Stable topology of moduli spaces of holomorphic curves in $\mathbb{C}P^n$

David Ayala*

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Abstract

This paper is centered around a homotopy theoretic approximation to $\mathcal{M}_g^d(\mathbb{C}P^n)$, the moduli space of degree d holomorphic maps from genus g Riemann surfaces into $\mathbb{C}P^n$. There results a calculation of the integral cohomology ring $H^*(\mathcal{M}_g^d(\mathbb{C}P^n))$ for $* << g << d$. The arguments follow those from a paper of G. Segal ([Seg79]) on the topology of the space of rational functions.

Conventions

Throughout this paper, all maps between topological spaces will be taken to be continuous and all diagrams commutative unless otherwise stated. Sets of continuous maps are topologized with the compact open topology. The symbol $H_q(-)$ will be used to denote the q^{th} integral homology group of $-$.

For G a topological group acting continuously on a space X, the *homotopy orbit space*, denoted $X//G$, is the standard orbit space $(EG \times X)/G$ where EG is a chosen contractible space with a free action of G and G acts on the product by the diagonal action. Observe that if the action of G on X is free then the projection $X//G \to X/G$ is a homotopy equivalence.

Introduction

0.1 Background and statement of main result

A Riemann surface is a pair (F_g, J) where $F_g \subset \mathbb{R}^\infty$ is a smooth oriented surface of genus g embedded in Euclidean space, and J is a complex structure on F_g which agrees with the given orientation on F_g. Write $\mathcal{J}_g$ for the space of such complex structures on F_g; a description of the topology of $\mathcal{J}_g$ is postponed to §1.1 where it is also shown to be contractible.

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For (F_g, J) a Riemann surface, write $Map^d(F_g, \mathbb{C}P^n)$ for the space of continuous maps from F_g to $\mathbb{C}P^n$ having homological degree $d \in H_2(\mathbb{C}P^n) \cong \mathbb{Z}$. Write $Hol^d((F_g, J), \mathbb{C}P^n)$ for the subspace of $Map^d(F_g, \mathbb{C}P^n)$ consisting of holomorphic maps from (F_g, J) to $\mathbb{C}P^n$.

Theorem 0.1.1 ([Seg79])**.** *For $g > 0$, the inclusion $Hol^d((F_g, J), \mathbb{C}P^n) \hookrightarrow Map^d(F_g, \mathbb{C}P^n)$ induces an isomorphism in $H_q(-;\mathbb{Z})$ for $q < (d-2g)(2n-1)$.*

This paper is devoted to proving such a theorem as the complex structure J is allowed to vary. This will amount to a statement about the moduli space of Riemann surfaces which will be defined presently.

For a fixed surface F_g, consider the set of pairs

$$\mathcal{J}_g^d(\mathbb{C}P^n) := \{(J, h) \mid J \in \mathcal{J}_g \text{ and } h \in Hol^d((F_g, J), \mathbb{C}P^n)\}.$$

Topologize $\mathcal{J}_g^d(\mathbb{C}P^n) \subset \mathcal{J}_g \times Map^d(F_g, \mathbb{C}P^n)$ as a subspace. Let $Diff_g^+$ be the topological group of orientation preserving diffeomorphisms of F_g endowed with the Whitney C^∞ topology. There is a continuous action of $Diff_g^+$ on $\mathcal{J}_g^d(\mathbb{C}P^n)$ given by pulling back the complex structure and precomposing maps. Define the (coarse) *moduli space of degree d genus g holomorphic curves in* $\mathbb{C}P^n$ as the resulting quotient

$$\mathcal{M}_g^d(\mathbb{C}P^n)^c := \mathcal{J}_g^d(\mathbb{C}P^n)/Diff_g^+.$$

Similarly, consider the orbit space

$$\mathcal{MT}_g^d(\mathbb{C}P^n)^c := (\mathcal{J}_g \times Map^d(F_g, \mathbb{C}P^n))/Diff_g^+;$$

referred to as the *(coarse) topological moduli space of degree d genus g curves in* $\mathbb{C}P^n$.

More well-behaved are the homotopy orbit spaces

$$\mathcal{M}_g^d(\mathbb{C}P^n) := \mathcal{J}_g^d(\mathbb{C}P^n)//Diff_g^+$$

and

$$\mathcal{MT}_g^d(\mathbb{C}P^n) := (\mathcal{J}_g \times Map^d(F_g, \mathbb{C}P^n))//Diff_g^+$$

which will be referred to as the *moduli space of degree d genus g curves in* $\mathbb{C}P^n$, and its topological counterpart, respectively. Note that the contracitbility of $\mathcal{J}_g$ implies the natural projection $\mathcal{MT}_g^d(\mathbb{C}P^n) \to Map^d(F_g, \mathbb{C}P^n)//Diff_g^+$ is a homotopy equivalence.

