unique morphisms $* \to X_* \to *$. Moreover, this zero-object $*$ is the unit of the symmetric monoidal structure $\bigvee$ on $\mathcal{Z}\mathcal{H}\mathsf{aus}$.

Example 1.6.3. For X a locally compact Hausdorff topological space, adjoining a disjoint base point X_+, as well as one-point compactifying X^+, both give examples of zero-pointed spaces.

Example 1.6.4. There is a continuous map $[0, \infty] \to \mathsf{ZEmb}(\overrightarrow{\mathbb{R}}, \overrightarrow{\mathbb{R}})$, given by $t \mapsto (x \mapsto x - t)$, from the identity morphism to the constant map at the base point.

Example 1.6.5. The only morphism $(\mathbb{R}^n)^+ \to \mathbb{R}^n_+$ in $\mathcal{Z}\mathcal{H}\mathsf{aus}$ is the constant map at the base point. There is a continuous based map $(\mathbb{R}^n)^+ \to \mathsf{ZEmb}\big(\mathbb{R}^n_+, (\mathbb{R}^n)^+\big)$ depicted as $v \mapsto (x \mapsto x + v)$.

Example 1.6.6. We follow up on Corollary 1.5.3. Let $\overline{M}$ be a cobordism. That is to say, $\overline{M}$ is a compact topological manifold with boundary, and the boundary $\partial\overline{M} = \partial_L \amalg \partial_R$ is partitioned.

Write M for the interior of $\overline{M}$. Through Corollary 1.5.3, there results the two pointed extensions of M

$$M_* := * \coprod_{\partial_L} (\overline{M} \smallsetminus \partial_R) \qquad \text{and} \qquad M_*^{\neg} := * \coprod_{\partial_R} (\overline{M} \smallsetminus \partial_L) ~.$$

Both of these are *well-pointed* extensions of M, and they are related as indicated by the negation in the superscript.

Here is an immediate consequence of Lemma 1.3.2.

Lemma 1.6.7 (Negation)**.** *Negation implements an isomorphism of symmetric monoidal topological categories*

$$\neg \colon \mathcal{ZH}\mathsf{aus} \cong \mathcal{ZH}\mathsf{aus}^{\mathsf{op}} \colon \neg ~.$$

Convention 1.6.8. We followup on Construction 1.4.1, suited for zero-pointed spaces. Consider a construction which outputs a pointed extension from input data that included of a collection of other pointed extensions. A priori, this output pointed extension need not be well-pointed, even if the input pointed extensions are *well-pointed.* (For instance, the construction **Quot**.) And so, the construction is not one among zero-pointed spaces. We patch this by postcomposing with well-pointed replacement, $\neg\neg$. In this way, we regard the construction as one among zero-pointed spaces. From this point onwards, we use will implicitly adopt this convention.

Remark 1.6.9. We comment on the implementation of Convention 1.6.8. Should a hypothetical construction send well-pointed extensions to well-pointed extensions, then the named modification of Convention 1.6.8 does nothing. For some constructions, there is a sense of functoriality, or associativity (for instance, **Smash**), and postcomposing by well-pointed replacement might destroy this aspect. In general, there is not much to be done to remedy this. Even so, we point out that well-pointed replacement, $\neg\neg$, is a left adjoint thereby preserving colimits. So should a hypothetical construction too be comprised appropriately as a left adjoint, then then it interacts well with well-pointed replacement.

The following result is an immediate application of Corollary 1.5.4.

Corollary 1.6.10. *Let X_* and Y_* be zero-pointed spaces, let $O \subset X$ be an open subset, and let i be a finite cardinality. There are the relationships*

$$O^{X_*^{\neg}} = (O_{X_*})^{\neg} \qquad \textit{and} \qquad X_*^{\neg} \wedge Y_*^{\neg} = (X_* \wedge Y_*)^{\neg} ~.$$

1.7. **Relationship with ordinary spaces.** We relate zero-pointed topological spaces to ordinary topological spaces. The main statement here is Proposition 1.7.3.

Definition 1.7.1 ($\mathcal{H}\mathsf{aus}$)**.** The symmetric monoidal topological category $\mathcal{H}\mathsf{aus}$ has as objects locally compact Hausdorff topological spaces and morphism spaces given by open embeddings equipped with the compact-open topology. The symmetric monoidal structure is given by disjoint union.

There are functors

$$(-)_+ \colon \mathcal{H}\mathsf{aus} \hookrightarrow \mathcal{ZH}\mathsf{aus} \hookleftarrow \mathcal{H}\mathsf{aus}^{\mathsf{op}} \colon (-)^+ \tag{3}$$

given by adjoining a disjoint basepoint, and by one-point compactification, respectively. These inclusions are faithful in the sense that each map of Hom-spaces is an inclusion of path components. Notice that these inclusions (3) are symmetric monoidal.

Definition 1.7.2. We denote the three full sub-topological categories of $\mathcal{ZH}\mathsf{aus}$

$$\mathcal{H}\mathsf{aus}_+ ~\subset~ \mathcal{H}\mathsf{aus}_+^+ ~\supset~ \mathcal{H}\mathsf{aus}^+ \tag{4}$$

the leftmost which is generated by the image of $(-)_+$, the rightmost which is generated by the image of $(-)^+$, and the middle which is generated by the images of both $(-)_+$ and $(-)^+$. These subcategories are symmetric monoidal.

This isomorphism of Lemma 1.6.7 restricts to an isomorphism $\neg \colon \mathcal{H}\mathsf{aus}_+^{\mathsf{op}} \cong \mathcal{H}\mathsf{aus}^+ \colon \neg$.

Proposition 1.7.3. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category. There are canonical equivalences of ∞-categories:*

$$\mathsf{Fun}^{\otimes}\big(\mathcal{H}\mathsf{aus}_+,\mathcal{V}\big)\xrightarrow{\simeq}\mathsf{Fun}^{\otimes,\mathsf{aug}}\big(\mathcal{H}\mathsf{aus},\mathcal{V}\big)$$

over $\mathsf{Fun}^{\otimes}\big(\mathcal{H}\mathsf{aus},\mathcal{V}\big)$*, and*

$$\mathsf{Fun}^{\otimes}\big(\mathcal{H}\mathsf{aus}^+,\mathcal{V}\big)\xrightarrow{\simeq}\mathsf{Fun}^{\otimes,\mathsf{aug}}\big(\mathcal{H}\mathsf{aus}^{\mathsf{op}},\mathcal{V}\big)$$

over $\mathsf{Fun}^{\otimes}\big(\mathcal{H}\mathsf{aus}^{\mathsf{op}},\mathcal{V}\big)$.

Proof. The two statements are equivalent, as seen by replacing $\mathcal{V}$ by $\mathcal{V}^{\mathsf{op}}$, so we consider only the first.

Consider the morphism among symmetric monoidal topological categories $(-)_+\colon \mathcal{H}\mathsf{aus}\to\mathcal{H}\mathsf{aus}_+$ – it is a bijection on objects. It is enough to show that this symmetric monoidal functor is universal among all such those whose target has a zero-object which is also the unit. This is to say, each such symmetric monoidal functor $\mathcal{H}\mathsf{aus}\to\mathcal{V}$ canonically extends to a symmetric monoidal functor from $\mathcal{H}\mathsf{aus}_+$. Such an extension is already exhibited on objects. An extension to morphisms is determined through the following natural expression for the space of morphisms between two objects of $\mathcal{H}\mathsf{aus}_+$:

$$\mathsf{ZEmb}\Big(\bigvee_{i\in I}(X_i)_+\ ,\ \bigvee_{j\in J}(Y_j)_+\Big)\ \cong\ \coprod_{I_+\xrightarrow{f}J_+}\ \prod_{j\in J}\mathsf{Emb}\Big(\bigsqcup_{f(i)=j}X_i\ ,\ Y_j\Big)\ .$$

□

1.8. **Conically finite.** We single out those zero-pointed spaces that are *locally tame* in the sense that they admit the struture of a stratified space in the sense of [AFT1], and that are *globally tame* in the sense that, as stratified spaces, they admit a compactification to a stratified space with corners. We call these zero-pointed space *conically finite*. In practice, it is the conically finite zero-pointed spaces that are tractable through the invariants considered in this article.

Definition 1.8.1 (Conically smooth, conically finite)**.** A *conical smoothing* of a zero-pointed space X_* is a continuous map to a poset

$$X_*\xrightarrow{S}P_*$$

with a distinguished minimum $*\in P_*$ for which $*=S^{-1}*$, together with the structure of a stratified space on this data (also known as a maximal conically smooth atlas) in the sense of [AFT1]. A *conically smoothed* zero-pointed space is a zero-pointed space X_* together with a conical smoothing. We say a zero-pointed space X_* is *conically smoothable* if it admits a conical smoothing. We say a zero-pointed space X_* is *conically finite* if it admits a conical smoothing to a stratified space that is the interior of a compact stratified space with corners in the sense of [AFT1]. The ∞-category of *conically finite* zero-pointed spaces is the full ∞-subcategory

$$\mathcal{Z}\mathcal{H}\mathsf{aus}^{\mathsf{fin}}\ \subset\ \mathcal{Z}\mathcal{H}\mathsf{aus}$$

consisting of the conically finite ones.

Example 1.8.2. Let $\overline{X}$ be a compact stratified space with boundary (in the sense of [AFT1]), with boundary partitioned $\partial\overline{X}=\partial_L\sqcup\partial_R$. For instance, $\overline{X}$ could be a smooth cobordism. Then the based topological spaces

$$(X_L)_*\ :=\ *\coprod_{\partial_L}(\overline{X}\smallsetminus\partial_R)\qquad\text{and}\qquad(X_R)_*\ :=\ *\coprod_{\partial_R}(\overline{X}\smallsetminus\partial_L)$$

are conically finite zero-pointed spaces, and they are negations of each other: $(X_L)^{\neg}_*\cong(X_R)_*$.

Remark 1.8.3. Let X_* be zero-pointed space. A conical smoothing of X_* determines a stratified space with boundary $\overline{X}:=\mathsf{Unzip}_*(X_*)$ which is the blow-up at $*$ (termed "unzip" in [AFT1]), equipped with a based homeomorphism from the mapping cone as well as an open cover involving $\overline{X}$

$$\ast \coprod_{\partial\overline{X}} \overline{X} ~\cong~ X_\ast ~\cong~ \mathsf{C}(\partial\overline{X}) \bigcup_{(0,1)\times\partial\overline{X}} X ~.$$

The latter isomorphism is supported through a main result of [AFT1] that guarantees the existence of conically smooth tubular neighborhoods along each stratum.

We point out that a zero-pointed map $X_\ast \to Y_\ast$ is vastly different from an embedding $\overline{X} \to \overline{Y}$ that respects the boundary in any sense.

Remark 1.8.4. We point out that not every zero-pointed space is conically smoothable. For M a topological manifold that does not admit a smooth structure, then M_+ illustrates this phenomenon; so does the one-point compactification of an infinite genus surface. Moreover, there are potentially many non-equivalent stratified space structures on a given zero-pointed space. More subtly, in the situation of Example 1.8.2, the stratified space structures on $M_\ast$ that agree with the smooth structure on the interior M are parametrized by the h-cobordisms of $\partial\overline{M}$.

Remark 1.8.5. As useful as Example 1.8.2 is for producing examples, we point out that the theory of zero-pointed *manifolds* is quite different from that of manifolds with boundary, as it is unplagued by potential pseudo-isotopy difficulties. Indeed, for $M_\ast$ a zero-pointed space for which M is a topological n-manifold, the homotopy type at infinity, by which we mean the path space $\partial M_\ast := \{[0,1] \xrightarrow{\gamma} M_\ast \mid \gamma(t) = \ast \iff t = 0\}$, need not even be homotopy finitely dominated (as witnessed by a one-point compactification of an infinite genus surface). Provided $M_\ast$ is conically finite, then $\partial M_\ast$ is finitely dominated (see Remark 1.8.3) and, through Lefschetz duality, this homotopy type is a Poincaré duality space of dimension $(n-1)$ equipped with a lift τ of the classifying map for its Spivak normal bundle to BTop. The structure space of $(\partial M_\ast, \tau)$, which has no preferred element, parametrizes the collection of manifolds with boundary $\overline{M}$ equipped with a based homeomorphism $\ast \coprod_{\partial\overline{M}} \overline{M} \cong M_\ast$.

The key advantage of zero-pointed spaces that are conically finite is that they admit open handlebody decompositions. Recall from [AFT1] the notion of a *collar-gluing* among stratified spaces. We simply record the next result, which is a reformulation of a main result of [AFT1].

Lemma 1.8.6 ([AFT1])**.** *Let $X_\ast$ be a conically finite zero-pointed space. Then $X_\ast$ can be witnessed as a finite iteration of collar-gluings conically finite zero-pointed spaces from basic singularity types.*

Remark 1.8.7. Each collar-gluing $X_\ast \cong Y_\ast \bigcup_{\mathbb{R}_+\wedge W_\ast} Z_\ast$ determines a pushout diagram of underlying based topological spaces

$$\begin{tikzcd}
\mathbb{R}_+ \wedge W_\ast \arrow[r] \arrow[d] & Z_\ast \arrow[d] \\
Y_\ast \arrow[r] & X_\ast ~.
\end{tikzcd}$$

In other words, the forgetful functor $\mathcal{ZH}\mathsf{aus} \to \mathsf{Spaces}_\ast$ to based spaces sends collar-gluing diagrams to pushout diagrams. After Lemma 1.8.6, we conclude that this functor restricts as a functor $\mathcal{ZH}\mathsf{aus}^{\mathsf{fin}} \to \mathsf{Spaces}_\ast^{\mathsf{fin}}$ to *finite* pointed spaces (hence the term) which is the smallest full subcategory of pointed spaces containing $\ast_+$, the two-element pointed set, and closed under pushouts, and is equivalent to finite pointed CW complexes.

1.9. **Configuration zero-pointed spaces.** We define two natural zero-pointed spaces of configurations associated to a zero-pointed space.

Through Construction 1.4.1, and with Convention 1.6.8, we make the following definitions.

Definition 1.9.1. Let i be a finite cardinality. Let $X_\ast$ be a zero-pointed space. The open subspace $\mathsf{Conf}_i(X) \subset X^i \subset (X_\ast)^{\wedge i}$ of the i-fold smash product is comprised of those maps $\{1,\dots,i\} \xrightarrow{c} X$

that are injective. Construction 1.4.1(**Sub**) and Construction 1.4.1(**Quot**), each applied to this open subspace, yield the respective zero-pointed spaces that universally fit into the sequence of zero-pointed embeddings:

$$\mathsf{Conf}_i(X_*) \longrightarrow (X_*)^{\wedge i} \longrightarrow \mathsf{Conf}_i^{\neg}(X_*) ~ .$$

Each of $\mathsf{Conf}_i(X_*)$ and $\mathsf{Conf}_i^{\neg}(X_*)$ is equipped with an action of Σ_i that is free away from the base point. Thereafter, Construction 1.4.1(**Coinv**) yields the zero-pointed spaces that universally fit into the sequence of zero-pointed embeddings:

$$\mathsf{Conf}_i(X_*)_{\Sigma_i} \longrightarrow (X_*)^{\wedge i}_{\Sigma_i} \longrightarrow \mathsf{Conf}_i^{\neg}(X_*)_{\Sigma_i} ~ .$$

Intuition. Consider an injection $c\colon \{1,\dots,i\} \hookrightarrow X$. Intuitively, c is near "infinity" in $\mathsf{Conf}_i(X_*)$ if one of its members is; while c is near "infinity" in $\mathsf{Conf}_i^{\neg}(X_*)$ if, either one of its members is, or at least two of its members are near each other.

Here is an important though immediate consequence of Corollary 1.5.4.

Proposition 1.9.2. *Let i be a finite cardinality. Let X_* be a zero-pointed space. Then there is a canonical Σ_i-equivariant identification*

$$\mathsf{Conf}_i^{\neg}(X_*^{\neg}) ~\cong~ \mathsf{Conf}_i(X_*)^{\neg} ~ ,$$

and thereafter a canonical identification of the coinvariants

$$\mathsf{Conf}_i^{\neg}(X_*^{\neg})_{\Sigma_i} ~\cong~ \mathsf{Conf}_i(X_*)^{\neg}_{\Sigma_i} ~ .$$

Lemma 1.9.3. *Let X_* be a conically finite zero-pointed space, and let i be a finite cardinality. Then the Σ_i-zero-pointed space $\mathsf{Conf}_i(X_*)$, as well as the coinvariants $\mathsf{Conf}_i(X_*)_{\Sigma_i}$, is conically finite.*

Proof. Fix a conical smoothing of X_*. The two statements are similar, so we only elaborate on that concerning coinvariants.

Notice the cofiber sequence of stratified topological spaces

$$\{c\colon I \hookrightarrow X_* \mid * \in c(I)\}_{\Sigma_i} ~\hookrightarrow~ \{c\colon I \hookrightarrow X_*\}_{\Sigma_i} \longrightarrow \mathsf{Conf}_i(X_*)_{\Sigma_i} ~ .$$

In [AFT1] it is explained that the middle term has the structure of a stratified space. By inspection, the inclusion is conically smooth and constructible. A result of [AFT1] gives that this cofiber inherits the structure of a stratified space.

Now consider a compactification $X_* \subset \overline{X}_*$ to a stratified space with corners. In [AFT1] we define, for each stratified space Z, a stratified space $\mathsf{Ran}_{\leq i}(Z)$ of subsets $T \subset Z$ with bounded cardinality $|T| \leq i$ for which the map to the connected components $T \to [Z]$ is surjective; and we consider variations thereof. There is a conically smooth constructible inclusion

$$\mathsf{Ran}^{\mathsf{red}}_{\leq i}(\overline{X}_*) ~\subset~ \mathsf{Ran}_{\leq i}(\overline{X}_*)$$

from the locus of those finite subsets that contain $* \in \overline{X}_*$. There results the conically smooth constructible inclusion from the blow-ups (termed "unzip" construction in [AFT1]):

$$\mathsf{Unzip}_{\mathsf{Ran}^{\mathsf{red}}_{<i}(\overline{X}_*)}\big(\mathsf{Ran}^{\mathsf{red}}_{\leq i}(\overline{X}_*)\big) ~\hookrightarrow~ \mathsf{Unzip}_{\mathsf{Ran}_{<i}(\overline{X}_*)}\big(\mathsf{Ran}_{\leq i}(\overline{X}_*)\big) ~ .$$

Because $\overline{X}_*$ is compact, then so are both of the above blow-ups, and they each inherit a corner-structure. From the same result finishing the previous paragraph, the cofiber of this map inherits the structure of a stratified space, and it is compact with corners. The cofiber sequence in the previous paragraph maps as a conically smooth open embedding into the cofiber just mentioned, and it does so as the inclusion of interiors of compact stratified spaces with corners. This proves the result for $\mathsf{Conf}_i(X_*)_{\Sigma_i}$.

□

We record the following homological coconnectivity bound for configurations of points in a zero-pointed manifold.

Proposition 1.9.4. *Let M_* be a conically finite zero-pointed space with at most ℓ components, and let i be a finite cardinality. Suppose M is an n-manifold. Then the reduced singular homology*

$$\overline{\mathsf{H}}_q\big(\,\mathsf{Conf}_i(M_*);A\big) = 0$$

vanishes for any local system of abelian groups A for $q > n\ell + (n-1)(i-\ell)$ and $i >> 0$.

Proof. First, consider the case in which M_* is open, i.e., has no compact component. In this case we show by induction on i the vanishing of $\overline{\mathsf{H}}_q\big(\,\mathsf{Conf}_i(M_*);A\big)$ given $q > (n-1)i$. Consider the base case of $i = 1$, in which case the conically finite assumption implies that M_* has the homotopy type of a CW complex with no q-cells for $q > n-1$. From this it is immediate that $\overline{\mathsf{H}}_q(M_*,A)$ vanishes for $q > n-1$.

We proceed by induction on $i > 1$. Consider the map $\pi\colon \mathsf{Conf}_i(M_*) \to \mathsf{Conf}_{i-1}(M_*)$ given by sending a non-base element $\big(c\colon \{1,\dots,i\} \hookrightarrow M\big)$ to the restriction $\big(c_{|}\colon \{1,\dots,i-1\} \hookrightarrow M\big)$. Recall that the E^2 term of the Leray spectral converging to $\overline{\mathsf{H}}_{p+q}\big(\,\mathsf{Conf}_i(M_*);A\big)$ is a direct sum of groups of the form $\overline{\mathsf{H}}_p\big(\mathsf{Conf}_i(M_*);\mathsf{R}_q\pi_*A\big)$. By induction it is enough to show that the push-forward $\mathsf{R}_q\pi_*A$ vanishes for $q > n-1$; equivalently, every point c in $\mathsf{Conf}_{i-1}(M_*)$ has a neighborhood U_x such that $\overline{\mathsf{H}}_q(\pi^{-1}U_c, A_{|\pi^{-1}U_c})$ vanishes for $q > n-1$. Since π is a fiber bundle away from the base point, there are two cases to check. First, if $c = (\{1,\dots,i\} \hookrightarrow M)$ is not the base point, the result follows from the vanishing of $\overline{\mathsf{H}}_q(M_* \smallsetminus c(\{1,\cdots,i-1\})\};A)$ for $q > n-1$ considered in the base case of the induction. Second, if c is the base point, then the inverse image $\pi^{-1}c$ has a contractible neighborhood (from the conically finite, and therefore conically smoothable, assumption).

To prove the general case, consider $M_* \cong M_{1,*} \vee \ldots \vee M_{\ell,*}$, where each $M_{j,*}$ is connected. Notice the isomorphism

$$\mathsf{Conf}_i(M_*) \quad \cong \quad \bigvee_{\{1,\ldots,i\}\xrightarrow{f}\{1,\ldots,\ell\}} \quad \mathsf{Conf}_{i_1}(M_{1,*}) \wedge \ldots \wedge \mathsf{Conf}_{i_\ell}(M_{\ell,*})$$

where here we have used the notation i_r for the cardinality of the set $f^{-1}r$. So it suffices to bound the degree of the non-vanishing homology of each term. Because each $M_{j,*}$ is e have already concluded that $\mathsf{Conf}_{i_r}(M_{r,*})$ has homology bounded by the sum of the bounds for $M_{j,*} \smallsetminus \{c(\{1,\dots,i\})\}$ and $\mathsf{Conf}_{i_r-1}(M_{j,*})$, for a point $c \in \mathsf{Conf}_{i_j-1}(M_{j,*})$ a non-base point. We assess these separately. Should the zero-pointed manifold $M_{r,*}$ be such that $M_r := M_{r,*} \smallsetminus *$ is compact, then M_r has a the degree of the non-vanishing homology is bounded of n. The topological space $M_{j,*} \smallsetminus \{x\}$ is not compact, and so $\mathsf{Conf}_{i_r-1}(M_{r,*} \smallsetminus \{c\})$ has a bound of $(n-1)(i_j-1)$. The result follows by summing over the i_r.

□

2. ZERO-POINTED STRUCTURED SPACES

We expand our definitions to accommodate zero-pointed *structured* spaces. As an instance of particular interest, we have *zero-pointed n-manifolds*, as well as zero-pointed n-manifolds with tangential smoothing, and zero-pointed *framed* n-manifolds. (By smoothing theory, for $n \neq 4$, our zero-pointed n-manifolds with tangential smoothing is equivalent to the evident notion of zero-pointed smooth manifolds. See Remark 2.5.7.) We explain that all of the results in this article have an evident *structured* version.

2.1. **Formal setup.** The result of this section is a construction which outputs an inner fibration $\mathcal{C}\mathsf{ov}\big(\mathcal{H}(\mathcal{F})\big) \to \mathcal{C}\mathsf{ov}(\mathcal{H})$ among simplicial spaces given input data a sheaf $\mathcal{H}(\mathcal{F}) \to \mathcal{H}$ over an ∞-category with a Grothendieck topology.

We denote by

$$\mathsf{Site}_\infty$$

the (large) ∞-category whose objects are ∞-categories $\mathcal{H}$ equipped with a Grothendieck topology (which will be without notation), and for which the space of maps

$$\mathsf{Site}_\infty(\mathcal{H},\mathcal{H}') \ \subset\ \mathsf{Cat}_\infty(\mathcal{H},\mathcal{H}')$$

is comprised of those components represented by functors $F\colon \mathcal{H} \to \mathcal{H}'$ for which, for each covering sieve $\mathcal{U}^{\triangleright} \to \mathcal{H}$, the sieve generated from the composition $\mathcal{U}^{\triangleright} \to \mathcal{H} \xrightarrow{F} \mathcal{H}'$ is a covering sieve.

Construction 2.1.1. For P a set, use the notation $\mathcal{P}_0(P)$ for the poset of non-empty subsets of P, ordered by inclusion. We regard any such poset as equipped with a standard Grothendieck topology, where a covering sieve $\mathcal{U}^{\triangleright} \to \mathcal{P}_0(P)$ is one that witnesses a colimit. For $p \geq 0$ a non-negative integer, denote the set $\underline{p} := \{0, \dots, p\}$ which is the underlying set of the linearly ordered set $[p]$. And so, we have a standard functor

$$\mathsf{sd}\colon \mathbf{\Delta} \longrightarrow \mathsf{Site}_{\infty} \tag{5}$$

given by $[p] \mapsto \mathcal{P}_{\neq\emptyset}(\underline{p})$

Definition 2.1.2 ($\mathcal{C}\mathsf{ov}$)**.** Through Construction 2.1.1, there is the restricted Yoneda functor

$$\mathcal{C}\mathsf{ov}\colon \mathsf{Site}_{\infty} \longrightarrow \mathsf{PShv}(\mathbf{\Delta})$$

to simplicial spaces.

Remark 2.1.3. The simplicial space $\mathcal{C}\mathsf{ov}(\mathsf{Haus})$ agrees with Cov of §1.1.1.

Lemma 2.1.4. *Consider a Grothendieck site $\mathcal{H}$, together with a right fibration $\mathcal{H}(\mathcal{F}) \to \mathcal{H}$ which is a sheaf. Then the map of simplicial spaces $\mathcal{C}\mathsf{ov}\big(\mathcal{H}(\mathcal{F})\big) \to \mathcal{C}\mathsf{ov}(\mathcal{H})$ is an inner fibration.*

Proof. Let $p \geq 2$ and let $0 \leq i \leq p$. We will explain why each diagram of simplicial spaces

$$\begin{tikzcd}
\Lambda_i[p] \arrow[r] \arrow[d] & \mathcal{C}\mathsf{ov}\big(\mathcal{H}(\mathcal{F})\big) \arrow[d] \\
\Delta[p] \arrow[r] \arrow[ur, dashed] & \mathcal{C}\mathsf{ov}(\mathcal{H})
\end{tikzcd}$$

can be filled. Unraveling the meaning of this diagram, the problem is to construct a filler for a diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{P}_{i\notin S\neq\emptyset}(\underline{p}) \arrow[r] \arrow[d] & \mathcal{H}(\mathcal{F}) \arrow[d] \\
\mathcal{P}_0(\underline{p}) \arrow[r] \arrow[ur, dashed] & \mathcal{H}
\end{tikzcd}$$

in where the bottom horizontal arrow sends covers to covers. By inspection, the left vertical map is a covering sieve of $\underline{p}$. Because the right fibration $\mathcal{H}(\mathcal{F}) \to \mathcal{H}$ is a *sheaf* then such a filler exists, and is in fact essentially unique.

□

2.2. **Symmetric monoidal sites.** We make some quick observations to fit the above section to one where symmetric monoidal structures are present and compatible with Grothendieck topologies.

Definition 2.2.1. A *symmetric monoidal site* is a symmetric monoidal ∞-category $\mathcal{H}$ together with a Grothendieck topology on its underlying ∞-category, such that, for each finite sequence of objects $(X_i)_{i\in I}$ of $\mathcal{H}$, the sieve generated by the diagram

$$\mathcal{P}_0(I) \longrightarrow \mathcal{H}_{/\bigotimes\limits_{i\in I} X_i}~, \qquad \Big(T \mapsto \big(\bigotimes_{i\in T} X_i \to \bigotimes_{i\in I} X_i\big)\Big)$$

given through the unit maps, is a covering sieve.

Observation 2.2.2. Write $\mathsf{Cat}_{\infty}^{\otimes}$ for the (large) ∞-category of symmetric monoidal ∞-categories. There is a forgetful functor

$$\mathsf{Cat}_{\infty}^{\otimes} \to \mathsf{Cat}_{\infty}$$

to the (large) ∞-category of ∞-categories. This functor is conservative and both preserves and detects limits. Also, provided an endofunctor $F\colon \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$ preserves finite products there is a preferred filler in the diagram among (large) ∞-categories

$$\begin{tikzcd}
\mathsf{Cat}_\infty^\otimes \arrow[r, dashed, "F"] \arrow[d] & \mathsf{Cat}_\infty^\otimes \arrow[d] \\
\mathsf{Cat}_\infty \arrow[r, "F"] & \mathsf{Cat}_\infty
\end{tikzcd}$$

In particular, for $\mathcal{V}$ a symmetric monoidal ∞-category, then the opposite of the underlying ∞-category $\mathcal{V}^{\mathsf{op}}$ canonically inherits a symmetric monoidal structure. category.

Observation 2.2.3. Let $\mathcal{H}$ be a symmetric monoidal site, and let $\mathcal{H}(\mathcal{F}) \to \mathcal{H}$ be a right fibration over the underlying ∞-category which is a sheaf. We explain here that $\mathcal{H}(\mathcal{F})$ canonically inherits the structure of a symmetric monoidal ∞-category over $\mathcal{H}$. By definition, there is a pullback diagram among ∞-categories

$$\begin{tikzcd}
\mathcal{H}(\mathcal{F}) \arrow[r] \arrow[d] & \mathsf{Shv}(\mathcal{H})_{/\mathcal{H}(\mathcal{F})} \arrow[d] \\
\mathcal{H} \arrow[r] & \mathsf{Shv}(\mathcal{H})
\end{tikzcd}$$

So our explanation is complete upon explaining that the bottom horizontal arrow, as well as the right vertical arrow, is the restrictions to underlying ∞-categories of symmetric monoidal functors.

Here, we take the coCartesian symmetric monoidal structure on $\mathsf{Shv}(\mathcal{H})$, after which it follows that the relevant slice ∞-category canonically inherits a symmetric monoidal structure with respect to which the downward projection is symmetric monoidal. Precisely because $\mathcal{H}$ is a symmetric monoidal Grothendieck site, the sheafification of the Yoneda functor $\mathcal{H} \to \mathsf{Shv}(\mathcal{H})$ canonically extends as a symmetric monoidal functor.

Example 2.2.4. Examples of symmetric monoidal sites are abound. For instance, Haus with disjoint union is an example. And therefore, each sheaf $\mathsf{Haus}(\mathcal{F})$ on Haus is an example. In particular, Mfld_n with disjoint union is an example, and likewise for manifolds with various B-structures.

Observation 2.2.5. Notice that $\mathcal{C}\mathsf{ov}$ preserves finite products. It follows that, for $\mathcal{H}$ a symmetric monoidal site, then the simplicial space $\mathcal{C}\mathsf{ov}(\mathcal{H})$ canonically inherits the structure of a symmetric monoidal simplicial space, and from there an E_∞-algebra in simplicial spaces (with Cartesian symmetric monoidal structure).

2.3. **Zero-pointed embeddings compared to covers.** We compare the topological category $\mathcal{ZH}\mathsf{aus}$ to a category of covers.

Definition 1.6.1 defines the topological category $\mathcal{ZH}\mathsf{aus}$, which we promptly regarded as an ∞-category and we did so without choosing a specific model. We would like to directly compare the ∞-category $\mathcal{ZH}\mathsf{aus}$ with the simplicial space $\mathcal{C}\mathsf{ov}(\mathcal{H}\mathsf{aus})$. We are implored, then, to find a common home for these two entities.

Notation 2.3.1. We use the notation

$$\Delta^\bullet\colon \mathbf{\Delta} \longrightarrow \mathsf{Top}$$

for the standard cosimplicial locally compact Hausodrff topological space. Its value on $[p]$ is $\bigl\{\{0,\dots,p\} \xrightarrow{t} [0,1] \mid \sum\limits_{0\leq i\leq p} t_i = 1\bigr\}$.

Definition 2.3.2 (Horizontal submersion)**.** A *horizontal submersion* is a continuous map $E \to D$ between topological spaces for which open embeddings over D

$$\{O \times D \hookrightarrow E\}$$

forms an open cover of E.

Definition 2.3.3. We consider three Cartesian fibrations over the simplex category $\mathbf{\Delta}$; each has a natural fiberwise symmetric monoidal structure, and the first two have natural fiberwise Grothendieck topologies that are symmetric monoidal.

(1) $\mathsf{Haus}^\bullet$: An object of this category is a pair $(X,[r])$ consisting of locally compact Hausdorff topological space together with an object $[r]\in\mathbf{\Delta}$. A morphism from $(X,[r])$ to $(Y,[s])$ a morphism $[r]\xrightarrow{\rho}[s]$ in $\mathbf{\Delta}$ together with a continuous map

$$f\colon X\times\Delta^r\longrightarrow Y\times\Delta^s$$

over $\Delta^r\xrightarrow{\rho_*}\Delta^s$ for which, for each $t\in\Delta^r$, the restriction $f_t\colon X_*\to Y_*$ is a zero-pointed embedding; composition is evident. The evident projection

$$\mathsf{Haus}^\bullet\longrightarrow\mathbf{\Delta}$$

is directly seen to be a Cartesian fibration, and we denote the fiber over $[r]$ as Haus^{Δ^r}. Fiberwise disjoint union makes $\mathsf{Haus}^\bullet$ into a symmetric monoidal Cartesian fibration over $\mathbf{\Delta}$. The standard notion of open cover makes each fiber into a symmetric monoidal Grothendieck site, and the Cartesian transformations send covers to covers.

(2) $\mathsf{Haus}_\bullet$: An object of this category is an object $[r]\in\mathbf{\Delta}$ together with a submersion $X\to\Delta^r$ from a locally compact Hausdorff topological space. A morphism from $([r],X\to\Delta^r)$ to $([s],Y\to\Delta^s)$ a morphism $[r]\xrightarrow{\rho}[s]$ in $\mathbf{\Delta}$ together with a continuous map

$$f\colon X\longrightarrow Y$$

over $\Delta^r\xrightarrow{\rho_*}\Delta^s$ for which, for each $t\in\Delta^r$, the restriction $f_t\colon X_{|\{t\}}\to Y_{\{t\}}$ is an open embedding; composition is evident. The evident projection

$$\mathsf{Haus}_\bullet\longrightarrow\mathbf{\Delta}$$

is directly seen to be a Cartesian fibration, and we denote the fiber over $[r]$ as Haus_{Δ^r}. Fiberwise disjoint union makes $\mathsf{Haus}_\bullet$ into a symmetric monoidal Cartesian fibration over $\mathbf{\Delta}$. The standard notion of open cover makes each fiber into a symmetric monoidal Grothendieck site, and the Cartesian transformations send covers to covers.

(3) $\mathsf{ZHaus}^\bullet$: An object of this category is a pair $(X_*,[r])$ consisting of zero-pointed space together with an object $[r]\in\mathbf{\Delta}$. A morphism from $(X_*,[r])$ to $(Y_*,[s])$ a morphism $[r]\xrightarrow{\rho}[s]$ in $\mathbf{\Delta}$ together with a continuous map

$$f\colon X_*\times\Delta^r\longrightarrow Y_*\times\Delta^s$$

over and under $\Delta^r\xrightarrow{\rho_*}\Delta^s$ for which, for each $t\in\Delta^r$, the restriction $f_t\colon X_*\to Y_*$ is a zero-pointed embedding; composition is evident. The evident projection

$$\mathsf{ZHaus}^\bullet\longrightarrow\mathbf{\Delta}$$

is directly seen to be a Cartesian fibration, and we denote the fiber over $[r]$ as $\mathsf{ZHaus}^{\Delta^r}$.

Lemma 2.3.4. *Each of the fiberwise symmetric monoidal Cartesian fibrations over $\mathbf{\Delta}$ of Definition 2.3.3 is a Kan object.*

Proof. The argument is the same for each of the three situations, so we only explain the first. Observe that the each of the fiberwise symmetric monoidal Cartesian fibrations over $\mathbf{\Delta}$ is naturally the restriction of one over Top. The standard retraction $\Delta^r\to\Lambda^r_i$ then shows that each of the functors

$$\mathsf{Haus}^{\Delta^p}\longrightarrow\mathsf{Haus}^{\Lambda^p_i}\quad\text{and}\quad\mathsf{Haus}_{\Delta^p}\longrightarrow\mathsf{Haus}_{\Lambda^p_i}\quad\text{and}\quad\mathsf{ZHaus}^{\Delta^p}\longrightarrow\mathsf{ZHaus}^{\Lambda^p_i}$$

is essentially surjective.

□

Observation 2.3.5. Consider the straightenings

$$\mathsf{Haus}^\bullet\colon \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{Cat}^\otimes \qquad \text{and} \qquad \mathsf{Haus}_\bullet\colon \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{Cat}^\otimes \qquad \text{and} \qquad \mathsf{ZHaus}^\bullet\colon \mathbf{\Delta}^{\mathsf{op}} \longrightarrow \mathsf{Cat}^\otimes ~.$$

After Lemma 2.3.4, each of these simplicial symmetric monoidal categories is adjoint to a symmetric monoidal category internal to Kan complexes. So we regard each as a symmetric monoidal ∞-category, in a standard manner. As so, there are canonical identifications

$$\mathcal{H}\mathsf{aus} ~\simeq~ \mathsf{Haus}^\bullet \qquad \text{and} \qquad \mathcal{Z}\mathcal{H}\mathsf{aus} ~\simeq~ \mathsf{ZHaus}^\bullet ~,$$

and for consistency we invent the notation

$$\underline{\mathcal{H}\mathsf{aus}} ~:=~ \mathsf{Haus}_\bullet$$

as the ∞-category. To see this, let $[p] \in \mathbf{\Delta}$, let I_+ be a based finite set, and let $[r] \in \mathbf{\Delta}$. Notice the isomorphisms (large) sets

$$\mathsf{Fun}\big([p] \times I, \mathsf{Haus}^{\Delta^r}\big) ~\cong~ \coprod_{(X_{j,i})} \prod_{1 \leq j \leq p} \mathsf{Map}\big(\Delta^r, \mathsf{Haus}(X_{j-1,i}, X_{j,i})\big)$$

and

$$\mathsf{Fun}\big([p] \times I, \mathsf{ZHaus}^{\Delta^r}\big) ~\cong~ \coprod_{\big((X_{j,i})_*\big)} \prod_{1 \leq j \leq p} \mathsf{Map}\big(\Delta^r, \mathsf{ZHaus}((X_{j-1,i})_*, (X_{j,i})_*)\big)$$

where the left coproduct is indexed by $\{0, \dots, p\} \times I$-indexed sequences of locally compact Hausdorff topological spaces, and the right coproduct is indexed by $\{0, \dots, p\} \times I$-indexed sequences of zero-pointed spaces. Notice also that these isomorphisms are functorial among morphisms in the argument $[p] \in \mathbf{\Delta}$ by way of composition, identities, and source-target maps; that they are functorial among morphisms in the argument $I_+ \in \mathsf{Fin}_*$ by way of disjoint unions/wedge sums, units, and projections; and that they are functorial among morphisms in the argument $[r] \in \mathbf{\Delta}$ by way of precomposition.

Lemma 2.3.6. *There is a map of symmetric monoidal simplicial spaces*

$$\mathcal{Z}\mathcal{H}\mathsf{aus} \longrightarrow \mathcal{C}\mathsf{ov}\big(\underline{\mathcal{H}\mathsf{aus}}\big) ~.$$

Proof. We will construct a map of fiberwise symmetric monoidal Cartesian fibrations over $\mathbf{\Delta}$

$$\mathsf{ZHaus}^\bullet \longrightarrow \mathsf{Cov}\big(\mathsf{Haus}_\bullet\big)$$

that sends Cartesian edges to Cartesian edges – here we are using $\mathsf{Cov}(-)$ for the fiberwise application of $\mathcal{C}\mathsf{ov}$ of Definition 2.1.2, which in this case agrees with the ordinary version introduced as Definition 1.1.1. Explicitly, we must construct, for each $[r] \in \mathbf{\Delta}$, each $[p] \in \mathbf{\Delta}$, and each sequence of fiberwise zero-pointed embeddings

$$(X_0)_* \times \Delta^r \xrightarrow{f_1} (X_1)_* \times \Delta^r \xrightarrow{f_2} \dots \xrightarrow{f_p} (X_p)_* \times \Delta^r$$

over and under Δ^r, a functor $X_-\colon \mathcal{P}_0(\underline{p}) \to \mathsf{Haus}^{\Delta^r}$ witnessing an open cover of $X_{\underline{p}}$. That the construction is functorial among the above named arguments will be clear from this construction.