Theorem 0.1.2 (Main Theorem)**.** *The map*

$$\mathcal{M}_g^d(\mathbb{C}P^n) \to \mathcal{MT}_g^d(\mathbb{C}P^n)$$

induced by the natural inclusion induces an isomorphism in $H_q(-)$ for $q < (d-2g)(2n-1)$.

Remark 0.1.3. Theorem 0.1.2 can be viewed as a statement about families. Indeed, an isomorphism in H_* implies an isomorphism in oriented bordism MSO_*. As so Theorem 0.1.2 says that any q-dimensional family of continuous curves in $\mathbb{C}P^n$ is cobordant to a family of holomorphic curves provided $q < (d-2g)(2n-1)$.

Recall the action of $Diff_g^+$ on $\mathcal{J}_g$. Via Teichmuller theory, the isotropy subgroups $(Diff_g^+)_J$ are finite for each $J \in \mathcal{J}_g$. Through a standard spectral sequence argument and the contractibility of $\mathcal{J}_g$, the product $EDiff_g^+ \times \mathcal{J}_g$, with the diagonal action of $Diff_g^+$, becomes a *rational* model for $EDiff_g^+$. It follows that the projection maps

$$Map^d(F_g, \mathbb{C}P^n)//Diff_g^+ \simeq \mathcal{MT}_g^d(\mathbb{C}P^n) \to \mathcal{MT}_g^d(\mathbb{C}P^n)^c$$

and

$$\mathcal{M}_g^d(\mathbb{C}P^n) \to \mathcal{M}_g^d(\mathbb{C}P^n)^c$$

induce isomorphisms on rational singular homology $H_*(-;\mathbb{Q})$. There is an immediate corollary.

Corollary 0.1.4. *Induced by the obvious inclusion, the map*

$$\mathcal{M}_g^d(\mathbb{C}P^n)^c \to \mathcal{MT}_g^d(\mathbb{C}P^n)^c$$

induces an isomorphism in $H_q(-;\mathbb{Q})$ for $q < (d-2g)(2n-1)$.

Remark 0.1.5. As described in §1.1, the space $\mathcal{J}_g$ of complex structures on F_g is canonically homeomorphic to the space of almost-complex structures on F_g. In this way, the construction of $\mathcal{MT}_g^d(\mathbb{C}P^n)$ can be regarded as homotopy theoretic.

Corollary 0.1.6. *Induced by a zig-zag of maps of spaces is an isomorphism*

$$H_q(\mathcal{M}_g^d(\mathbb{C}P^n)) \cong H_q(Map^d(F_g, \mathbb{C}P^n)//Diff_g^+)$$

for $q < (d-2g)(2n-1)$.

There is the following useful group completion result due to Cohen and Madsen.

Theorem 0.1.7 ([CM06])**.** *For X simply connected and for h_* any connective homology theory, there is a map*

$$Map(F_g, X)//Diff^+(F_g) \to \Omega^\infty(\mathbb{C}P_{-1}^\infty \wedge X_+)$$

which induces an isomorphism in h_q for $q > (g-5)/2$.

Corollary 0.1.8. *Induced by a zig-zag of maps of spaces is an isomorphism*

$$H_q(\mathcal{M}_g^d(\mathbb{C}P^n)) \cong H_q(\Omega^\infty(\mathbb{C}P_{-1}^\infty \wedge \mathbb{C}P^n))$$

for $q < (d-2g)(2n-1)$ and $q < (g-5)/2$.

The rational (co)homology of Ω^∞-spaces being reasonably well-understood, there is the following corollary which requires some notation to state. For V a graded vector space over $\mathbb{Q}$, denote by $A(V)$ the free graded-commutative $\mathbb{Q}$-algebra generated by V. Let $\mathcal{K}$ be the graded vector space over $\mathbb{Q}$ generated by the set $\{k_i\}_{i \geq -1}$ where $|k_i| = 2i$. Let W another graded vector space over $\mathbb{Q}$. Denote by $(\mathcal{K} \otimes W)_+$ the positively graded summand of the vector space $\mathcal{K} \otimes W$. Recall that $H^*(\mathbb{C}P^n) \cong \mathbb{Q}[c]/c^{n+1}$ where $|c| = 2$.

Corollary 0.1.9. *There is an isomorphism of graded rings*

$$H^*(\mathcal{M}_g^d(\mathbb{C}P^n);\mathbb{Q}) \cong A((\mathcal{K} \otimes \mathbb{Q}[c])_+).$$

through the range $* < (d-2g)(2n-1)$ and $q < (g-5)/2$.