Choose such parameters as in the previous paragraph. Denote the evident coequalizer topological space:

$$X_{\underline{p}} ~:=~ \mathsf{cEq}\Big(\coprod_{0<i\leq p} f_i^{-1}(X_i \times \Delta^r) \underset{\mathsf{inc}_i}{\overset{f_i}{\rightrightarrows}} \coprod_{0 \leq i \leq p} (X_i \times \Delta^r)\Big) ~.$$

This topological space is manifestly locally compact and lies over Δ^r; and it is Hausdorff precisely because each map f_i is zero-pointed over Δ^r. For each subset $T \subset \underline{p}$ denote the open subspace

$$X_T ~:=~ \bigcup_{i \in T} X_i ~\subset~ X_{\underline{p}}$$

which is the union of the images of the canonical open embeddings. The assignment $\mathcal{P}_0(\underline{p}) \ni T \mapsto X_T \in \mathsf{Haus}^{\Delta^r}$ is the desired open cover of $X_{\underline{p}}$.

□

Observation 2.3.7 (Zero-pointed embeddings as two-term open covers)**.** From the proof of Lemma 2.3.6, we extract a conceptually useful reformulation of zero-pointed embeddings. Let X_* and Y_* be well-pointed extensions, and let D be a locally compact Hausdorff topological space (the case $D = *$ is particularly clarifying). There is a bijection between the set of continuous maps $\bigl\{D \to \mathsf{ZEmb}(X_*, Y_*)\bigr\}$ and the set of isomorphism classes of cospans of Hausdorff horizontal submersions over D,

$$X \times D \xrightarrow{f} W \xleftarrow{g} Y \times D \ ,$$

witnessing an open cover of W, and such that the following condition is satisfied:

> Let $K \subset (X_* \times D)\coprod_D (Y_*^\neg \times D)$ be a compact subspace. Then the map from the complement $f \amalg g \colon K \smallsetminus D \longrightarrow W$ is closed.

Furthermore, this bijection is compatible with the contravariant functoriality in the argument D. Also, this bijection is compatible with composition of zero-pointed embeddings on the one hand, and pushouts on the other; as well as fiberwise wedge sum on the one hand and fiberwise disjoint union on the other.

2.4. **Structures.** We refer to presheaves on $\mathcal{H}\mathsf{aus}$ that restrict as sheaves on Haus simply as *structures*:

Definition 2.4.1. A *(continuous) structure* on $\mathcal{H}\mathsf{aus}$ is a right fibration

$$\mathcal{H}\mathsf{aus}(\mathcal{F}) \to \mathcal{H}\mathsf{aus}$$

whose restriction to Haus is a (space-valued) sheaf. We usually denote such a structure solely by the symbol $\mathcal{F}$, and for each locally compact Hausdorff space X we will typically use the notation

$$\mathcal{F}(X) \ := \ \mathcal{H}\mathsf{aus}(\mathcal{F})_{|X}$$

for the space which is the fiber over X.

Observation 2.4.2. There is an evident fiberwise symmetric monoidal map of Cartesian fibrations over $\mathbf{\Delta}$

$$\mathsf{Haus}^\bullet \longrightarrow \mathsf{Haus}_\bullet \ .$$

This functor is fully faithful and sends open cover diagrams to open cover diagrams. The Definition 2.3.2 of *horizontal submersion* is designed so that this fully faithful functor is a basis for the fiberwise Grothendieck topology on the target. It follows that the functor $\mathcal{H}\mathsf{aus} \to \underline{\mathcal{H}\mathsf{aus}}$ induces an equivalence of ∞-categories of continuous structures:

$$\mathsf{Shv}(\mathcal{H}\mathsf{aus}) \ \simeq \ \mathsf{Shv}(\underline{\mathcal{H}\mathsf{aus}}) \ .$$

In particular, each *structure* $\mathcal{H}\mathsf{aus}(\mathcal{F}) \to \mathcal{H}\mathsf{aus}$ canonically determines one $\underline{\mathcal{H}\mathsf{aus}}(\mathcal{F}) \to \underline{\mathcal{H}\mathsf{aus}}$.

Recall from Lemma 2.3.6 the map of simplicial spaces $\mathcal{Z}\mathcal{H}\mathsf{aus} \to \mathfrak{Cov}\bigl(\underline{\mathcal{H}\mathsf{aus}}\bigr)$.

Definition 2.4.3. Let $\mathcal{F}$ be a structure on $\mathcal{H}\mathsf{aus}$, which we regard as a sheaf $\underline{\mathcal{H}\mathsf{aus}}(\mathcal{F}) \to \underline{\mathcal{H}\mathsf{aus}}$ by way of Observation 2.4.2. Define the pullback simplicial space

$$\begin{tikzcd}
\mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F}) \arrow[r] \arrow[d] & \mathfrak{Cov}\bigl(\underline{\mathcal{H}\mathsf{aus}}(\mathcal{F})\bigr) \arrow[d] \\
\mathcal{Z}\mathcal{H}\mathsf{aus} \arrow[r] & \mathfrak{Cov}(\underline{\mathcal{H}\mathsf{aus}}) \ .
\end{tikzcd}$$

After Lemma 2.1.4, we regard $\mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F})$ as an ∞-category over $\mathcal{Z}\mathcal{H}\mathsf{aus}$. We will sometimes refer to an object of $\mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F})$ as a *zero-pointed $\mathcal{F}$-space.* Explicitly, a zero-pointed $\mathcal{F}$-space is a zero-pointed space X_* together with a section $g \in \mathcal{F}(X)$; and a morphism $(X_*, g) \to (Y_*, h)$ between two is a zero-pointed embedding $X_* \xrightarrow{f} Y_*$ and a path $g_{|f^{-1}Y} \underset{\gamma}{\simeq} h_{|f^{-1}Y}$ in $\mathcal{F}(f^{-1}Y)$.

We will use the notations

$$\mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F})_+ ~:=~ \mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F})_{|\mathcal{Z}\mathcal{H}\mathsf{aus}_+} ~\subset~ \mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F}) ~\supset~ \mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F})^+ ~=:~ \mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F})_{|\mathcal{Z}\mathcal{H}\mathsf{aus}^+}$$

for the full symmetric monoidal ∞-categories; we notate similarly for other evident ∞-subcategories of $\mathcal{Z}\mathcal{H}\mathsf{aus}$.

Theorem 2.4.4. *Let $\mathcal{F}$ be a structure on $\mathcal{H}\mathsf{aus}$. Let $\mathcal{V}$ be a symmetric monoidal ∞-category. The following statements are true which concern the symmetric monoidal ∞-category $\mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F})$ of zero-pointed $\mathcal{F}$-spaces.*

(1) *The zero-pointed $\mathcal{F}$-space $*$ is a zero-object, as well as a symmetric monoidal unit.*

(2) *Negation implements an equivalence*

$$\neg\colon~\mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F}) \cong \mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F})^{\mathsf{op}}\colon \neg~.$$

(3) *The symmetric monoidal functor $\mathcal{H}\mathsf{aus}(\mathcal{F}) \xrightarrow{(-)_+} \mathcal{H}\mathsf{aus}(\mathcal{F})_+$ implements an equivalence of ∞-categories*

$$\mathsf{Fun}^{\otimes}\big(\mathcal{H}\mathsf{aus}(\mathcal{F})_+,\mathcal{V}\big) \xrightarrow{\simeq} \mathsf{Fun}^{\otimes,\mathsf{aug}}\big(\mathcal{H}\mathsf{aus}(\mathcal{F}),\mathcal{V}\big)$$

over $\mathsf{Fun}^{\otimes}(\mathcal{H}\mathsf{aus}(\mathcal{F}),\mathcal{V})$.

2.5. **Examples of structures.** Here are just a few examples of structures, targeted toward the case of manifolds.

Construction 2.5.1. Let $\mathcal{B}$ be a collection of topological spaces such that for each $U\in\mathcal{B}$ the collection $\{\phi(V)\subset U \mid V\in\mathcal{B},~\phi\colon V\hookrightarrow U\}$, indexed by open embeddings from members of $\mathcal{B}$, is a basis for the topology of U. Consider the full ∞-subcategory

$$\mathcal{H}\mathsf{aus}(\mathcal{F}_{\mathcal{B}}) \longrightarrow \mathcal{H}\mathsf{aus}$$

consisting of those X for which the collection of open embedddings $\{U\hookrightarrow X \mid U\in\mathcal{B}\}$ forms a basis for the topology of X. Then this full ∞-subcategory is a right fibration, and is a sheaf, by design. In this case, $\mathcal{F}_{\mathcal{B}}(X)\simeq *$ is terminal precisely if it lies in this ∞-subcategory, and $\mathcal{F}(X)=\emptyset$ otherwise.

Example 2.5.2. A case of particular interest is $\mathcal{B}=\{\mathbb{R}^n\}$ so that the full ∞-subcategory of $\mathcal{H}\mathsf{aus}$

$$\mathcal{H}\mathsf{aus}(\mathcal{F}_{\{\mathbb{R}^n\}}) ~\simeq~ \mathcal{M}\mathsf{fld}_n$$

is identified as that of topological n-manifolds and open embeddings among them.

Definition 2.5.3 (Zero-pointed manifolds)**.** The ∞-category of *zero-pointed manifolds* is the case of Construction 2.5.1 applied to $\mathcal{B}=\{\mathbb{R}^n\mid n\geq 0\}$:

$$\mathcal{Z}\mathcal{M}\mathsf{fld} ~:=~ \mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F}_{\{\mathbb{R}^n\mid n\geq 0\}})~.$$

For $n\geq 0$, there is the ∞-category of *zero-pointed n-manifolds*:

$$\mathcal{Z}\mathcal{M}\mathsf{fld}_n ~:=~ \mathcal{Z}\mathcal{H}\mathsf{aus}(\mathcal{F}_{\{\mathbb{R}^n\}})~.$$

Consider a collection $\mathcal{B}$ as in Construction 2.5.1, and regard it as a full ∞-subcategory of $\mathcal{H}\mathsf{aus}$. Then there is the restricted Yoneda functor

$$\tau\colon~\mathcal{H}\mathsf{aus}(\mathcal{F}_{\mathcal{B}}) \longrightarrow \mathsf{PShv}\big(\mathcal{H}\mathsf{aus}(\mathcal{F}_{\mathcal{B}})\big) \longrightarrow \mathsf{PShv}(\mathcal{B}) ~\simeq~ \mathsf{RFib}_{\mathcal{B}}$$

where the last equivalence is making specific the right fibration model for presheaves. For $\mathcal{E}\to\mathcal{B}$ a right fibration that is a sheaf, we denote the pullback among ∞-categories as

$$\begin{tikzcd}
\mathcal{H}\mathsf{aus}(\mathcal{F}_{\mathcal{E}}) \arrow[r] \arrow[d] & \mathsf{PShv}(\mathcal{B})_{/\mathcal{E}} \arrow[d] \\
\mathcal{H}\mathsf{aus}(\mathcal{F}_{\mathcal{B}}) \arrow[r, "\tau"] & \mathsf{PShv}(\mathcal{B})~.
\end{tikzcd}$$

By construction, the projection $\mathcal{H}\mathsf{aus}(\mathcal{F}_{\mathcal{E}}) \longrightarrow \mathcal{H}\mathsf{aus}(\mathcal{F}_{\mathcal{B}})$ is a map between structure on $\mathcal{H}\mathsf{aus}$.

Recall the following classical result, the Kister-Mazur theorem.

Theorem 2.5.4 ([Ki])**.** *Let $n \geq 0$. The morphism of topological monoids*

$$\mathsf{Top}(n) \xrightarrow{\simeq} \mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n) ~,$$

from self-homeomorphisms of $\mathbb{R}^n$ to self-embeddings of $\mathbb{R}^n$, is a weak homotopy equivalence of underlying topological spaces.

Corollary 2.5.5. *Let $n \geq 0$. There is a fully faithful embedding $\mathsf{BTop}(n) \to \mathcal{H}\mathsf{aus}$ whose essential image is equivalent to the full ∞-subcategory consisting of $\mathbb{R}^n$. Furthermore, the functor τ implements an equialence of ∞-catgories*

$$\mathsf{Shv}\big(\mathcal{M}\mathsf{fld}_n\big) ~\simeq~ \mathsf{PShv}\big(\mathsf{BTop}(n)\big) ~\simeq~ \mathsf{Mod}_{\mathsf{Top}(n)}(\mathsf{Spaces}) ~\simeq~ \mathsf{Spaces}_{/\,\mathsf{BTop}(n)}$$

– the latter two equivalence being standard.

Definition 2.5.6 (B-structures)**.** Let $B \to \mathsf{BTop}(n)$ be a map of spaces. By way of Corollary 2.5.5, we regard B as a structure $\mathcal{F}_B$ on $\mathcal{H}\mathsf{aus}$ over the structure $\mathcal{F}_{\{\mathbb{R}^n\}}$. We use the notations

$$\mathcal{M}\mathsf{fld}_n^B ~:=~ \mathcal{H}\mathsf{aus}\big(\mathcal{F}_B\big) \qquad \text{and} \qquad \mathcal{Z}\mathcal{M}\mathsf{fld}_n^B ~:=~ \mathcal{Z}\mathcal{H}\mathsf{aus}\big(\mathcal{F}_B\big) ~.$$

As the case $B = *$ we will use the notation

$$\mathcal{M}\mathsf{fld}_n^{\mathsf{fr}} ~:=~ \mathcal{M}\mathsf{fld}_n^{*} \qquad \text{and} \qquad \mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{fr}} ~:=~ \mathcal{Z}\mathcal{M}\mathsf{fld}_n^{*} ~.$$

Remark 2.5.7. Via smoothing theory ([KS]), the ∞-category $\mathcal{M}\mathsf{fld}_n^{\mathsf{BO}(n)}$ is canonically equivalent as that of *smooth* n-manifolds and smooth open embeddings among them (with the compact-open C^∞ Whitney topologies on the mapping sets), provided $n \neq 4$. Likewise, for $G \subset \mathsf{GL}(\mathbb{R}^n) \subset \mathsf{BTop}(n)$ a topological subgroup (with $n \neq 4$), a zero-pointed $\mathsf{B}G$-manifold is a zero-pointed n-manifold M_* together with a smooth structure on M as well as a reduction of the structure group of M from $\mathsf{GL}(\mathbb{R}^n)$ to G.

2.6. **Zero-pointed structured stratified spaces.** We end this section by pointing out a variation of zero-pointed structured spaces: zero-pointed *structured stratified spaces*. This subsection makes implicit reference to [AFT1].

We will make use of the standard cosimplicial smooth manifold given by

$$[p] \mapsto \Delta^p_e := \big\{\{0,\dots,p\} \xrightarrow{t} \mathbb{R} \mid \sum_{0\leq i\leq p} t_i = 1\big\}$$

– the cosimplicial structure maps are visible.

Recall from [AFT1] the ∞-category

$$\mathcal{S}\mathsf{nglr}$$

associated to the Kan-enriched category for which an object is a stratified space (which is to say a locally compact Hausdorff topological space $X \to P$ equipped with a map to a poset, together with a *conically smooth atlas* by *basics*), and for which a p-simplex in the Kan complex of morphisms from X to Y is a conically smooth open embedding

$$f\colon X \times \Delta^p_e \hookrightarrow Y \times \Delta^p_e$$

over Δ^p_e – the simplicial structure maps are evident, as is composition.

Definition 2.6.1 ($\mathcal{Z}\mathcal{S}\mathsf{nglr}$)**.** The ∞-category of *zero-pointed stratified spaces* is that associated to the Kan-enriched category

$$\mathcal{Z}\mathcal{S}\mathsf{nglr}$$

whose objects are zero-pointed spaces X_* together with a conical smoothing, and whose Kan complex of morphisms $\mathsf{ZEmb}^{\mathsf{sm}}(X_*, Y_*)$ is that for which a p-simplex is a map conically smooth map

$$f\colon X_* \times \Delta^p_e \longrightarrow Y_* \times \Delta^p_e$$

over and under Δ^p_e such that, for each $t \in \Delta^p_e$, the restriction $f_|\colon f^{-1}(Y \times \{t\}) \to Y \times \{t\}$ is a conically smooth open embedding; the simplicial structure maps are evident, as is composition.

There is a version of Lemma 2.3.6, whose proof is the same and so we omit it.

Lemma 2.6.2. *There is a functor*

$$\mathcal{Z}\mathsf{Snglr} \longrightarrow \mathcal{C}\mathsf{ov}(\mathcal{S}\mathsf{nglr}) ~.$$

A main result in [AFT1] is the classification of (continuous) sheaves on $\mathcal{S}\mathsf{nglr}$ –

$$\mathsf{Shv}(\mathcal{S}\mathsf{nglr}) ~\simeq~ \mathsf{RFib}_{\mathcal{B}\mathsf{sc}}$$

– as right fibrations on the full ∞-subcategory $\mathcal{B}\mathsf{sc} \subset \mathcal{S}\mathsf{nglr}$ consisting of the basic singularity types: $\mathbb{R}^i \times \mathsf{C}(Z)$, where Z is a compact stratified space. We call such a right fibration $\mathcal{B} = (\mathcal{B} \to \mathcal{B}\mathsf{sc})$ a *category of bascis*. Given a cateogy of bascis $\mathcal{B}$, there results the right fibration

$$\mathcal{M}\mathsf{fld}(\mathcal{B}) \longrightarrow \mathcal{S}\mathsf{nglr}$$

of $\mathcal{B}$-structured stratified spaces, termed $\mathcal{B}$-*manifolds* for short.

Example 2.6.3. Various specializations of a category of basics $\mathcal{B}$ give these ∞-categories:

- $\mathcal{Z}\mathcal{M}\mathsf{fld}_n^B$ of zero-pointed smooth B-manifolds, where $B \to \mathsf{BO}(n)$ is a map of spaces; in particular, of zero-pointed *framed* n-manifolds.
- $\mathcal{Z}\mathcal{M}\mathsf{fld}_n^\partial$ of zero-pointed smooth n-manifolds with boundary, and likewise $\mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\partial,\mathsf{fr}}$ of zero-pointed framed smooth n-manifolds with boundary.
- $\mathcal{Z}\mathcal{M}\mathsf{fld}_{k\subset n}$ of zero-pointed smooth n-manifolds equipped with properly embedded zero-pointed k-submanifolds.

Notation 2.6.4. We follow up on Remark 2.5.7. Let $B \to \mathsf{BO}(n)$ be a map of spaces. On the one hand, Definition 2.5.6 provides an ∞-category $\mathcal{Z}\mathcal{M}\mathsf{fld}_n^B$ by way of the composite map $B \to \mathsf{BO}(n) \to \mathsf{BTop}(n)$. On the otherhand, Example 2.6.3 provides an ∞-category with the same notation. While it appears that this notation $\mathcal{Z}\mathcal{M}\mathsf{fld}_n^B$ is double booked, smoothing theory ([KS]) grants that there is only a distinction in the case $n = 4$. In this case, we will take the convention that the ∞-category $\mathcal{Z}\mathcal{M}\mathsf{fld}_n^B$ is understood as that of Example 2.6.3.

Definition 2.6.5. Let $\mathcal{B}$ be a category of basics. The ∞-category of *zero-pointed* $\mathcal{B}$-*manifolds* is the pullback

$$\begin{tikzcd}
\mathcal{Z}\mathcal{M}\mathsf{fld}(\mathcal{B}) \arrow[r] \arrow[d] & \mathcal{C}\mathsf{ov}\big(\mathcal{M}\mathsf{fld}(\mathcal{B})\big) \arrow[d] \\
\mathcal{Z}\mathcal{S}\mathsf{nglr} \arrow[r] & \mathcal{C}\mathsf{ov}(\mathcal{S}\mathsf{nglr}) ~.
\end{tikzcd}$$

3. REDUCED FACTORIZATION (CO)HOMOLOGY

For coefficients in an augmented algebra, we extend factorization homology to zero-pointed manifolds; likewise for augmented coalgebras and factorization cohomology.

In this section we fix the following parameters.

- A dimension n and hereafter focus our attention on $\mathcal{Z}\mathcal{M}\mathsf{fld}_n$, zero-pointed *$n$-manifolds.*
- A symmetric monoidal ∞-category $\mathcal{V}$.

Terminology 3.0.6 ($\otimes$-sifted cocomplete)**.** Let $\mathcal{V}$ be a symmetric monoidal ∞-category. We say $\mathcal{V}$ is $\otimes$-*sifted cocomplete* if its underlying ∞-category admits sifted colimits, and its symmetric monoidal structure distributes over sifted colimits. We say $\mathcal{V}$ is $\otimes$-cosifted complete is $\mathcal{V}^{\mathsf{op}}$ is $\otimes$-sifted cocomplete.

Remark 3.0.7 (B-structures)**.** We comment that, for $B \to \mathsf{BTop}(n)$ a map of spaces, every result in this section is valid after an evident modification that accounts for a B-structures. Likewise, for $\mathcal{B} = (\mathcal{B} \to \mathcal{B}\mathsf{sc})$ a category of basics, every notion in this section is valid after an evident modificaiton for (zero-pointed) $\mathcal{B}$-manifolds, and $\mathcal{D}\mathsf{isk}(\mathcal{B})$ in place of $\mathcal{D}\mathsf{isk}_n$. We choose to not carry the discussion with such additional notation present.

3.1. **Disks.** We consider the symmetric monoidal subcategory of zero-pointed manifolds generated by Euclidean spaces under disjoint union. We characterize augmented n-disk algebras in terms of $\mathcal{D}\mathsf{isk}_{n,+}$.

Definition 3.1.1 ($\mathcal{D}\mathsf{isk}_n$)**.** We denote the full sub-symmetric monoidal topological category

$$\mathcal{D}\mathsf{isk}_n \ \subset \ \mathcal{M}\mathsf{fld}_n$$

of finite disjoint unions of n-dimensional Euclidean spaces, and open embeddings among them. Likewise, we denote the three full sub-symmetric monoidal topological categories of $\mathcal{Z}\mathcal{M}\mathsf{fld}_n$

$$\mathcal{D}\mathsf{isk}_{n,+} \ \subset \ \mathcal{Z}\mathcal{D}\mathsf{isk}_n \ \supset \ \mathcal{D}\mathsf{isk}_n^+ \tag{6}$$

the leftmost which is generated by $\mathbb{R}^n_+$, the rightmost which is generated by $(\mathbb{R}^n)^+$, and the middle which is generated by $\mathbb{R}^n_+$ and $(\mathbb{R}^n)^+$.

Through Lemma 1.6.7, negation implements an isomorphism of symmetric monoidal topological categories $\mathcal{Z}\mathcal{D}\mathsf{isk}_n \cong \mathcal{Z}\mathcal{D}\mathsf{isk}_n^{\mathsf{op}}$ that restricts to a further isomorphism $\mathcal{D}\mathsf{isk}_{n,+} \cong (\mathcal{D}\mathsf{isk}_n^+)^{\mathsf{op}}$.

The assignment $\bigvee\limits_I \mathbb{R}^n_+ \mapsto I_+$ of connected components defines a continuous symmetric monoidal functor

$$[-]\colon \ \mathcal{D}\mathsf{isk}_{n,+} \to \mathsf{Fin}_* \tag{7}$$

to based finite sets, with wedge sum. The following result follows immediately from Theorem 2.5.4.

Corollary 3.1.2. *As a functor among ∞-categories, $[-]\colon \mathcal{D}\mathsf{isk}_{n,+} \to \mathsf{Fin}_*$ is conservative. In other words, the maximal ∞-subgroupoid of $\mathcal{D}\mathsf{isk}_{n,+}$ is canonically identified as any of the following ∞-groupoids*

$$\mathcal{D}\mathsf{isk}^{\mathsf{bij}}_{n,+} := (\mathcal{D}\mathsf{isk}_{n,+})_{|\,\mathsf{Fin}_*^{\mathsf{bij}}} \quad \simeq \quad \coprod_{i\geq 0} \mathcal{D}\mathsf{isk}^{=i}_{n,+} \quad \simeq \quad \mathsf{B}(\Sigma \wr \mathsf{Top}(n)) \quad \simeq \quad \coprod_{i\geq 0} \mathsf{B}(\Sigma_i \wr \mathsf{Top}(n)) \ ,$$

the first which is the subcategory consisting of those morphisms which are bijections on connected components, the second which is the coproduct over finite cardinalities of the subcategories with a fixed cardinality of components and bijections thereof, and the others which are classical.

Recall Definition 3.1.1 of the symmetric monoidal topological category $\mathcal{D}\mathsf{isk}_n$ and its variants.

Definition 3.1.3 (Alg_n)**.** Let $\mathcal{V}$ be a symmetric monoidal ∞-category. We denote the ∞-categories of symmetric monoidal functors

$$\mathsf{Alg}_n(\mathcal{V}) := \mathsf{Fun}^{\otimes}(\mathcal{D}\mathsf{isk}_n, \mathcal{V}) \qquad \text{and} \qquad \mathsf{cAlg}_n(\mathcal{V}) := \mathsf{Fun}^{\otimes}(\mathcal{D}\mathsf{isk}_n^{\mathsf{op}}, \mathcal{V})$$

the latter which is alternatively identified as $\big(\mathsf{Fun}^{\otimes}(\mathcal{D}\mathsf{isk}_n, (\mathcal{V}^{\mathsf{op}})^{\otimes})\big)^{\mathsf{op}}$. We refer to objects in the left as *n-disk algebras* and to objects in the right as *n-disk coalgebras.*

The next result follows easily from Proposition 1.7.3.

Corollary 3.1.4. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category. There are canonical equivalences of categories*

$$\mathsf{Fun}^{\otimes}(\mathcal{D}\mathsf{isk}_{n,+}, \mathcal{V}) \ \simeq \ \mathsf{Alg}_n^{\mathsf{aug}}(\mathcal{V}) \qquad \textit{and} \qquad \mathsf{Fun}^{\otimes}(\mathcal{D}\mathsf{isk}_n^+, \mathcal{V}) \ \simeq \ \mathsf{cAlg}_n^{\mathsf{aug}}(\mathcal{V})$$

over $\mathsf{Alg}_n(\mathcal{V})$ and $\mathsf{cAlg}_n(\mathcal{V})$, respectively.

Remark 3.1.5. There is a close relationship between the topological operad $\mathcal{E}_n$ of little n-cubes and the symmetric monoidal topological category $\mathcal{D}\mathsf{isk}_n$: the manifest representation $\mathcal{E}_n \to \mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}$ induces an equivalence of symmetric monoidal ∞-categories $\mathsf{Env}(\mathcal{E}_n) \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}$ from the symmetric monoidal envelope. Another articulation of this relationship is that there is pullback diagram among symmetric monoidal ∞-categories

$$\begin{tikzcd}
\mathsf{Env}(\mathcal{E}_n) \arrow[r] \arrow[d] & \mathsf{Fin}^{\sqcup} \arrow[d] \\
\mathcal{D}\mathsf{isk}_n \arrow[r] & \big(\mathsf{Fin} \wr \mathsf{BTop}(n)\big)^{\sqcup} ~.
\end{tikzcd}$$

Yet another articulation: there is a standard action of $\mathsf{Top}(n)$ on $\mathcal{E}_n$ with respect to which there is an identification of the coinvariants

$$\mathsf{Env}(\mathcal{E}_n)_{\mathsf{Top}(n)} \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}_n~,$$

and therafter a canonical identification of invariants

$$\mathsf{Alg}_n(\mathcal{V}) \xrightarrow{\simeq} \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{V})^{\mathsf{Top}(n)}~.$$

3.2. **Homology and cohomology.** We extend factorization homology and cohomology to zero-pointed manifolds.

Definition 3.2.1 (Factorization (co)homology for zero-pointed manifolds)**.** Let $\mathcal{V}$ be a symmetric monoidal ∞-category. Let M_* be a zero-pointed n-manifold. Let $A\colon \mathcal{D}\mathsf{isk}_{n,+} \to \mathcal{V}$ and $C\colon \mathcal{D}\mathsf{isk}_n^+ \to \mathcal{V}$ be functors. Whenever they exist, we define the objects of $\mathcal{V}$

$$\int_{M_*} A \quad := \quad \mathsf{colim}\Big(\big(\mathcal{D}\mathsf{isk}_{n,+}\big)_{/M_*} \to \mathcal{D}\mathsf{isk}_{n,+} \xrightarrow{A} \mathcal{V}\Big) \quad = \quad \underset{U_+\to M_*}{\mathsf{colim}}\, A(U_+) \tag{8}$$

and

$$\int^{M_*} C \quad := \quad \mathsf{lim}\Big(\big(\mathcal{D}\mathsf{isk}_n^+\big)^{M_*^-/} \to \mathcal{D}\mathsf{isk}_n^+ \xrightarrow{C} \mathcal{V}\Big) \quad = \quad \underset{M_*^-\to V^+}{\mathsf{lim}}\, C(V^+) \tag{9}$$

and refer to the first as the *factorization homology of M_* (with coefficients in A)*, and the second as the *factorization cohomology of M_* (with coefficients in C)*.

We point out that the above notion of factorization homology agrees with that considered in previous work [AF1].

Lemma 3.2.2. *Let M be an n-manifold and let A be an augmented n-disk algebra. Consider symmetric monoidal functor $A_+\colon \mathcal{D}\mathsf{isk}_{n,+} \to \mathcal{V}$ determined by A through Lemma 3.1.4. There is a canonical equivalence*

$$\int_M A \xrightarrow{\simeq} \int_{M_+} A_+~.$$

Proof. The functor $(-)_+\colon \big(\mathcal{D}\mathsf{isk}_n\big)_{/M} \to \big(\mathcal{D}\mathsf{isk}_{n,+}\big)_{/M_+}$ admits a left adjoint, given by $\big(\bigvee\limits_{i\in I}\mathbb{R}^n_+ \xrightarrow{e} M_+\big) \mapsto \big(e^{-1}M \xrightarrow{e_|} M\big)$. The resulting adjunction is a localization, so in particular the functor $(-)_+$ is final.

□

We conclude this subsection by stating a universal property that factorization (co)homology satisfies. Recall Conditions 3.0.6 that a symmetric monoidal ∞-category might satisfy. We prove the following result contingent on the key technical result Proposition 3.3.3.

Theorem 3.2.3. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category. Consider the diagram of solid arrows*

$$\begin{tikzcd}
\mathsf{Alg}_n^{\mathsf{aug}}(\mathcal{V}) \arrow[d] \arrow[r, dashed, bend left, "\int_{-}", "(\mathrm{Sftd}^{\otimes})"'] & \mathsf{Fun}^{\otimes}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{fin}}, \mathcal{V}\big) \arrow[l] \arrow[r] \arrow[d] & \mathsf{cAlg}_n^{\mathsf{aug}}(\mathcal{V}) \arrow[d] \arrow[l, dashed, bend left, "\int^{-}", "(\mathrm{cSftd}^{\otimes})"'] \\
\mathsf{Fun}\big(\mathcal{D}\mathsf{isk}_{n,+}, \mathcal{V}\big) \arrow[r, dashed, bend left, "\int_{-}", "(\mathrm{Sftd})"'] & \mathsf{Fun}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{fin}}, \mathcal{V}\big) \arrow[l] \arrow[r] & \mathsf{Fun}\big(\mathcal{D}\mathsf{isk}_n^{+}, \mathcal{V}\big) \arrow[l, dashed, bend left, "\int^{-}", "(\mathrm{cSftd})"']
\end{tikzcd} \tag{10}$$

in which the downward arrows are conservative, given by restricting to underlying ∞-categories, and the unlabeled horizontal arrows are restrictions. Provided the additional (marked) Conditions 3.0.6 on $\mathcal{V}$, then factorization homology determines the indicated dashed arrows which are fully faithful left adjoints making the left dashed square commute; and factorization cohomology determines the indicated dashed arrows which are fully faithful right adjoints making the right dashed square commute.

Remark 3.2.4. Even without the distribution assumptions on $\otimes$, we understand that the dashed arrows in diagram (10) are always defined on some full, possibly empty, subcategories of the respective domain (co)algebra categories.

We give the proof of Theorem 3.2.3 here, though it depends on results in the coming subsection.

Proof of Theorem 3.2.3. We only concern ourselves with the left side of the diagram, for the right side is dual. Corollary 3.3.5, together with Lemma 3.3.2, gives that the values of factorization homology can be computed over a sifted ∞-category. By inspection, the evident functor

$$\mathcal{D}\mathsf{isk}_+(M_*) \times \mathcal{D}\mathsf{isk}_+(M'_*) \to \mathcal{D}\mathsf{isk}_+(M_* \vee M'_*)$$

is an equivalence of ∞-categories. After these observations, the proposition follows from a key result of [AFT2].

□

3.3. **Exiting disks.** The slice ∞-category $\mathcal{D}\mathsf{isk}_{n,+/M_*}$ appears in the defining expression for factorization homology. We give a variant of this ∞-category $\mathcal{D}\mathsf{isk}_+(M_*)$, of *exiting disks* in M_*, which offers several conceptual and technical advantages. Heuristically, objects of $\mathcal{D}\mathsf{isk}_+(M_*)$ are embeddings from finite disjoint unions of basics into M, while morphisms are isotopies of such to embeddings with some of these isotopies slide disks off to infinity where they are forgotten – disks are not allowed to be created at infinity, unlike in $\mathcal{D}\mathsf{isk}_{n,+/M_*}$.
We make light use of some theory of stratified spaces as developed in [AFT1], and of some results thereabout in [AFT2].

For this subsection, fix a conically smooth zero-pointed manifold M_*. In [AFT2] we define, for each stratified space X, the ∞-category

$$\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/X}$$

of finite disjoint unions of basics embedding into X. This is a stratified version of $\mathcal{D}\mathsf{isk}_{n/M}$.

Definition 3.3.1 ($\mathcal{D}\mathsf{isk}_+(M_*)$)**.** The ∞-category of *exiting disks* of M_* is the full ∞-subcategory

$$\mathcal{D}\mathsf{isk}_+(M_*) \ \subset\ \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/M_*}$$

consisting of those $V \hookrightarrow M_*$ whose image contains $*$. Explicitly, an object of $\mathcal{D}\mathsf{isk}_+(M_*)$ is a conically smooth open embedding $B \sqcup U \hookrightarrow M_*$ where $B \cong \mathsf{C}(L)$ is a cone-neighborhood of $* \in M_*$

and U is abstractly diffeomorphic to a finite disjoint union of Euclidean spaces, and a morphism is a isotopy to an embedding among such. We use the notation

$$\mathcal{D}\mathsf{isk}^{+}(M_*^{-}) \ := \ \big(\mathcal{D}\mathsf{isk}_{+}(M_*)\big)^{\mathsf{op}} \ .$$

Lemma 3.3.2. *The ∞-category $\mathcal{D}\mathsf{isk}_{+}(M_*)$ is sifted.*

Proof. Let $\mathsf{C}(L) \hookrightarrow M_*$ be a basic centered at the base point. In [AFT1] it is shown that any conically smooth open embedding from a basic $U \hookrightarrow M_*$ whose image contains $*$ is canonically isotpic to one that factors through an isomorphism $U \cong \mathsf{C}(L) \hookrightarrow M_*$. We conclude that the projection from the slice

$$\big(\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/M_*}\big)^{\mathsf{C}(L)/} \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}_{+}(M_*) \tag{11}$$

is an equivalence of ∞-categories.

A main result of [AFT2] is that the ∞-category $\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/M_*}$ is sifted. It follows from the identification (11) that $\mathcal{D}\mathsf{isk}_{+}(M_*)$ is sifted.

□

The unique zero-pointed embedding $* \to M_*$ induces the functor

$$\mathcal{D}\mathsf{isk}_{n,+} = \mathcal{D}\mathsf{isk}_{n,+/*} \longrightarrow \mathcal{D}\mathsf{isk}_{n,+/M_*} \ .$$

We denote the subcategory

$$\mathsf{Fin}_*^{\mathsf{inrt}} \ \subset \ \mathsf{Fin}_*$$

of based finite sets that consists of the same objects but only those based maps $I_+ \xrightarrow{f} J_+$ for which the restriction $f_{|}\colon f^{-1}J \to J$ is an isomorphism. Taking connected components gives a functor $\mathcal{D}\mathsf{isk}_{+}(M_*) \to \mathsf{Fin}_*$ to finite based sets where the component containing $*$ is assigned to the base point. Denote the ∞-subcategory

$$\mathcal{J} \ := \ \big(\mathcal{D}\mathsf{isk}_{+}(M_*)\big)_{|\,\mathsf{Fin}_*^{\mathsf{inrt}}} \ \subset \ \mathcal{D}\mathsf{isk}_{+}(M_*)$$

which is the restriction to the inert maps among based finite sets. We will prove the next result as §3.5.

Proposition 3.3.3. *There is a natural functor to the quotient ∞-category*

$$\mathcal{D}\mathsf{isk}(M_*) \longrightarrow \big(\mathcal{D}\mathsf{isk}_{n,+/M_*}\big)/\big(\mathcal{D}\mathsf{isk}_{n,+}\big) \tag{12}$$

whose value on $\big(B \sqcup U \hookrightarrow M_*\big)$ is represented by $(U_+ \hookrightarrow M_*) \in \mathcal{D}\mathsf{isk}_{n,+/M_*}$. Furthermore, this functor witnesses a localization

$$\mathcal{D}\mathsf{isk}(M_*)[\mathcal{J}^{-1}] \ \simeq \ \big(\mathcal{D}\mathsf{isk}_{n,+/M_*}\big)/\big(\mathcal{D}\mathsf{isk}_{n,+}\big) \ .$$

In particular, the functor (12) is final.

Consider the composite functor

$$\mathsf{Alg}_n^{\mathsf{aug}}(\mathcal{V}) \to \mathsf{Fun}(\mathcal{D}\mathsf{isk}_{n,+/M_*}, \mathcal{V}) \xrightarrow{\mathsf{LKan}} \mathsf{Fun}\big(\big(\mathcal{D}\mathsf{isk}_{n,+/M_*}\big)/\big(\mathcal{D}\mathsf{isk}_{n,+}\big), \mathcal{V}\big) \to \mathsf{Fun}\big(\mathcal{D}\mathsf{isk}_{+}(M_*), \mathcal{V}\big) \ : \tag{13}$$

the first arrow is restriction along the projection $\mathcal{D}\mathsf{isk}_{n,+/M_*} \to \mathcal{D}\mathsf{isk}_{n,+}$; the second arrow is left Kan extension along the quotient functor $\mathcal{D}\mathsf{isk}_{n,+/M_*} \to \big(\mathcal{D}\mathsf{isk}_{n,+/M_*}\big)/\big(\mathcal{D}\mathsf{isk}_{n,+}\big)$; the third arrow is restriction along the functor of Proposition 3.3.3.

Notation 3.3.4. Given an augmented n-disk algebra $A\colon \mathcal{D}\mathsf{isk}_{n,+} \to \mathcal{V}$, we will use the same notation $A\colon \mathcal{D}\mathsf{isk}_{+}(M_*) \to \mathcal{V}$ for the value of the functor (13) on A.

We content ourselves with this Notation 3.3.4 because of the immediate corollary of Proposition 3.3.3.

Corollary 3.3.5. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category whose underlying ∞-category admits sifted colimits. Let $A\colon \mathcal{D}\mathsf{isk}_{n,+} \to \mathcal{V}$ be an augmented n-disk algebra, and let $C\colon \mathcal{D}\mathsf{isk}_n^+ \to \mathcal{V}$ be an augmented n-disk coalgebra. There are canonical identifications in $\mathcal{V}$:*

$$\int_{M_*} A \quad\simeq\quad \mathsf{colim}\Big(\mathcal{D}\mathsf{isk}_+(M_*) \xrightarrow{A} \mathcal{V}\Big) \quad\simeq \underset{(B\sqcup U\hookrightarrow M_*)\in\mathcal{D}\mathsf{isk}_+(M_*)}{\mathsf{colim}} A(U_+)~,$$

and

$$\int^{M_*} C \quad\simeq\quad \mathsf{lim}\Big(\mathcal{D}\mathsf{isk}^+(M_*^\neg) \xrightarrow{C} \mathcal{V}\Big) \quad\simeq \underset{(B\sqcup V\hookrightarrow M_*)\in\mathcal{D}\mathsf{isk}^+(M_*^\neg)}{\mathsf{lim}} C(V^+)~.$$

3.4. **Reduced homology theories.** We use zero-pointed manifolds to articulate additional functorialities of *reduced* homology theories.

We use the notation

$$\mathcal{M}\mathsf{fld}_n^{\partial,\mathsf{sm},\mathsf{fin}}$$

for the topological category of finitary *smoothed* n-manifolds with boundary and smooth open embeddings among them. The concept of a homology theory for smooth n-manifolds with boundary is defined in [AFT2] – this is a symmetric monoidal functor $H\colon \mathcal{M}\mathsf{fld}_n^{\partial,\mathsf{sm},\mathsf{fin}} \to \mathcal{V}$ satisfying an $\otimes$-excision axiom. We will be concerned with the pointed variant, a functor $H : \mathcal{M}\mathsf{fld}_{n,+}^{\partial,\mathsf{sm},\mathsf{fin}} \to \mathcal{V}$.