0.2 Variants of the main theorem

There are two easy variations of the main theorem. See §6 for precise definitions.

Write $\mathcal{M}^d_{g,k}(\mathbb{C}P^n)$ for the moduli space of degree d curves in $\mathbb{C}P^n$ with k marked points. Write $\mathcal{MT}^d_{g,k}(\mathbb{C}P^n)$ for the topological counterpart.

Theorem 0.2.1. *The standard map*

$$\mathcal{M}^d_{g,k}(\mathbb{C}P^n) \to \mathcal{MT}^d_{g,k}(\mathbb{C}P^n)$$

induces an isomorphism in $H_q(-)$ for $q < (d-2g)(2n-1)$.

Fix a finite collection of marked points $p_1, \dots, p_k \in F_g$ and an equivalence relation $\sim$ on $\{p_i\}$. Denote this data by $F := (F_g, \{p_i\}, \sim)$. Denote the cardinality $n(F) := |\{p_i\}_{/\sim}|$ and the number $g(F)$ such that

$$\chi((F_g)_{/\sim}) = 2 - 2g(F) + n(F).$$

Write $\mathcal{M}^d_{[F]}(\mathbb{C}P^n)$ for the moduli space of degree d genus g holomorphic marked curves $[(J, h)]$ in $\mathbb{C}P^n$ satisfying $h(p_i) = h(p_j)$ when $p_i \sim p_j$. Write $\mathcal{MT}^d_{[F]}(\mathbb{C}P^n)$ for the topological counterpart.

Theorem 0.2.2. *The standard map*

$$\mathcal{M}^d_{[F]}(\mathbb{C}P^n) \to \mathcal{MT}^d_{[F]}(\mathbb{C}P^n)$$

induces an isomorphism in $H_q(-)$ for $q < (d - 2(g(F) - n(F) + 1))(2n-1)$.

1 Preliminaries

1.1 The topology of $\mathcal{J}_g$

Let $\mathrm{Gr}_2^+(\mathbb{R}^N)$ denote the Grassmann manifold of 2-dimensional oriented subspaces of the vector space $\mathbb{R}^N$, and $\mathrm{Gr}_1(\mathbb{C}^N)$ the Grassmann manifold of 1-dimensional complex subspaces of $\mathbb{C}^N$. There is an obvious directed system

$$\cdots \hookrightarrow Gr_2^+(\mathbb{R}^N) \hookrightarrow Gr_2^+(\mathbb{R}^{N+1}) \hookrightarrow \dots .$$

Take the colimit of this directed system as a model for $BSO(2)$. Similarly, take the colimit

$$BU(1) := colim(\cdots \hookrightarrow Gr_1(\mathbb{C}^N) \hookrightarrow Gr_1(\mathbb{C}^{N+1}) \hookrightarrow \dots)$$

as a model for $BU(1)$. Denote by $\theta : BU(1) \to B\mathrm{SO}(2)$ the induced map on colimits given by forgetting complex structure. The embedding $F_g \hookrightarrow \mathbb{R}^\infty$ induces a map $F_g \xrightarrow{\tau_F} BSO(2)$, given by $p \mapsto T_pF_g \subset \mathbb{R}^\infty$, which classifies the tangent bundle τ_{F_g} of F_g.

An *almost-complex structure* on F_g is a map J in the commutative diagram

$$\begin{array}{ccc} & & BU(1) \\ & \overset{J}{\nearrow} & \downarrow{\scriptstyle \theta} \\ F_g & \xrightarrow{\tau_F} & BSO(2). \end{array}$$

Note that the set of almost-complex structures is in bijection with the set of bundle automorphisms $J : \tau_{F_g} \xrightarrow{\cong} \tau_{F_g}$ such that $J^2 = -id$. In particular, the set $\mathcal{J}_g$ of complex structures on F_g (which agree with the given orientation on F_g) inherits the subspace topology $\mathcal{J}_g \subset Map(F_g, BU(1))$.

For dimension reasons, the Nijenhuis tensor on F_g will always vanish (see [NN57]). Therefore every almost-complex structure on an oriented surface is integrable and thus comes from an honest complex structure. In this way, endow the set of complex structures on F_g, $\mathcal{J}_g \subset Map(F_g, BU(1))$ with the subspace topology. This works well provided we choose a good model for $BSO(2)$ and $BU(1)$ which we will do presently.

Note that the homotopy equivalence $\mathrm{GL}_2^+(\mathbb{R}) \simeq \mathrm{SO}(2) = \mathrm{U}(1) \simeq \mathrm{GL}_1(\mathbb{C})$ yeilds the space $\mathcal{J}_g$ as contractible.