Definition 3.4.1 (Reduced homology theories)**.** The ∞-category of *reduced* homology theories is the full ∞-subcategory

$$\mathbf{H}^{\mathsf{aug}}_{\mathsf{red}}\big(\mathcal{M}\mathsf{fld}_n^{\partial,\mathsf{sm},\mathsf{fin}},\mathcal{V}\big) \;\subset\; \mathbf{H}^{\mathsf{aug}}\big(\mathcal{M}\mathsf{fld}_n^{\partial,\mathsf{sm},\mathsf{fin}},\mathcal{V}\big)$$

consisting of those H which are *reduced*, which is to say that the canonical map

$$\mathbb{1}_{\mathcal{V}} \simeq H(+) \xrightarrow{\simeq} H\big((\mathbb{R}_{\geq 0}\times N)_+\big)$$

is an equivalence in $\mathcal{V}$ for every closed smooth $(n-1)$-manifold N.

The following is an immediate consequence of some of our previous work ([AFT2]).

Proposition 3.4.2. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category that is $\otimes$-sifted cocomplete. There is an equivalence of ∞-categories*

$$\mathsf{Alg}^{\mathsf{aug}}_{\mathcal{D}\mathsf{isk}_n^{\mathsf{sm}}}(\mathcal{V}) \xrightarrow{\simeq} \mathbf{H}^{\mathsf{aug}}_{\mathsf{red}}\big(\mathcal{M}\mathsf{fld}_n^{\partial,\mathsf{sm},\mathsf{fin}},\mathcal{V}\big)~.$$

Proof. There is the evident fully faithful symmetric monoidal functor

$$i\colon\ \mathcal{D}\mathsf{isk}_{n,+}^{\mathsf{sm}} \longrightarrow \mathcal{D}\mathsf{isk}_{n,+}^{\partial,\mathsf{sm}}~.$$

There is the symmetric monoidal functor

$$q\colon\ \mathcal{D}\mathsf{isk}_{n,+}^{\partial,\mathsf{sm}} \longrightarrow \mathcal{D}\mathsf{isk}_{n,+}^{\mathsf{sm}}$$

determined by declaring $q\colon (\mathbb{R}_{\geq 0}\times\mathbb{R}^{n-1})_+ \mapsto +$. There is the evident natural transformation $\mathsf{id}\to i\circ q$ witnessing q as a left adjoint to i. It follows that, for $W := q^{-1}\big(\mathcal{D}\mathsf{isk}_{n,+}^{\mathsf{sm}}\big)^{\sim}$ the preimage of the maximal ∞-subgroupoid, then the canonical map from the localization

$$\overline{q}\colon W^{-1}\,\mathcal{D}\mathsf{isk}_{n,+}^{\partial,\mathsf{sm}} \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}_{n,+}^{\mathsf{sm}}$$

is an equivalence of symmetric monoidal ∞-categories. By inspection, $W\subset\mathcal{D}\mathsf{isk}_{n,+}^{\partial,\mathsf{sm}}$ is the smallest sub-symmetric monoidal category containing the isomorphisms as well as the morphism $\big(*\to(\mathbb{R}_{\geq 0}\times\mathbb{R}^{n-1})_+\big)$. Through the main results (†) of [AFT2], we conclude that

$$q^*\colon\ \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{D}\mathsf{isk}_n^{\mathsf{sm}}}(\mathcal{V}) \xrightarrow{\qquad\qquad} \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{D}\mathsf{isk}_n^{\partial,\mathsf{sm}}}(\mathcal{V}) \xrightarrow[\underset{(\dagger)}{\simeq}]{\int_-} \mathbf{H}^{\mathsf{aug}}(\mathcal{M}\mathsf{fld}_n^{\partial,\mathsf{sm},\mathsf{fin}},\mathcal{V})$$

is fully faithful, with essential image $\mathbf{H}^{\mathsf{aug}}_{\mathsf{red}}\big(\mathcal{M}\mathsf{fld}_n^{\partial,\mathsf{sm},\mathsf{fin}},\mathcal{V}\big)$.

□

Recall from Definition 2.5.6 the ∞-category $\mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{BO}(n)}$ of zero-pointed *smoothed* n-manifolds.

Theorem 3.4.3. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category that is $\otimes$-sifted cocomplete. There is a fully faithful horizontal functor making the diagram among ∞-categories*

$$\begin{tikzcd} & \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{D}\mathsf{isk}_n^{\mathsf{sm}}}(\mathcal{V}) \arrow[dl, "\int_-"'] \arrow[dr, "\int_-"] & \\ \mathbf{H}^{\mathsf{aug}}_{\mathsf{red}}\bigl(\mathcal{M}\mathsf{fld}_n^{\partial,\mathsf{sm},\mathsf{fin}},\mathcal{V}\bigr) \arrow[rr] & & \mathsf{Fun}^{\otimes}\bigl(\mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{sm},\mathsf{fin}},\mathcal{V}\bigr) \end{tikzcd}$$

and the essential image of the rightmost ∞-category consists of those functors that satisfy $\otimes$-excision.

Proof. Theorem 3.2.3 offers the existence of the rightward factorization homology functor as a left adjoint to the restriction, and it is fully faithful because $\mathcal{D}\mathsf{isk}^{\mathsf{sm}}_{n,+} \to \mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{sm},\mathsf{fin}}$ is fully faithful. Proposition 3.4.2 states that the leftward factorization homology functor exists and is an equivalence. This establishes the triangle, and it remains to show that this triangle commutes. This is to say, for each finitary smooth n-manifold with boundary $\overline{M}$, with associated zero-pointed manifold $M_* = * \coprod_{\partial\overline{M}} \overline{M}$, and each augmented n-disk algebra, there is a canonical equivalence

$$\int_{\overline{M}} A \quad\simeq\quad \int_{M_*} A~.$$

Observe that the quotient map $\pi : \overline{M} \to M_*$ is constructible. Consequently applying the push-forward formula for factorization homology from [AFT2], we obtain an equivalence

$$\int_{\overline{M}} A \simeq \underset{\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/M_*}}{\mathsf{colim}} \pi_*A$$

where

$$\pi_*A\colon\ \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/M_*} \xrightarrow{\pi^{-1}} \mathcal{M}\mathsf{fld}^{\partial}_{n/\overline{M}} \xrightarrow{\int A} \mathcal{V}~.$$

The inclusion $\mathcal{D}\mathsf{isk}_+(M_*) \to \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/M_*}$ is final by an immediate application of Quillen's theorem A. That is, we can verify for each $V \in \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/M_*}$ the contractibility of $\mathcal{D}\mathsf{isk}_+(M_*)^{V/}$ by noting that this slice ∞-category has an initial object: if $V \hookrightarrow M_*$ contains the cone-point in its image, then $V \hookrightarrow M_*$ is itself is the initial object; if V does not, adjoin a small conical neighborhood $\mathsf{C}(\partial\overline{M})$ of the cone-point of M_* to V, and the disjoint union $V \sqcup \mathsf{C}(\partial\overline{M}) \hookrightarrow M_*$ is initial in $\mathcal{D}\mathsf{isk}_+(M_*)^{V/}$.

As a consequence, we can calculate the factorization homology $\int_{\overline{M}} A$ as the colimit of the restriction of π_*A to $\mathcal{D}\mathsf{isk}_+(M_*)$. Since $\mathcal{D}\mathsf{isk}_+(M_*) \to \bigl(\mathcal{D}\mathsf{isk}_{n,+/M_*}\bigr)/\,\mathcal{D}\mathsf{isk}_{n,+}$ is a localization by morphisms on which π_*A is an equivalence, therefore the functor π_*A canonically factors through $\bigl(\mathcal{D}\mathsf{isk}_{n,+/M_*}\bigr)/\,\mathcal{D}\mathsf{isk}_{n,+}$. By inspection, it is immediate that this factorized functor

$$\pi_*A : \bigl(\mathcal{D}\mathsf{isk}_{n,+/M_*}\bigr)/\,\mathcal{D}\mathsf{isk}_{n,+} \to \mathcal{V}$$

is equivalent to the left Kan extension of $A : \mathcal{D}\mathsf{isk}_{n,+/M_*} \to \mathcal{V}$ along $\mathcal{D}\mathsf{isk}_{n,+/M_*} \to \bigl(\mathcal{D}\mathsf{isk}_{n,+/M_*}\bigr)/\,\mathcal{D}\mathsf{isk}_{n,+}$. Consequently, we obtain the equivalence

$$\int_{\overline{M}} A \simeq \underset{\mathcal{D}\mathsf{isk}_+(M_*)}{\mathsf{colim}} \pi_*A \simeq \underset{\bigl(\mathcal{D}\mathsf{isk}_{n,+/M_*}\bigr)/\,\mathcal{D}\mathsf{isk}_{n,+}}{\mathsf{colim}} \pi_*A \simeq \int_{M_*} A~.$$

Finally, it is direct from the commutativity of the triangle that the essential image in the righthand ∞-category consists of symmetric monoidal functors that satisfy $\otimes$-excision. That this is precisely the essential image follows quickly because the conically finite stratified spaces are generated by collar-gluings; see [AFT1] for a precise statement.

□

Remark 3.4.4. Theorem 3.4.3 implies that a reduced homology theory for n-manifolds with boundary has notable additional functorialities, namely extension-by-zero maps. For instance, for a properly embedded codimension-zero submanifold $\overline{U} \subset M$ possibly with boundary, and with quotient $U_* := * \coprod_{\partial \overline{U}} \overline{U}$, then for a reduced augmented homology theory H there is a natural map

$$H(M) \longrightarrow H(U_*) \simeq \mathbb{1}_{\mathcal{V}} \underset{H(\mathbb{R}\times\partial\overline{U})}{\otimes} H(U)\ .$$

We extract the following statement from Theorem 3.4.3 and its proof.

Corollary 3.4.5 (Reduced factorization (co)homology satisfies $\otimes$-(co)excision)**.** *Let $\mathcal{V}$ be a symmetric monoidal ∞-category. Let $\overline{M} \cong \overline{M}_L \bigcup_{\mathbb{R}\times\overline{M}_0} \overline{M}_R$ be a collar-gluing among smooth n-manifolds with boundary. Consider the associated zero-pointed n-manifolds M_* and M_{L*} and M_{R*} and M_{0*}. Provided $\mathcal{V}$ is $\otimes$-sifted cocomplete, then, for each augmented n-disk algebra A in $\mathcal{V}$, there is a canonical equivalence in $\mathcal{V}$*

$$\int_{M_{L*}} A \bigotimes_{\int_{M_{0*}} A} \int_{M_{R*}} A \ \simeq\ \int_{M_*} A$$

among reduced factorization homologies, from a two-sided bar construction. Likewise, provided $\mathcal{V}^{\mathsf{op}}$ is $\otimes$-sifted cocomplete, then, for each augmented n-coalgebra C in $\mathcal{V}$, there is a canonical equivalence in $\mathcal{V}$

$$\int^{M_*} C \ \simeq\ \int^{M_{L*}} C \bigotimes^{\int^{M_{0*}} C} \int^{M_{R*}} C$$

among reduced factorization cohomologies, to a two-sided cobar construction.

Example 3.4.6. Let $\mathcal{V}$ be a $\otimes$-sifted cocomplete symmetric monoidal ∞-category, and let A be an augmented $\mathcal{D}\mathsf{isk}_n$-algebra in $\mathcal{V}$. As in Remark 1.8.3, there is a constructible map $\overline{M} \to M_*$ from a manifold with corners, which restricts to the interior as a homeomorphism onto M. Corollary 3.4.5 gives the identification

$$\int_{M_*} A \ \simeq\ \mathbb{1}_{\mathcal{V}} \bigotimes_{\int_{\partial\overline{M}} A} \int_M A\ .$$

Remark 3.4.7 (Pushforward for reduced factorization homology)**.** We explain here that there is a pushforward formula for factorization homology of *topological* manifolds (this discussion is only topical for the case $n \neq 4$). The key result in [AFT2] that proves this pushforward formula is the finality of the functor

$$\mathsf{Disk}_{n/M} \longrightarrow \mathcal{D}\mathsf{isk}_{n/M}\ .$$

The proof of this finality is tantamount to the fact that the map of topological monoids $\mathsf{Diff}(\mathbb{R}^n) \to \mathsf{Emb}^{\mathsf{sm}}(\mathbb{R}^n, \mathbb{R}^n)$ is a weak homotopy equivalence of topological spaces – there, we made use of a much more general fact, concerning automorphisms versus endomorphisms of basic singularity types, not just the singularity type $\mathbb{R}^n$. So we point out here that the argument verifying finality of the above functor is valid even in this topological setting, because of the Theorem 2.5.4 of Kister-Mazur ([Ki]). And thereafter, that the reduced factorization homology introduced in this article, too, satisfies the pushforward formula.

3.5. **Proof of Proposition 3.3.3.** This is the most technical part of this paper.

The strategy of this proof is captured by the diagram among ∞-categories

$$\begin{tikzcd}
\mathcal{D}\mathsf{isk}_{n,+} \arrow[r, "="] \arrow[d] & \mathcal{D}\mathsf{isk}_{n,+} \arrow[r, "="] \arrow[d] & \mathcal{D}\mathsf{isk}_{n,+} \arrow[d] \\
\mathsf{cEnd}_{\mid \mathsf{Fin}_*^{\mathsf{inrt}}} \arrow[r] \arrow[d] & \mathsf{cEnd} \arrow[r, "\mathrm{Loc}"] \arrow[d, "\mathrm{Quot}"] & \mathcal{D}\mathsf{isk}_{n,+/M_*} \\
\mathcal{D}\mathsf{isk}_+(M_*)_{\mid \mathsf{Fin}_*^{\mathsf{inrt}}} \arrow[r] & \mathcal{D}\mathsf{isk}_+(M_*) & .
\end{tikzcd} \tag{14}$$

in where the right horizontal functors are localizations on the image of the left horizontal functors, and the vertical sequences are cofiber sequences. Because the classifying space $\mathsf{B}\big(\mathcal{D}\mathsf{isk}_{n,+}\big) \simeq *$ is weakly contractable, the quotient at hand, too are localizations. The proposition then follows upon the construction of such a diagram, because iterated localizations too are localizations. Let us now construct this diagram.

Fix a weakly constructible map $f\colon M_* \to [0,1]$ between stratified spaces for which $* = f^{-1}0$. Denote the submanifold $L := f^{-1}(\frac{1}{2}) \subset M$ – it is a compact smooth $(n-1)$-dimensional submanifold.

Consider the map of colored operads

$$\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]} \xrightarrow{f^{-1}} \mathsf{ZMfld}_{n/M_*}~, \qquad [0,1] \supset O \mapsto f^{-1}(U)_*$$

where $f^{-1}(U)_*$ is declared to be the sub-zero-pointed manifold $f^{-1}(U)_* \subset M_*$ if $0 \in U$ and as $f^{-1}(U)_+$ if $0 \notin U$. And so we have the composite map of ∞-operads

$$f_*\mathbb{E}_+\colon~ \mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]} \xrightarrow{f^{-1}} \mathsf{ZMfld}_{n/M_*} \xrightarrow{\mathcal{D}\mathsf{isk}_{n,+/-}} \mathsf{PShv}_*\big(\mathcal{D}\mathsf{isk}_{n,+}\big)_{/~\mathcal{D}\mathsf{isk}_{n,+/M_*}}$$

where here we are using the model for pointed presheaves of right fibrations with a section model, so that (symmetric monoidal) Yoneda is given through slice ∞-categories, as indicated. There results a canonical functor between pointed presheaves on $\mathcal{D}\mathsf{isk}_{n,+}$ from the colimit

$$\underset{(U\hookrightarrow[0,1])\in\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}}{\mathsf{colim}} \mathcal{D}\mathsf{isk}_{n,+/f^{-1}(U)_*} \quad =: \quad \int_{[0,1]} f_*\mathbb{E}_+ \;\longrightarrow\; \mathcal{D}\mathsf{isk}_{n,+/M_*}~. \tag{15}$$

Lemma 3.5.1. *The map (15) of pointed presheaves on $\mathcal{D}\mathsf{isk}_{n,+}$ is an equivalence.*

Proof. Being a map between right fibrations over $\mathcal{D}\mathsf{isk}_{n,+}$, it is enough to show, for each finite set I, that the induced map of spaces

$$\underset{U\hookrightarrow[0,1]}{\mathsf{colim}}\, \mathsf{ZEmb}\big((\mathbb{R}^n_+)^{\bigvee I}, f^{-1}(U)_*\big) \longrightarrow \mathsf{ZEmb}\big((\mathbb{R}^n_+)^{\bigvee I}, M_*\big)$$

is an equivalence.

Let I be a finite set. Consider a zero-pointed n-manifold Z_*. For $\mathbb{D}^n \subset \mathbb{R}^n$ the closed n-disk, consider the topological space $\mathsf{ZEmb}\big((\mathbb{D}^n_+)^{\bigvee I}, Z_*\big)$ which is the subspace of the space of pointed maps (with the compact-open topology) consisting of those $f\colon (\mathbb{D}^n_+)^{\bigvee I} \to Z_*$ for which the restriction $f_|\colon f^{-1}Z \to Z$ is a topological embedding. In a standard manner, the evident restriction $\mathsf{ZEmb}\big((\mathbb{R}^n_+)^{\bigvee I}, Z_*\big) \xrightarrow{\simeq} \mathsf{ZEmb}\big((\mathbb{D}^n_+)^{\bigvee I}, Z_*\big)$ is a weak homotopy equivalence, and it is functorial in the argument Z_*. So it is enough to argue that the likewise map of spaces as displayed above inwhere each instance of $\mathbb{R}^n$ is replaced by one of $\mathbb{D}^n$, is an equivalence of spaces.

Each map $f^{-1}(U)_* \hookrightarrow M_*$ appearing in the above colimit is an open embedding. It follows from the compact-open topology on the set of zero-pointed embeddings that the collection

$$\Big\{\mathsf{ZEmb}\big((\mathbb{D}^n_+)^{\bigvee I}, f^{-1}(U)_*\big) \longrightarrow \mathsf{ZEmb}\big((\mathbb{D}^n_+)^{\bigvee I}, M_*\big) \mid (U \hookrightarrow [0,1]) \in \mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}\Big\} \tag{16}$$

is comprised of open embeddings. Consider the union $\mathcal{U} \subset \mathsf{ZEmb}\big((\mathbb{D}^n_+)^{\bigvee I}, M_*\big)$ of these open embeddings.

Each finite subset of $[0,1]$ is contained in a member of the collection of open embeddings $\{(U \hookrightarrow [0,1])\}$ indexed by the objects of $\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}$. Thereofre, the same is true for the collection of open embeddings $\{f^{-1}(U)_* \hookrightarrow M_*\}$ indexed by this same set. Now, fix a compact family $K \times (\mathbb{D}^n_+)^{\bigvee I} \xrightarrow{f} M_*$ of zero-pointed embeddings. The previous paragraph guarantees the existence of a positive number ϵ for which pre-scaling by ϵ

$$K \times (\mathbb{D}^n_+)^{\bigvee I} \xhookrightarrow{\epsilon \cdot} K \times (\mathbb{D}^n_+)^{\bigvee I} \xrightarrow{f} M_*$$

has the property that, for each $x \in K$, the restriction $(\epsilon \cdot f)_x \colon (\mathbb{D}^n_+)^{\bigvee I} \to M_*$ through $f^{-1}(U)_*$ for some term in the given indexing set. We conclude that the inclusion $\mathcal{U} \hookrightarrow \mathsf{ZEmb}\big((\mathbb{D}^n_+)^{\bigvee I}, M_*\big)$ is a weak homotopy equivalence.

It remains to show that the union $\mathcal{U}$ is a homotopy colimit of its terms. The collection of open embeddings $\big\{U \hookrightarrow [0,1]\big\}$, indexed by the objects of $\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}$, is a hypercover of $[0,1]$. It follows that the collection $\{f^{-1}(U)_* \hookrightarrow M_*\}$, too is a hypercover; and thereafter that the collection (16) too is a hypercover of $\mathcal{U}$. It follows from the results of [DI] that the canonical map of topological spaces from the homotopy colimit

$$\underset{U \hookrightarrow [0,1]}{\mathsf{hocolim}}\, \mathsf{ZEmb}\big((\mathbb{D}^n_+)^{\bigvee I}, f^{-1}(U)_*\big) \longrightarrow \mathcal{U}$$

is a weak homotopy equivalence.

□

Now, consider the likewise composite representation

$$f_*\mathbb{E}\colon\ \mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]} \xrightarrow{f^{-1}} \mathsf{Snglr}_{n/M_*} \xrightarrow{\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/-}} \mathsf{PShv}\big(\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})\big)\ .$$

A main result of [AFT2] ($\otimes$-excision) states that the likewise map of presheaves over $\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})$ from the colimit

$$\underset{(U \hookrightarrow [0,1]) \in \mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}}{\mathsf{colim}} \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}(U)} \quad =: \quad \int_{[0,1]} f_*\mathbb{E} \xrightarrow{\ \simeq\ } \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/M_*} \tag{17}$$

is an equivalence. We highlight the following consequence of this equivalence. Denote the full subcategory

$$\mathsf{Disk}^{\partial,\mathsf{or}}_{1/(0 \in [0,1])} \subset \mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}$$

consisting of those $U \hookrightarrow [0,1]$ for which $0 \in U$.

Lemma 3.5.2. *The equivalence (17) restricts as an equivalence of presheaves on* $\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})$*:*

$$\underset{(0 \in U \hookrightarrow [0,1]) \in \mathsf{Disk}^{\partial,\mathsf{or}}_{1/(0 \in [0,1])}}{\mathsf{colim}} \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}(U)} \quad =: \quad \int_{0 \in [0,1]} f_*\mathbb{E} \xrightarrow{\ \simeq\ } \mathcal{D}\mathsf{isk}_+(M_*) \tag{18}$$

Proof. Because $\mathsf{Disk}^{\partial,\mathsf{or}}_{1/(0 \in [0,1])} \subset \mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}$ is fully faithful, then so is the functor between colimits $\underset{0 \in U}{\mathsf{colim}}\, \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}(U)} \hookrightarrow \underset{U}{\mathsf{colim}}\, \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}(U)}$. By inspection, the functor from this restricted colimit factors through $\mathcal{D}\mathsf{isk}_+(M_*)$, and does so essentially surjectiely. The result follows because the functor (17) is an equivalence of ∞-categories, and is in particualr fully faithful.

□

Consider the functor

$$[\bullet]\ :\ \big(\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}\big)^{\mathsf{op}} \longrightarrow \mathsf{Cat}\quad , \qquad (U \hookrightarrow [0,1]) \mapsto \big((0,1] \smallsetminus U, \leq\big)$$

where $\big((0,1] \smallsetminus U, \leq\big)$ is the category associated to the set $(0,1] \setminus U$ with the linear order inherited from the standard one on $[0,1]$. Let us use the shorthand notation

$$\mathsf{cEnd} := \mathsf{cEnd}_{\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}}\big(f_*\mathbb{E}_+, [\bullet]\big)$$

for the ∞-category which is the coend as indicated. The unique natural transformation $[\bullet] \to *$ determines a functor between coends

$$\mathsf{cEnd} \longrightarrow \mathsf{cEnd}_{\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}}\big(f_*\mathbb{E}_+,*\big) \underset{(\text{Lem } 3.5.1)}{\simeq} \mathcal{D}\mathsf{isk}_{n,+/M_*} ~. \tag{19}$$

Lemma 3.5.3. *The functor (19) is a localization of ∞-categories, localized at $\mathsf{cEnd}_{|\,\mathsf{Fin}^{\mathsf{inrt}}_*}$.*

Proof. Because linearly ordered sets have contractable classifying spaces, the natural transformation $[\bullet] \to *$ is by localizations. Therefore the functor

$$\mathsf{cEnd} \longrightarrow \mathsf{cEnd}_{\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}}\big(f_*\mathbb{E}_+,*\big)$$

is a localization. Inspecting the coends at hand, there is a filler (which is necessarily essentially unique) in the diagram of ∞-categories

$$\begin{tikzcd}
\mathsf{cEnd}_{|\,\mathsf{Fin}^{\mathsf{inrt}}_*} \arrow[r, dashed] \arrow[d] & \Big(\mathsf{cEnd}_{\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}}\big(f_*\mathbb{E}_+,*\big)\Big)^{\sim} \arrow[d] \\
\mathsf{cEnd} \arrow[r] & \mathsf{cEnd}_{\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}}\big(f_*\mathbb{E}_+,*\big) ~,
\end{tikzcd}$$

making the diagram pullback.

□

We now construct a functor

$$\mathsf{cEnd} \longrightarrow \mathsf{cEnd}_{\mathsf{Disk}^{\partial,\mathsf{or}}_{1/(0\in[0,1])}}\big(f_*\mathbb{E},*\big) \underset{(\text{Lem } 3.5.2)}{\simeq} \mathcal{D}\mathsf{isk}_{+}(M_*) ~. \tag{20}$$

The domain and codomain of this functor (20) being coends, it is enough to construct a natural transformation $\big(\phi_{U,t}\big)_{\{(U,t)\}}$ making a diagram of ∞-categories

$$\begin{tikzcd}
\big(\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}\big) \wr [\bullet] \arrow[rr, "\mathcal{D}\mathsf{isk}_{n,+/f^{-1}(-)_*}"] \arrow[dr] & & \mathsf{Cat}_\infty \\
& \mathsf{Disk}^{\partial,\mathsf{or}}_{1/(0\in[0,1])} \arrow[ur, "\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}(-)}"'] & .
\end{tikzcd}$$

commute. Here, the top left term is the Grothendieck construction on the functor $[\bullet]$, and so the proported downward arrow is a functor between ordinary categories. The downward arrow is given on objects as $(U \hookrightarrow [0,1], t) \mapsto \Big([0,t) \sqcup \big(U \setminus [0,t)\big) \hookrightarrow [0,1]\Big)$, and given on morphisms evidently. We take the natural transformation

$$\phi_{U,t}\colon\ \mathcal{D}\mathsf{isk}_{+,n/f^{-1}(U)_*} \longrightarrow \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}U}$$

to be the composition

$$\begin{aligned}
\mathcal{D}\mathsf{isk}_{+,n/f^{-1}(U)_*} \quad & \xrightarrow{\simeq} \quad \mathcal{D}\mathsf{isk}_{+,n/f^{-1}(U\cap[0,t))_*} \times \mathcal{D}\mathsf{isk}_{n/f^{-1}(U\smallsetminus[0,t))} \\
& \xrightarrow{\mathsf{pr}\times\mathsf{id}} \quad * \times \mathcal{D}\mathsf{isk}_{n/f^{-1}(U\smallsetminus[0,t))} \\
& \xrightarrow{\{f^{-1}[0,t)\}\times\mathsf{id}} \quad \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}[0,t)} \times \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}(U\smallsetminus[0,t))} \\
& \xrightarrow{\simeq} \quad \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}\big([0,t)\bigsqcup U\smallsetminus[0,t)\big)} ~.
\end{aligned}$$

Each of these arrows is manifestly functorial in the argument $(U,t) \in \mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]} \wr [\bullet]$, and so we have constructed the functor (20).

The verification of the diagram (14), and therefore the proof of Proposition 3.3.3, is complete after the final result.

Lemma 3.5.4. *The functor (20) witnesses an identification from the quotient ∞-category:*

$$\mathsf{cEnd}/\,\mathcal{D}\mathsf{isk}_{n,+} \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}_+(M_*)~.$$

Proof. Consider the functor

$$\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}\wr[\bullet] \longrightarrow \mathsf{Cat}_\infty~, \qquad (U,t)~\mapsto~\mathcal{D}\mathsf{isk}_{n,+/f^{-1}(U\cap[0,t))_*}~. \tag{21}$$

Visible in the construction of the natural transformation $(\phi_{U,t})_{(U,t)}$ is that each component functor induces an identification from the quotient ∞-category:

$$\phi_{U,t}\colon \big(\mathcal{D}\mathsf{isk}_{+,n/f^{-1}(U)_*}\big)/\big(\mathcal{D}\mathsf{isk}_{n,+/f^{-1}(U\cap[0,t))_*}\big)~\simeq~\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}U}~.$$

From the construction of the functor (20) as one between colimits, it is enough to show that, for each $(U,t)\in\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]})\wr[\bullet]$, the functor $\phi_{U,t}$ witnesses an identification from the quotient ∞-category

$$\phi_{U,t}\colon \big(\mathcal{D}\mathsf{isk}_{+,n/f^{-1}(U)_*}\big)/\big(\mathcal{D}\mathsf{isk}_{n,+/f^{-1}(U\cap[0,t))_*}\big)~\simeq~\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/f^{-1}\big([0,t)\bigsqcup U\smallsetminus[0,t)\big)}~.$$

The proof is complete upon showing the canonical natural transformation from the constant functor at $\mathcal{D}\mathsf{isk}_{n,+}$ to the functor (21) induces an equivalence to the colimit. Consider the full subcategory of $\mathcal{D}\subset\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}\wr[\bullet]$ consisting of those (U,t) for which $U\cap[0,t)$ is a half-open interval containing 0. Let (W,s) be an object of $\mathsf{Disk}^{\partial,\mathsf{or}}_{1/[0,1]}\wr[\bullet]$, and consider the slice category $\mathcal{D}^{(W,s)/}$. We will show this slice category has contractable classifying space by argueing that it is filtered. Let $(U_\bullet,t_\bullet)\colon K\to\mathcal{D}^{(W,s)/}$ be a finite diagram. Consider the open subset $0\in U_0:=\bigcap\limits_{x\in K}U_x\subset[0,1]$ and the number $0<t_0:=\mathsf{Min}\{t_x\mid x\in K\}\notin U_0$. There is a canonical morphism $(W,s)\to(U_0,t_0)$, thereby witnessing a cone on the given finite diagram.

□

4. Duality

Our setup is ripe for depicting a number of dualities: we will see Koszul duality among n-disk (co)algebras, as well as Poincaré duality among manifolds. Here, we recover a twisted version of Atiyah duality.

In this section we fix the following parameters.

- A dimension n.
- A symmetric monoidal ∞-category $\mathcal{V}$ whose underlying ∞-category admits sifted colimits and cosifted limits.

4.1. Poincaré/Koszul duality map.

We now construct the *Poincaré/Koszul duality map.*

For each map of pointed spaces $B\to\mathsf{BTop}(n)$ there is a span of ∞-categories

$$\mathsf{Fun}(\mathcal{D}\mathsf{isk}^B_{n,+},\mathcal{V}) \xleftarrow{(-)_+} \mathsf{Fun}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}^{B,\mathsf{fin}}_n,\mathcal{V}\big) \xrightarrow{(-)^+} \mathsf{Fun}(\mathcal{D}\mathsf{isk}^{B,+}_n,\mathcal{V}) \tag{22}$$

given by the evident restrictions. By way of Theorem 3.2.3, the assumption that the underlying ∞-category of $\mathcal{V}$ is cocomplete and complete grants that the left functor has a left adjoint and the right functor has a right adjoint. In particular, there results a functor involving the twisted arrow ∞-category of $\mathcal{V}$ (see §5.2.1 of [Lu2]) for a definition)

$$\mathsf{Fun}^{\otimes}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}^{B,\mathsf{fin}}_n,\mathcal{V}\big)\longrightarrow\mathsf{Fun}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}^{B,\mathsf{fin}}_n,\mathsf{TwAr}(\mathcal{V})\big)$$

whose value on $\mathcal{A}$ evaluates on a conically finite zero-pointed B-manifold M_* as the arrow in $\mathcal{V}$

$$\boxed{\int_{M_*}\mathcal{A}_+ \quad\longrightarrow\quad \int^{M_*^{\neg}}\mathcal{A}^+} \tag{23}$$

termed the *Poincaré/Koszul duality map*, which is a composition of the counit $\int_{M_*} \mathcal{A}_+ \to \mathcal{A}(M_*)$ followed by the unit $\mathcal{A}(M_*) \to \int^{M_*^\neg} \mathcal{A}^+$. In the same way, for each category of basics $\mathcal{B}$ there is the Poincaré/Koszul duality functor

$$\mathsf{Fun}^{\otimes}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}^{\mathsf{fin}}(\mathcal{B}), \mathcal{V}\big) \longrightarrow \mathsf{Fun}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}^{\mathsf{fin}}(\mathcal{B}), \mathsf{TwAr}(\mathcal{V})\big) ~,$$

whose value on $\mathcal{A}$ evaluates on a conically finite zero-pointed $\mathcal{B}$-manifold M_* as the arrow in $\mathcal{V}$

$$\boxed{\int_{M_*} \mathcal{A}_+ \quad \longrightarrow \quad \int^{M_*^\neg} \mathcal{A}^+} \tag{24}$$

which is a composition of a counit morphism followed by a unit morphism.

The following question drives this work and the sequel [AF2].

Question 4.1.1. What conditions on $\mathcal{A}$ guarantee that the Poincaré/Koszul duality map (23)/(24) is an equivalence?

Remark 4.1.2 (No choices)**.** The definitions present in our work culminate as the duality map (23) above, for the universality of this map involves no choices.

Remark 4.1.3. We point out that the Poincaré/Koszul duality map(s) are utterly ambidextrous in the background symmetric monoidal ∞-category $\mathcal{V}$ in the sense that these maps are equivalences if and only if they are when $\mathcal{V}$ is replaced by $\mathcal{V}^{\mathsf{op}}$.

Remark 4.1.4 (Scanning)**.** The special case of the Poincaré/Koszul duality map in the case $\mathcal{V} = (\mathsf{Spaces}, \times)$ is equivalent to the scanning map of [Mc], [Se2], [Bö]. In those works the map is defined one manifold at a time, in compact families, upon making contractible choices; this makes the establishment of continuous functoriality in the manifold a nuisance to verify. To verify the relation, for simplicity, we fix a smooth framed n-manifold equipped with a complete Riemannian metric for which there is a uniform radius of injectivity $\epsilon > 0$. Again for simplicity, consider A to be a (discrete) commutative group. In this case, we can identify the defining colimit for factorization homology as a labeled configuration space:

$$\int_{M_*} A ~\simeq~ \underset{S\in\mathsf{Ran}(M_*)}{\mathsf{colim}}\, A^S \simeq \Big(\bigvee_{i\geq 0} {M_*}^{\times i} \underset{\Sigma_i}{\wedge} A^{\times i}\Big)_{/\sim} \quad \Big(= ~\{(S \subset M \text{ finite},~ S \xrightarrow{l} A)\}~\Big)$$

where the equivalence relation of the third term is determined by declaring $[(x_1, a_1), \ldots, (x, a), (x, b)] \sim [(x_1, a_1), \ldots, (x, a + b)]$, and the fourth term is just a convenient description of the underlying set of the third space. Dold-Thom theory gives that the homotopy groups of this space are identified as the reduced homology of M_*. Through the same theory, we know $\int_{(\mathbb{R}^n)^+} A \simeq \mathsf{B}^n A \simeq K(A, n)$ is a model for the Eilenberg-MacLane space. Because we are working in spaces, and using that M is framed, factorization cohomology

$$\int^{M_*^\neg} \mathsf{B}^n A ~\simeq~ \mathsf{Map}_*\big(M_*^\neg, K(A, n)\big)$$

is weakly equivalent to the based mapping space. Tracing through these identifications, the map (23) is weakly equivalent to the assignment

$$\big(M \supset S \xrightarrow{l} A\big) \mapsto \Big(M \ni x \mapsto \big(B_\epsilon(x) \cap S \xrightarrow{l_|} A\big) \in \int_{B_\epsilon(x)^*} A \simeq K(A, n)\Big)$$

– here $x \in B_\epsilon(x) \subset M$ is the ϵ-ball about x. This assignment is continuous, and is the *scanning map* as mentioned.

4.2. **Koszul duality.** Evaluating the Poincaré/Koszul duality map (23) on pointed Euclidean spaces provokes a meaningful examination: Koszul duality. Here we geometrically define a procedure for assigning to an augmented n-disk algebra an augmented n-disk coalgebra (and vice-versa) – this is the Bar-coBar adjunction.

Definition 4.2.1 (Koszul duality)**.** Consider a symmetric monoidal ∞-category $\mathcal{V}$. Say a symmetric monodical functor $\mathcal{A}\colon \mathcal{Z}\mathcal{D}\mathsf{isk}_n \longrightarrow \mathcal{V}$ is a *Koszul duality* if it has the following two properties.

- $\mathcal{A}$ is initial among all such whose restriction to $\mathcal{D}\mathsf{isk}_{n,+}$ is $\mathcal{A}_+$. That is to say, it is initial in the ∞-category that is the fiber over $\mathcal{A}_+$ of the restriction $\mathsf{Fun}^{\otimes}\big(\mathcal{Z}\mathcal{D}\mathsf{isk}_n,\mathcal{V}\big)\xrightarrow{(-)_+}\mathsf{Alg}_n^{\mathsf{aug}}(\mathcal{V})$.
- $\mathcal{A}$ is terminal among all such whose restriction to $\mathcal{D}\mathsf{isk}_n^+$ is $\mathcal{A}^+$. That is to say, it is terminal in the ∞-category that is the fiber over $\mathcal{A}^+$ of the restriction $\mathsf{Fun}^{\otimes}\big(\mathcal{Z}\mathcal{D}\mathsf{isk}_n,\mathcal{V}\big)\xrightarrow{(-)^+}\mathsf{cAlg}_n^{\mathsf{aug}}(\mathcal{V})$.

Remark 4.2.2. The key feature of a Koszul duality $\mathcal{A}$ is that it is determined by its restriction to either $\mathcal{D}\mathsf{isk}_{n,+}$ or to $\mathcal{D}\mathsf{isk}_n^+$. Corollary 4.2.9 makes this explicit.

Remark 4.2.3. For $\mathcal{B}$ a category of basics, there is an evident variation of Definition 4.2.1 for each instance of $\mathcal{D}\mathsf{isk}_n$ replaced by $\mathcal{D}\mathsf{isk}(\mathcal{B})$. Just as the case of Definition 4.2.1, this variation is motivated by investigating the Poincaré/Koszul duality map (24) in the case that X_* is a basic $\mathcal{B}$-manifold.

Remark 4.2.4. The notion of a Koszul duality has been developed in other works, such as [GK], [GJ], and [Lu3]. We will leave to another work to explain the relationship between the notion presented here and that of [Lu3].

The diagram (10) provokes the following definition, the notation for which is justified as Theorem 4.3.1 to come.

Definition 4.2.5 (Bar-coBar)**.** Consider a symmetric monoidal ∞-category $\mathcal{V}$ that is sifted cocomplete and cosifted complete. Define the composite functors

$$\mathsf{Bar}^n\colon\ \mathsf{Fun}(\mathcal{D}\mathsf{isk}_{n,+},\mathcal{V})\xrightarrow{\int_-}\mathsf{Fun}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{fin}},\mathcal{V}\big)\longrightarrow\mathsf{Fun}\big(\mathcal{D}\mathsf{isk}_n^+,\mathcal{V}\big)$$

and

$$\mathsf{cBar}^n\colon\ \mathsf{Fun}(\mathcal{D}\mathsf{isk}_n^+,\mathcal{V})\xrightarrow{\int^-}\mathsf{Fun}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{fin}},\mathcal{V}\big)\longrightarrow\mathsf{Fun}\big(\mathcal{D}\mathsf{isk}_{n,+},\mathcal{V}\big)$$

in where the unlabeled arrows are restrictions.

In the following subsection §4.5, we will examine instances of this Bar and coBar construction.

Proposition 4.2.6. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category whose underlying ∞-category admits sifted colimits and cosifted limits. The pair of functors*

$$\mathsf{Bar}^n\colon\ \mathsf{Fun}(\mathcal{D}\mathsf{isk}_{n,+},\mathcal{V})\leftrightarrows\mathsf{Fun}(\mathcal{D}\mathsf{isk}_n^+,\mathcal{V})\colon\ \mathsf{cBar}^n \tag{25}$$

is an adjunction. Moreover, both the associated monad and the comonad are idempotent; which is to say, the unit and counit maps

$$\mathsf{cBar}^n\circ\mathsf{Bar}^n\xrightarrow{\simeq}\big(\mathsf{cBar}^n\circ\mathsf{Bar}^n\big)^{\circ 2}\qquad\text{and}\qquad\big(\mathsf{Bar}^n\circ\mathsf{cBar}^n\big)^{\circ 2}\xrightarrow{\simeq}\mathsf{Bar}^n\circ\mathsf{cBar}^n$$

are equivalences. If $\mathcal{V}$ further satisfies $\otimes$-sifted cocomplete, Bar^n *canonically factors through* $\mathsf{cAlg}^{\mathsf{aug}}_{\mathcal{D}\mathsf{isk}_n}(\mathcal{V})$*; and dually if $\mathcal{V}$ is $\otimes$-cosifted cocomplete. If $\mathcal{V}$ is $\otimes$-sifted cocomplete and $\otimes$-cosifted complete then the adjunction (25) restricts as an adjunction*

$$\mathsf{Bar}^n\colon\ \mathsf{Alg}_n^{\mathsf{aug}}(\mathcal{V})\leftrightarrows\mathsf{cAlg}_n^{\mathsf{aug}}(\mathcal{V})\colon\ \mathsf{cBar}^n\ . \tag{26}$$

Proof. The first statement is immediate from the pair of adjunctions (10). The idempotence follows because both functors $\mathcal{D}\mathsf{isk}_{n,+}\to\mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{fin}}\leftarrow\mathcal{D}\mathsf{isk}_n^+$ are fully faithful. The final statements regarding extensions as symmetric monoidal functors follow from Theorem 3.2.3.

□

Recall the notation for the functors displayed in (22). The following result is a simple rephrasing of universal properties, premised on Proposition 4.2.6.

Proposition 4.2.7. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category that is $\otimes$-sifted cocomplete and $\otimes$-cosifted complete. Let $\mathcal{A}\colon \mathcal{Z}\mathcal{D}\mathsf{isk} \to \mathcal{V}$ be a symmetric monoidal functor. The following statements are equivalent.*

(1) *$\mathcal{A}$ is a Koszul duality.*

(2) *Both of the universal arrows*

$$\mathsf{Bar}^n \mathcal{A}_+ \xrightarrow{\simeq} \mathcal{A}^+ \qquad \textit{and} \qquad \mathcal{A}_+ \xrightarrow{\simeq} \mathsf{cBar}^n \mathcal{A}^+$$

are equivalences.

(3) *The universal triangle in $\mathsf{Fun}^{\otimes}\big(\mathcal{Z}\mathcal{D}\mathsf{isk}, \mathcal{V}\big)$*

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dr, "\simeq"] & \\
\int_{(-)} \mathcal{A}_+ \arrow[ur, "\simeq"] \arrow[rr, "\simeq"] & & \int^{(-)^\neg} \mathcal{A}^+
\end{tikzcd}$$

is of equivalences.

Remark 4.2.8. Recall the central Question 4.1.1, asking for conditions for when the universal transformation $\int_{(-)} \mathcal{A}_+ \to \int^{(-)^\neg} \mathcal{A}^+$ is an equivalence. Evaluating on $\mathbb{R}^n_+$ and $(\mathbb{R}^n)^+$ we see that a necessary condition is that $\mathcal{A}$ is a Koszul duality. As we will see, in controlable enough situations, this is a sufficient condition.

Koszul dualities can be constructed by iterating Bar-coBar once:

Corollary 4.2.9. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category whose underlying ∞-category admits sifted colimits and cosifted limits. Let A be an augmented n-disk algebra in $\mathcal{V}$, and let C be an augmented n-disk coalgebra in $\mathcal{V}$. The functors*

$$\int^{-} \mathsf{Bar}^n A\colon \mathcal{Z}\mathcal{D}\mathsf{isk} \longrightarrow \mathcal{V} \qquad \textit{and} \qquad \int_{-} \mathsf{cBar}^n C\colon \mathcal{Z}\mathcal{D}\mathsf{isk} \longrightarrow \mathcal{V}$$

are each Koszul dualities if and only if they extend to symmetric monoidal functors.

Proof. This follows immediately from Lemma 4.2.6.

□

4.3. **The bar construction.** Let $\mathcal{V}$ be a symmetric monoidal ∞-category that is $\otimes$-sifted cocomplete. Recall from §5.2.1 of [Lu2] the bar construction $\mathsf{Bar}(A) \simeq \mathbb{1} \underset{A}{\otimes} \mathbb{1}$ of an augmented associative algebra $A \to \mathbb{1}$ in $\mathcal{V}$. There it is explained that $\mathsf{Bar}(A)$ is equivalent to the geometric realization of a simplicial object $\mathsf{Bar}_\bullet(A)$ in $\mathcal{V}$, which is a two-sided bar construction. Pointwise explicitly, the object of p-simplices is canonically equivalent to $A^{\otimes p}$, and through this identification the inner face maps can be identified as (a choice of) the associative multiplication map for A, the outer face maps can be identified as the augmentation of A, and the degeneracy maps can be identified as (a choice of) the unit of A.

There is a naive comultiplication

$$\mathbb{1} \underset{A}{\otimes} \mathbb{1} \simeq \mathbb{1} \underset{A}{\otimes} A \underset{A}{\otimes} \mathbb{1} \longrightarrow \mathbb{1} \underset{A}{\otimes} \mathbb{1} \underset{A}{\otimes} \mathbb{1}$$

given by the augmentation of A in the middle term. It is a classical result that, in chain complexes, one can choose a model specific representation which admits a strict coalgebra refinement of this homotopy associative map.

Now let A be an n-disk algebra in $\mathcal{V}$. Consider the symmetric monoidal functor

$$\mathcal{D}\mathsf{isk}_1 \times \mathcal{D}\mathsf{isk}_{n-1} \longrightarrow \mathcal{D}\mathsf{isk}_n\ , \qquad (U, V) \mapsto U \times V\ .$$

Restriction along this functor gives a symmetric monoidal functor $\mathcal{D}\mathsf{isk}_1 \times \mathcal{D}\mathsf{isk}_{n-1} \to \mathcal{V}$, which is adjoint to a 1-disk algebra in $\mathsf{Alg}_{n-1}(\mathcal{V})$, that we will again denote as A. Through induction, the *n-fold* bar construction is the object of $\mathcal{V}$

$$\mathsf{Bar}^n(A) \ := \ \mathsf{Bar}\big(\mathsf{Bar}^{n-1}(A)\big) \ .$$

(See §5.2.2 of [Lu2] for a thorough discussion of this iterated Bar construction.) Through similar considerations as the case $n = 1$, one can expect an n-disk coalgebra structure on $\mathsf{Bar}^n(A)$. The non-iterative nature of an n-disk (co)algebra puts tension against this expectation, particularly when considering the $\mathsf{Top}(n)$-module structure on the underlying objects of n-disk (co)algebras. The coming results validate this expectation.

Theorem 4.3.1. *Let A be an augmented n-disk algebra in a symmetric monoida ∞-category $\mathcal{V}$. Provided $\mathcal{V}$ is $\otimes$-sifted cocomplete, there is a canonical equivalence*

$$\int_{(\mathbb{R}^n)^+} A \ \simeq \ \mathsf{Bar}^n(A)$$

between the factorization homology of the 1-point compactification of $\mathbb{R}^n$ with coefficients in A, and the n-fold iteration of the bar construction applied to A.
Likewise, let C be an augmented n-disk coalgebra in $\mathcal{V}$. Provided $\mathcal{V}$ is $\otimes$-cosifted complete, there is a canonical equivalence

$$\int^{(\mathbb{R}^n)^+} C \ \simeq \ \mathsf{cBar}^n(C) \ .$$

Proof. The first statement implies the second by replacing $\mathcal{V}$ by $\mathcal{V}^{\mathsf{op}}$. Theorem 3.4.3 gives the canonical identification

$$\int_{(\mathbb{R}^n)^+} A \ \simeq \ \int_{\mathbb{D}^n} A \ .$$

We proceed by induction on n. Consider the base case $n = 1$. The conditions on $\mathcal{V}$ give that factorization homology satisfies $\otimes$-excision. Applying this $\otimes$-excision for manifolds with boundary from [AFT2] to the collar-gluing $\mathbb{D}^1 \cong [0,1) \cup_{(0,1)} (0,1]$, we have an identification

$$\int_{\mathbb{D}^1} A \ \simeq \ \int_{[0,1)} A \bigotimes_{\int_{(0,1)} A} \int_{(0,1]} A \ \simeq \mathbb{1} \bigotimes_{A} \mathbb{1} \ \simeq \ \mathsf{Bar}(A) \ .$$

Now, the standard projection $\mathbb{D}^n \xrightarrow{\mathsf{pr}} \mathbb{D}^1$ is weakly constructible, and the pushforward formula for factorization homology of [AFT2] gives

$$\int_{\mathbb{D}^n} A \ \simeq \ \int_{\mathbb{D}^1} \mathsf{pr}_* A$$

where $\mathsf{pr}_* A$ evaluates on $U \subset \mathbb{D}^1$ as $\int_{\mathsf{pr}^{-1}U} A$. By inspection, the $\otimes$-excision formula gives the identification

$$\int_{\mathbb{D}^1} \mathsf{pr}_* A \ \simeq \ \int_{[-1,1)} \mathsf{pr}_* A \bigotimes_{\int_{(-1,1)} \mathsf{pr}_* A} \int_{(-1,1]} \mathsf{pr}_* A \ .$$

We land at the identification

$$\int_{\mathbb{D}^n} A \ \simeq \ \mathbb{1} \bigotimes_{\int_{\mathbb{D}^{n-1}} A} \mathbb{1} \ \simeq \ \mathbb{1} \bigotimes_{\mathsf{Bar}^{n-1} A} \mathbb{1} \ \simeq \mathsf{Bar}^n A$$

where the left equivalence is by inspection that of the previous display, the middle equivalence is by induction, and the right equivalence is by definition of the iterated bar construction.

□

This result allows us to see the naive comultiplication above as exactly the fold map $(\mathbb{R}^n)^+ \to (\mathbb{R}^n)^+ \vee (\mathbb{R}^n)^+$, the Pontryagin-Thom collapse map of an embedding $\mathbb{R}^n \sqcup \mathbb{R}^n \hookrightarrow \mathbb{R}^n$.

Corollary 4.3.2. *For A an augmented n-disk algebra in $\mathcal{V}$, a symmetric monoidal ∞-category which is $\otimes$-sifted cocomplete, the n-times iterated bar construction $\mathsf{Bar}^n(A)$ carries a natural n-disk coalgebra structure.*

Proof. We exhibit an augmented n-disk coalgebra structure on $\int_{(\mathbb{R}^n)^+} A$. By the $\otimes$-sifted-complete condition, the factorization homology $\int_{\_} A$ exists and defines a symmetric monoidal functor on $\mathcal{ZM}\mathsf{fld}_n^{\mathsf{fin}}$, conically finite zero-pointed n-manifolds. Restricting this functor to $\mathcal{D}\mathsf{isk}_n^+$ gives the requisite n-disk coalgebra. □

Remark 4.3.3. For a general operad $\mathcal{O}$ together with a left $\mathcal{O}$-module A, such as an $\mathcal{O}$-algebra, and a right $\mathcal{O}$-module M, one can define an analogue of factorization homology

$$\int_M A := M \bigotimes_{\mathsf{Env}(\mathcal{O})} A$$

as the coend of A and M over the symmetric monoidal envelope of $\mathcal{O}$. If $\mathcal{O}$ is augmented, then one can construct a likewise analogue of the map

$$\int_M A \longrightarrow \int^{\mathsf{B}M} \mathsf{B}A \tag{27}$$

to the factorization cohomology (i.e., the end) of the left $\mathbb{1} \circ_{\mathcal{O}} \mathbb{1}$-module $\mathsf{B}A := |\mathsf{Bar}(\mathbb{1}, \mathcal{O}, A)|$ and the right $\mathbb{1} \circ_{\mathcal{O}} \mathbb{1}$-module $\mathsf{B}M := |\mathsf{Bar}(M, \mathcal{O}, \mathbb{1})|$. This map does not reflect a phenomenon of Poincaré duality, however. In the case $\mathcal{O} = \mathcal{E}_n$, Poincaré duality takes place in the identification of the Bar construction $\mathbb{1} \circ_{\mathcal{E}_n} \mathbb{1}$ with a stable shift of $\mathcal{E}_n$ and, thus, in the identification of the righthand side of (27) as factorization cohomology. In particular, an operadic approach would not obviously account for non-abelian Poincaré duality and the unstable Koszul self-duality of n-disk algebra provided by Proposition 4.5.3. However, should the map (27) be of interest, the same tools used here and in the sequel [AF2] to address when the Poincaré/Koszul duality map is equivalence also apply to it. In short, one requires certain (co)connectivity bounds on the objects A and M.

4.4. **Cartesian-presentable.** Here we aim toward an answer to Question 4.1.1 in the case that the underlying ∞-category of $\mathcal{V}$ is Cartesian-presentable and the symmetric monoidal structure of $\mathcal{V}$ is Cartesian. This generalization of the notion of an ∞-topos is crafted specifically toward this need.

Definition 4.4.1 (Definition 6.2.3.1 of [Lu2]). An ∞-category $\mathcal{S}$ is Cartesian-presentable if the following conditions are satisfied.

- $\mathcal{S}$ is presentable: $\mathcal{S}$ admits small colimits and every object of $\mathcal{S}$ is a κ-filtered colimit of κ-compact objects for some regular cardinal κ.
- Sifted colimits in $\mathcal{S}$ are universal: for each morphism $f\colon X \to Y$ in $\mathcal{S}$, the limit functor

  $$f^*\colon \mathcal{S}_{/Y} \longrightarrow \mathcal{S}_{/X} ~, \qquad (Z \to Y) \mapsto (X \underset{Y}{\times} Z \to X)$$

  preserves sifted colimits.
- Groupoid objects in $\mathcal{S}$ are effective. That is, let $\mathcal{G}\colon \mathbf{\Delta}^{\mathsf{op}} \to \mathcal{S}$ be a functor such that for each $L \in \mathbf{\Delta}^{\mathsf{op}}$, and each pair of subsets S, $T \subset L$ whose union is all of L and whose intersection is a single element $\{l\} \subset L$, then the diagram in $\mathcal{S}$

  $$\begin{tikzcd} \mathcal{G}(L) \arrow[r] \arrow[d] & \mathcal{G}(T) \arrow[d] \\ \mathcal{G}(S) \arrow[r] & \mathcal{G}(\{l\}) \end{tikzcd}$$

is a pullback. Then the canonical diagram in $\mathcal{S}$

$$\begin{tikzcd}
\mathcal{G}(\{0<1\}) \arrow[r] \arrow[d] & \mathcal{G}(\{1\}) \arrow[d] \\
\mathcal{G}(\{0\}) \arrow[r] & \underset{\mathbf{\Delta}^{\mathsf{op}}}{\mathsf{colim}}\ \mathcal{G}
\end{tikzcd}$$

is a pullback.

Example 4.4.2. Here some examples of Cartesian-presentable ∞-categories.

- A presentable stable ∞-category $\mathcal{S}$ is a Cartesian-presentable ∞-category. In particular, for k a ring spectrum, $\mathsf{Mod}_k(\mathsf{Spectra})$ is a Cartesian-presentable ∞-category.
- An ∞-topos $\mathcal{E}$ is a Cartesian-presentable ∞-category. In particular, for any small ∞-category $\mathcal{C}$, the ∞-category $\mathsf{PShv}(\mathcal{C})$ is Cartesian-presentable.

Convention 4.4.3. We will adopt the convention to regard a Cartesian-presentable ∞-category $\mathcal{S}$ as a symmetric monoidal ∞-category whose underlying ∞-category is $\mathcal{S}$ and whose symmetric monoidal structure is product.

Remark 4.4.4. In an extreme sense, not every symmetric monoidal ∞-category is an instance of Convention 4.4.3 – we view Cartesian-presentable ∞-categories as degenerate examples of symmetric monoidal ∞-categories because comultiplication is unique and commutative. For instance, for k a ring spectrum, tensor product over k does *not*, in general, distribute over totalizations. This case of considerable interest is the subject of the sequal [AF2].

Observation 4.4.5. Let $B \to \mathsf{BTop}(n)$ be a map of pointed spaces. Let $\mathcal{S}$ be a Cartesian-presentable ∞-category. Then, as a symmetric monoidal ∞-category, it is $\otimes$-sifted cocomplete and $\otimes$-cosifted complete. Theorem 3.2.3 ensures the existence of the two adjunctions

$$\begin{tikzcd}
\int_-\colon \mathsf{Alg}^{\mathsf{aug}}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V}) \arrow[r, bend left] & \mathsf{Fun}^{\otimes}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}_n^{\mathsf{fin},\mathsf{B}}, \mathcal{V}\big) \arrow[l] \arrow[r] & \mathsf{cAlg}^{\mathsf{aug}}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{V})\colon \int^- \arrow[l, bend left]
\end{tikzcd} .$$

4.5. **Koszul dualities in Cartesian-presentable ∞-categories.** We examine Koszul dualities in Cartesian-presentable ∞-categories.

For $X \in \mathcal{C}$ an object of an ∞-category, we will use the notation $\mathcal{C}_X$ for retractive objects over X. Consider a presentable ∞-category $\mathcal{C}$. For $X \in \mathcal{C}$ an object, we will use the notation $\mathcal{C}_X$ for the ∞-category of retractive objects over X. Consider a pointed space B. Coend over B gives the functor

$$-\bigotimes_B -\colon \mathsf{Spaces}_B \times \mathsf{Mod}_B(\mathcal{C})_* \longrightarrow \mathcal{C}$$

and end over B gives the functor

$$\mathsf{Map}_B\big(-,-\big)\colon \mathsf{Spaces}_B^{\mathsf{op}} \times \mathsf{Mod}_B(\mathcal{C})_* \longrightarrow \mathcal{C} .$$

For the next definition, we use the notation $\mathcal{D}\mathsf{isk}_{n,+}^{\leq 1} \subset \mathcal{Z}\mathcal{M}\mathsf{fld}_n$ for the full ∞-subcategory consisting of the two objects $*$ and $\mathbb{R}^n_+$.

Definition 4.5.1. The *frame bundle* functor is the restricted Yoneda functor

$$\mathsf{Fr}_-\colon \mathcal{Z}\mathcal{M}\mathsf{fld}_n \longrightarrow \mathsf{PShv}\big(\mathcal{Z}\mathcal{M}\mathsf{fld}_n\big)^{*/} \longrightarrow \mathsf{PShv}\big(\mathcal{D}\mathsf{isk}_{n,+}^{\leq 1}\big)^{*/} \simeq \mathsf{Mod}_{\mathsf{Top}(n)}(\mathsf{Spaces})^{*/} \simeq \mathsf{Spaces}_{\mathsf{BTop}(n)}$$

– the latter equivalence is facilitated through Corollary 2.5.5. For $B \to \mathsf{BTop}(n)$ a map of pointed spaces, there is a canonical lift

$$\mathsf{Fr}_-\colon \mathcal{Z}\mathcal{M}\mathsf{fld}_n^B \longrightarrow \mathsf{Spaces}_B .$$

Remark 4.5.2. In the case that $M_* = M_+$ is an n-manifold with disjoint basepoint, then Fr_{M_+} classifies the tangent microbundle of M (with a disjoint section added).

Proposition 4.5.3. *Let $\mathcal{S}$ be a Cartesian-presentable ∞-category. Let $B \to \mathsf{BTop}(n)$ be a map of pointed spaces. The following statements are true.*

1. *Denote by $B := B \otimes * \in \mathcal{S}$ the colimit of the constant functor $B \xrightarrow{*} \mathcal{S}$ at a terminal object. Consider the ∞-category $\mathcal{S}_B$ of retractive objects in $\mathcal{S}$ over B. There are canonical equivalences of ∞-categories*
$$\mathsf{Alg}^{\mathsf{aug}}_{\mathcal{D}\mathsf{isk}^B_n}(\mathcal{S}) \xrightarrow{\simeq} \mathsf{Alg}_{\mathcal{D}\mathsf{isk}^B_n}(\mathcal{S}) \qquad \text{and} \qquad \mathsf{cAlg}^{\mathsf{aug}}_{\mathcal{D}\mathsf{isk}^B_n}(\mathcal{S}) \xrightarrow{\simeq} \mathcal{S}_B ~.$$
2. *Let $R \leftrightarrows B$ be a retractive object in $\mathcal{S}$ over B, which we canonically extend as an augmented $\mathcal{D}\mathsf{isk}^B_n$-coalgebra C^R in $\mathcal{S}$. Let M_* be a conically finite zero-pointed B-manifold. There is a canonical identification*
$$\int^{M_*} C^R \xrightarrow{\simeq} \mathsf{Map}_B\big(\mathsf{Fr}_{M_*}, R\big) ~,$$
to the space of morphisms over and under B.
3. *Through the above equivalences, the Bar-coBar adjunction becomes the adjunction*
$$\mathsf{Bar}^n \colon \mathsf{Alg}_{\mathcal{D}\mathsf{isk}^B_n}(\mathcal{S}) \rightleftarrows \mathcal{S}_B \colon \Omega^n ~,$$
in which the values of the left adjoint are given as the n-fold Bar construction, and the values of the right adjoint are given as n-fold loops over B.
4. *An augmented $\mathcal{D}\mathsf{isk}^B_n$-algebra A in $\mathcal{S}$ belongs to a Koszul duality if and only if it is grouplike. This is to say that there is an open embedding $e \colon \mathbb{R}^n \sqcup \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ for which the two squares in the diagram in $\mathcal{S}$*
$$\begin{tikzcd} A(\mathbb{R}^n) \arrow[d] & A(\mathbb{R}^n) \times A(\mathbb{R}^n) \arrow[l, "A(i_0)"'] \arrow[r, "A(i_1)"] \arrow[d, "A(e)"] & A(\mathbb{R}^n) \arrow[d] \\ * & A(\mathbb{R}^n) \arrow[l] \arrow[r] & * \end{tikzcd}$$
are pullback – here we are identifying A through the equivalence of the first point of the lemma, and we are denoting the two inclusions $i_{0,1} \colon \mathbb{R}^n \hookrightarrow \mathbb{R}^n \sqcup \mathbb{R}^n$.
5. *An augmented $\mathcal{D}\mathsf{isk}^B_n$-coalgebra C in $\mathcal{S}$ belongs to a Koszul duality if and only if the morphism $C \to B$ in $\mathcal{S}$ is n-connective – here we are identifying C through the equivalence of the first point of the lemma.*

Proof. We prove statement (1). Because the symmetric monoidal structure of $\mathcal{S}$ is Cartesian, the symmetric monoidal unit is terminal, proving the first part of statement (1), and the restriction $\mathsf{cAlg}^{\mathsf{aug}}_n(\mathcal{S}) \xrightarrow{\simeq} \mathsf{Mod}_B(\mathcal{S}_*)$ is an equivalence. We next explain the following string of equivalences of ∞-categories
$$\mathsf{Mod}_B(\mathcal{S}) \xleftarrow{\simeq} \mathsf{Mod}_B(\mathcal{S}_{/*}) \xleftarrow{\simeq} \mathsf{Fun}^{\mathsf{colim}}\big(B^{\triangleright}, \mathcal{S}_{/*}\big) \xrightarrow{\simeq} \mathcal{S}_{/B} ~.$$
The first arrow is induced by the projection $\mathcal{S}_{/*} \to \mathcal{S}$, which is an equivalence because $*$ is terminal. The domain of the second arrow is the category of colimit diagrams; this second arrow is an equivalence by the definition of colimits, using that $\mathcal{S}$ admits B-shaped colimits. The third arrow is evaluation at the cone-point; this arrow is an equivalence precisely because $\mathcal{S}$ is a Cartesian-presentable ∞-category. Comparing ∞-categories of pointed objects of this string of equivalences, one arrives at an equivalence of ∞-categories $\mathsf{Mod}_B(\mathcal{S}_*) \simeq \mathcal{S}_B$.

Statement (2) follows from statement (1). The statement concerning the right adjoint of statement (3) follows from statement (2). The statement concerning the left adjoint of statement (3) follows form Theorem 4.3.1. Statement (5) follows from statement (3).

We now concentrate on statement (4). We make use of statement (3). Consider the unit morphism
$$A \to \Omega^n \mathsf{Bar}^n A ~. \tag{28}$$

There is a functor $B \otimes \mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \to \mathcal{D}\mathsf{isk}_n^B$ from the tensor among symmetric monoidal ∞-categories with the space B. There results the functor

$$\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^B}(\mathcal{S}) \xrightarrow{\simeq} \mathsf{Map}_{\mathsf{BTop}(n)}\big(B, \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}}(\mathcal{S})\big)$$

which is an equivalence. And so, this unit morphism is an equivalence if and only if its restriction $A_{|\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}} \to \Omega^n \mathsf{Bar}^n A_{|\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}}$ is an equivalence for each point $* \xrightarrow{b} B$. So it is enough to only consider the framed case: $\mathcal{D}\mathsf{isk}_n^{\mathsf{fr}}$.

If this unit morphism is an equivalence, then clearly A is grouplike. For the converse, we proceed by induction. In the case that $n = 1$, A grouplike implies (28) is an equivalence because groupoids are effective in $\mathcal{S}$. Suppose $n > 1$. Consider the symmetric monoidal restriction

$$A_{|}\colon\ \mathcal{D}\mathsf{isk}_1^{\mathsf{fr}} \times \mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{fr}} \longrightarrow \mathcal{D}\mathsf{isk}_n^{\mathsf{fr}} \xrightarrow{A} \mathcal{S}\ .$$

This functor is adjoint to a functor $A^\dagger\colon \mathcal{D}\mathsf{isk}_1^{\mathsf{fr}} \to \mathsf{Alg}_{\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{fr}}}(\mathcal{S})$. By inspection, A grouplike implies $A^\dagger$ is a grouplike $\mathcal{D}\mathsf{isk}_1^{\mathsf{fr}}$-algebra in grouplike $\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{fr}}$-algebras in $\mathcal{S}$. By induction, we conclude that

$$A^\dagger \xrightarrow{\simeq} \Omega\, \mathsf{Bar}\big(A_{|\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{fr}}}\big) \xrightarrow{\simeq} \Omega\, \mathsf{Bar}\big(\Omega^{n-1}\, \mathsf{Bar}^{n-1} A_{|\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{fr}}}\big)$$

where the outer $\Omega\,\mathsf{Bar}$ is understood to be taking place in the ∞-category $\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{fr}}}(\mathcal{S})$. Now, because the symmetric monoidal structure of $\mathcal{S}$ distributes over sifted colimits, then $\Omega\,\mathsf{Bar}\big(A_{|\mathcal{D}\mathsf{isk}_{n-1}^{\mathsf{fr}}}\big) \xrightarrow{\simeq} \Omega^n\, \mathsf{Bar}^n A$, as desired.

□

Proposition 4.5.4 (Linear)**.** *Let $\mathcal{S}$ be a stable presentable ∞-category. The following statements are true.*

(1) *The projections to underlying B-modules,*

$$\mathsf{Alg}_n^{\mathsf{aug}}(\mathcal{S}) \xrightarrow{\simeq} \mathsf{Mod}_B(\mathcal{S}) \xleftarrow{\simeq} \mathsf{cAlg}_n^{\mathsf{aug}}(\mathcal{S})\ ,$$

are equivalences of ∞-categories from augmented n-disk (co)algebras in $\mathcal{S}$.

(2) *Let E and F be B-modules in $\mathcal{S}$, which we respectively uniquely extend as an augmented $\mathcal{D}\mathsf{isk}_n^B$-algebra A_E in $\mathcal{S}$ and an augmented $\mathcal{D}\mathsf{isk}_n^B$-coalgebra C^F in $\mathcal{S}$. The canonical morphisms in $\mathcal{S}$*

$$\mathsf{Fr}_{M_*} \bigotimes_B E \xrightarrow{\simeq} \int_{M_*} A_E$$

and

$$\int^{M_*} C^F \xrightarrow{\simeq} \mathsf{Map}_B\big(\mathsf{Fr}_{M_*^\neg}, F\big)$$

are equivalences.

(3) *Through the above equivalences, the Bar-coBar adjunction becomes the adjunction*

$$(\mathbb{R}^n)^+ \otimes (-)\colon\ \mathsf{Mod}_B(\mathcal{S}) \rightleftarrows \mathsf{Mod}_B(\mathcal{S})\colon (-)^{(\mathbb{R}^n)^+}\ ,$$

the left adjoint with the diagonal B-module structure, and the right adjoint with the conjugation B-module structure. Because $\mathcal{S}$ is stable, both of these functors are equialences.

(4) *Every augmented $\mathcal{D}\mathsf{isk}_n^B$-algebra in $\mathcal{S}$ belongs to a Koszul duality.*

(5) *Every augmented $\mathcal{D}\mathsf{isk}_n^B$-coalgebra in $\mathcal{S}$ belongs to a Koszul duality.*

Proof. Each statement implies the next. The first two follow from Proposition 4.5.3 because $\mathcal{S}$ is a Cartesian-presentable ∞-category.

□

4.6. **Interval duality.** In this subsection we examine the Poincaré/Koszul duality map (24) in the special case of a closed interval. Following through with Remark 4.2.3, examining the values of the Poincaré/Koszul duality maps on basics gives rise to Koszul duality, definitionally.

In [AFT1] we described the category of basics $\mathsf{D}_1^{\partial,\mathsf{fr}}$ for whose manifolds are oriented 1-manifolds with boundary. In [AFT2] we proved that the symmetric monoidal ∞-category $\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{fr}}$ corepresents the data of an associative algebra A together with a unital right A-module P and a unital left A-module Q. Therefore the symmetric monoidal ∞-category $\mathcal{D}\mathsf{isk}_{1,+}^{\partial,\mathsf{fr}}$ corepresents the likewise augmented data. Likewise, $\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{fr},+}$ corepresents the data of a coaugmented coassociative algebra C together with a counital and coaugmented right C-comodule R and a counital and coaugmented left C-comodule S.

There is this direct result.

Lemma 4.6.1. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category that is $\otimes$-sifted cocomplete and $\otimes$-cosifted complete. Let*

$$\mathcal{A}\colon\ \mathcal{Z}\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{fr}} \longrightarrow \mathcal{V}$$

be a symmetric monoidal functor. Then $\mathcal{A}$ is a interval *Koszul duality if and only if the following canonical comparison maps are equivalences in $\mathcal{C}$, as indicated:*

$$A \xrightarrow{\simeq} \mathbb{1}\overset{C}{\bigotimes}\mathbb{1} \qquad \textit{and} \qquad \mathbb{1}\underset{A}{\bigotimes}\mathbb{1} \xrightarrow{\simeq} C\ ,$$

$$P \xrightarrow{\simeq} R\overset{C}{\bigotimes}\mathbb{1} \qquad \textit{and} \qquad P\underset{A}{\bigotimes}\mathbb{1} \xrightarrow{\simeq} R\ ,$$

$$Q \xrightarrow{\simeq} \mathbb{1}\overset{C}{\bigotimes}S \qquad \textit{and} \qquad \mathbb{1}\underset{A}{\bigotimes}Q \xrightarrow{\simeq} S\ .$$

Lemma 4.6.2. *Let $\mathcal{S}$ be a Cartesian-presentable ∞-category. Let $\mathcal{A}\colon\ \mathcal{Z}\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{fr}} \longrightarrow \mathcal{V}$ be a symmeric monoidal functor. Use the notation (P,A,Q) for the restriction $\mathcal{A}_{|\mathcal{D}\mathsf{isk}_{1,+}^{\partial,\mathsf{fr}}}$, and (R,C,S) for the restriction $\mathcal{A}_{|\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{fr},+}}$. Suppose $\mathcal{A}$ is an interval Koszul duality. Then A is grouplike, and the Poincaré/Koszul duality map*

$$\int_{[-1,1]_+}(P,A,Q) \quad \xrightarrow{\simeq} \quad \int^{[-1,1]^+}(R,C,S)$$

is an equivalence.

Proof. We must verify that the canonical morphism in $\mathcal{S}$

$$\mathsf{Bar}(P,A,Q) \ \simeq\ \int_{\overline{\mathbb{R}}_+}(P,A,Q) \quad\longrightarrow\quad \int^{\overline{\mathbb{R}}^+}(R,C,S) \ \simeq\ R\underset{C}{\times}S\ ,$$

from the two-sided bar construction to the pullback, is an equivalence. The datum (P,A,Q) determines the evident diagram $\mathcal{S}$

(29)
$$\begin{tikzcd}
\mathsf{Bar}(P,A,Q) \arrow[r] \arrow[d] & \mathsf{Bar}(\ast,A,Q) \arrow[d] \\
\mathsf{Bar}(P,A,\ast) \arrow[r] & \mathsf{Bar}(\ast,A,\ast)\ .
\end{tikzcd}$$

Inspecting the expressions displayed in Lemma 4.6.1, because $\mathcal{A}$ is taken to be an interval Koszul duality, the problem is to verify that this square (29) is a pullback.

The diagram is the geometric realization of the simplicial square-diagram in $\mathcal{S}$ whose value on $[p]$ is the square of projections

$$\begin{tikzcd} P \times A^{\times p} \times Q \arrow[r] \arrow[d] & A^{\times p} \times Q \arrow[d] \\ P \times A^{\times p} \arrow[r] & A^{\times p} , \end{tikzcd} \tag{30}$$

which is certainly pullback. Because $\mathcal{S}$ is an sifted-topos, then the square (29) is a pullback provided each arrow in (30) implements a *Cartesian transformation* among simplicial objects. We disect this last criterion in the case of the right vertical arrow, the cases of the other arrows are nearly identical. For the right vertical arrow in (30) to be a Cartesian transformation means the square

$$\begin{tikzcd} A^{\times q} \times Q \arrow[r] \arrow[d] & A^{\times p} \times Q \arrow[d] \\ A^{\times q} \arrow[r] & A^{\times p} \end{tikzcd}$$

pullback for every simplicial morphism $\rho\colon [p] \to [q]$. This is always the case for ρ degenerate. This is the case for ρ an arbitrary face map if and only if A acts invertibly on Q as well as on itself by both left and right translation. This is the case if and only if A is grouplike, which is implied by $\mathcal{A}$ being an interval Koszul duality. □

4.7. **Atiyah duality and non-abelian Poincaré duality.** We prove that the Poincaré/Koszul duality map is an equivalence for coefficients in a Koszul duality in a Cartesian-presentable ∞-category. This immediately implies the classical Atyiah duality, as well as the non-abelian Poincaré duality of [Lu2].

Theorem 4.7.1 (Non-abelian Poincaré duality)**.** *Let $B \to \mathsf{BTop}(n)$ be a map of pointed spaces, and let $\mathcal{A}\colon \mathcal{ZD}\mathsf{isk}_n^B \longrightarrow \mathcal{S}$ be a Koszul duality in a Cartesian-presentable ∞-category $\mathcal{S}$. Then for each conically finite zero-pointed B-manifold M_*, the Poincaré/Koszul duality map*

$$\int_{M_*} \mathcal{A}_+ \longrightarrow \int^{M_*^\neg} \mathcal{A}^+$$

is an equivalence.

Proof. Consider the collection of zero-pointed B-manifolds M_* for which the Poincaré/Koszul duality map is an equivalence. Because $\mathcal{A}$ is a Koszul duality, this collection contains the objects of $\mathcal{ZD}\mathsf{isk}_n^B$. A main result of [AFT1] states that the collection of stratified spaces that are the interiors of compact stratified spaces with corners is generated from basics by collar-gluings. From the very definition of *conically finite* (Definition 1.8.1), once we show this collection is closed under collar-gluings, the argument is complete. After Observation 4.4.5, and Theorem 3.4.3, it is enough to prove that

$$M_* \ \mapsto \ \int_{M_*} \mathcal{A}_+$$

satisfies $\otimes$-coexcision.

Choose a conical smoothing of a conically finite zero-pointed B-manifold M_*. Choose a weakly constructible map $M_* \xrightarrow{f} [\text{-}1, 1]$ so that f is constant in a neighborhood of $* \in M_*$. Because of the hypotheses on $\mathcal{V}$, Corollary 3.4.5 validates $\otimes$-excision for factorization homology and $\otimes$-coexcision for factorization cohomology. With this, the naturality of the Poincaré/Koszul duality map gives the diagram in $\mathcal{V}$:

$$\begin{tikzcd}
\int_{M_*} \mathcal{A}_+ \arrow[r, "(23)"] & \int^{M_*^{\neg}} \mathcal{A}^+ \arrow[d, "\simeq"] \\
\int_{f^{-1}[-1,1)_{M_*}} \mathcal{A}_+ \underset{\int_{f^{-1}(-1,1)_{M_*}} \mathcal{A}_+}{\bigotimes} \int_{f^{-1}(-1,1]_{M_*}} \mathcal{A}_+ \arrow[u, "\simeq"'] \arrow[r, "(23)"] & \int^{f^{-1}[-1,1)_{M_*^{\neg}}} \mathcal{A}^+ \overset{\int_{f^{-1}(-1,1)_{M_*^{\neg}}} \mathcal{A}^+}{\bigotimes} \int^{f^{-1}(-1,1]_{M_*^{\neg}}} \mathcal{A}^+
\end{tikzcd}$$

(here we have used the super- and sub-script notation for the zero-pointed manifolds of Construction 1.4.1, according to Convention 1.6.8). So we must show the bottom horizontal map is an equivalence. For this, we need only explain that we are in the situation of Lemma 4.6.2.

In [AFT2] we verified a pushforward formula for factorization homology, and a pullback formula for factorization cohomology follows dually. In [AFT2] we also showed that the ∞-category of $[-1,1]$-algebras is canonically identified as that of $\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{fr}}$-algebras, and so likewise for their augmented versions, as well as their dual versions. Through these means, the pushforward $f_*\mathcal{A}$ is canonically identified as a $\mathcal{Z}\mathcal{D}\mathsf{isk}_1^{\partial,\mathsf{fr}}$-algebra in $\mathcal{S}$. After Lemma 4.6.2, we need only explain that $f_*\mathcal{A}$ is an interval Koszul duality. Inspecting Lemma 4.6.1, this amounts to verifying that the canonical arrow

$$\mathbb{1} \underset{\int_{f^{-1}(-1,1)_{M_*}} \mathcal{A}_+}{\bigotimes} \int_{f^{-1}(-1,1]_{M_*}} \mathcal{A}_+ \longrightarrow \int^{f^{-1}(-1,1]_{M_*^{\neg}}} \mathcal{A}^+ , \tag{31}$$

is an equivalence, and likewise for the five other terms. Through $\otimes$-excision, we recognize the lefthand side of this expression (31) as

$$\mathbb{1} \underset{\int_{f^{-1}(-1,1)_{M_*}} \mathcal{A}_+}{\bigotimes} \int_{f^{-1}(-1,1]_{M_*}} \mathcal{A}_+ \simeq \int_{f^{-1}(-1,1]^{M_*}} \mathcal{A}_+ .$$

Through the negation relations of Corollary 1.6.10, we recognize $\big(f^{-1}(-1,1]^{M_*}\big)^{\neg} = f^{-1}(-1,1]_{M_*^{\neg}}$. And so, should the Poincaré/Koszul duality map be an equivalence for each of $f^{-1}(-1,1]_{M_*}$ and $f^{-1}[-1,1)_{M_*}$ and $f^{-1}(-1,1)_{M_*}$, then the Poincaré/Koszul duality map is an equivalence for M_*.

□

Corollary 4.7.2 (Linear Poincaré duality)**.** *Let $\mathcal{S}$ be a stable presentable ∞-category. Let $B \to \mathsf{BTop}(n)$ be a map of pointed spaces. Let $E, F\colon B \to \mathcal{S}$ be a pair of functors. Suppose there is an equivalence of functors $B \to \mathcal{S}$:*

$$(\mathbb{R}^n)^+ \otimes E \simeq F \qquad \text{or equivalently} \qquad E \simeq F^{(\mathbb{R}^n)^+} .$$

Let M_* be a conically finite zero-pointed B-manifold. There is an equivalence in $\mathcal{S}$

$$\mathsf{Fr}_{M_*} \underset{B}{\bigotimes} E \simeq \mathsf{Map}_B\big(\mathsf{Fr}_{M_*^{\neg}}, F\big) .$$

Corollary 4.7.3 (Atiyah duality)**.** *Let M_* be a conically finite zero-pointed n-manifold. There is an equivalence of spectra*

$$(M_*)^{-\tau_M} \simeq \mathbb{S}^{M_*^{\neg}}$$

between the Thom spectrum of the normal bundle of M_* and Spanier-Whithead dual of the based space $M_*^{\neg}$.

Proof. Apply Corollary 4.7.2 to the case that $\mathcal{S}$ is the ∞-category of spectra, and $F = \mathbb{S}$ is the sphere spectrum. Observe the equivalence $(M_*)^{\tau_M} \simeq \mathsf{Fr}_{M_*} \underset{\mathsf{Top}(n)}{\bigotimes} (\mathbb{R}^n)^+$, which gives as a consequence $(M_*)^{-\tau_M} \simeq \mathsf{Fr}_{M_*} \underset{\mathsf{Top}(n)}{\bigotimes} \mathbb{S}^{(\mathbb{R}^n)^+}$.

□

Here is an immediate corollary of Theorem 4.7.1, which is a gentle generalization of the nonabelian Poincaré duality of Lurie (see Theorem 5.5.6.6 of [Lu2]).

Corollary 4.7.4 (Poincaré/Koszul duality for ∞-topoi)**.** *Let $B \to \mathsf{BTop}(n)$ be a map of pointed spaces. Let $\mathcal{E}$ be an ∞-topos. Let A be a grouplike $\mathcal{D}\mathsf{isk}_n^B$-algebra in $\mathcal{E}$. Let $C \to B$ be an n-connective morphism in $\mathcal{E}$, equipped with a section. Let M_* be a conically finite zero-pointed B-manifold. There are canonical equivalences in $\mathcal{E}$*

$$\int_{M_*} A \xrightarrow{\simeq} \mathsf{Map}_B\big(\mathsf{Fr}_{M_*^\neg}, \mathsf{Bar}^n A\big) \qquad \text{and} \qquad \int_{M_*} \Omega^n C \xrightarrow{\simeq} \mathsf{Map}_B\big(\mathsf{Fr}_{M_*^\neg}, C\big)$$

to the cotensors over and under $B \in \mathcal{E}$. A case of particular interest is for $\mathcal{S} = \mathsf{Spaces}$ and $C = B \times Z \rightleftarrows B$ with Z a based space, for in this case the relative cotensor is simply a based mapping space $\mathsf{Map}_*\big(M_*^\neg, Z\big)$.

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