1.2 Symmetric products

Let X be a topological space. For each $k \geq 0$ there is a continuous action of the permutation group Σ_k on the product $X^{\times k}$ given by permuting the factors. Define the *d-fold symmetric product of X* to be the quotient topological space

$$Sp_d(X) := X^{\times d}/\Sigma_d.$$

Define the *symmetric product of X* as the disjoint union $Sp(X) := \amalg_{d\in\mathbb{N}} Sp_d(X)$. Given a base point $* \in X$, define $Sp(X, *) := Sp(X)/\sim$ where $\sim$ is the equivalence relation generated by $[(*, x_2, \ldots, x_d)] \sim [(x_2, \ldots, x_d)]$.

1.3 Strategy for proving the main theorem

The argument for the proof of Theorem 0.1.2 will follow that of Segal's ([Seg79]) for when the complex structure is fixed. Details will be supplied for $n = 1$ in which case the target complex manifold is $\mathbb{C}P^1 \approx S^2$. The general situation is not much more difficult as will be outlined later.

The idea is to regard a holomorphic map $(F_g, J) \to \mathbb{C}P^1$ as a rational function on (F_g, J), then to regard a rational function as a pair of divisors (η, ξ) given by its zeros and poles. The degree to which such a pair of divisors is realized in this way from a rational function is described by a theorem of Abel's. Abel's theorem results in a map from the space of divisors on (F_g, J) to the Jacobian of (F_g, J) whose fiber is the space of rational functions on (F_g, J). This Jacobian is then identified with a standard torus which is independent of J. The resulting sequence is a *homology fibration through a range.*

Using 'scanning maps', there is a comparison of this homology fibration to a homotopy theoretic fibration with fiber $Map(F_g, S^2)$. These scanning maps are shown to be equivalences from which it follows that the space of pairs (h, J), where h is a rational function on (F_g, J), is homology equivalent to $Map(F_g, S^2)$ through a range. With sufficient care, one has a family of such constructions parametrized by the space of complex structures. A simple spectral sequence argument is then in place to have a similar comparison on homotopy quotients by $Diff_g^+$ and the result follows.

2 Spaces of divisors

Once and for all, choose a base point $x_0 \in F_g$. Write $Sp_d := Sp_d(F_g \setminus x_0)$ and $Sp := \coprod_d Sp_d$. There is a bijection with the underlying set of Sp_d and the set of positive degree d divisors on $F_g \setminus x_0$. Define

$$Div_d := \{(\eta, \zeta) \mid \eta \cap \zeta = \emptyset\} \subset Sp_d \times Sp_d$$

to be the space of pairs (η, ζ) of *disjoint* positive divisors on $F_g \setminus x_0$ of bi-degree (d, d). The remainder of this section will be devoted to defining a larger space $Div_d \hookrightarrow \widehat{Div}_d$ which is better homotopically behaved.

Let $F_g \supset U_0 \supset U_1 \supset \ldots$ be a nested sequence of closed disk-neighborhoods of x_0 such that $Int(U_d) \supset U_{d+1}$ for each $d \in \mathbb{N}$. For each d consider the subspace $M_d \subset \mathcal{J}_g \times Map(F_g, \mathbb{C}P^1)$ consisting of those pairs (J, h) for which h is J-holomorphic of degree 1 and for u and v the unique zero and pole of h then $(u, v) \in U_d \times U_d \setminus (U_{d+1} \times U_{d+1})$. Once and for all, fix such a nested sequence so that for each d there is a continuous section of the projection $M_d \to \mathcal{J}_g$; for each d, fix such a continuous section $\sigma_d : \mathcal{J}_g \to M_d$ and write $(y_d, z_d) : \mathcal{J}_g \to Sp_d(U_d \setminus U_{d+1}) \times Sp_d(U_d \setminus U_{d+1})$ for the continuous map determined as the zero and pole of σ_d.

Regard $\mathbb{N}$ as a poset in the standard way. Consider a functor $\mathcal{D} : \mathbb{N} \to \mathsf{Top}$ into topological spaces given on objects by

$$\mathcal{D}(n) = \{(J, \eta, \zeta) \mid \eta \cap \zeta = \emptyset\} \subset \mathcal{J}_g \times Sp(F_g \setminus U_n) \times Sp(F_g \setminus U_n)$$

and given on a morphism $n \leq n'$ as

$$(J, \eta, \zeta) \mapsto (J, \eta + \sum_{i>n}^{n'} y_i(J) \ , \ \zeta + \sum_{i>n}^{n'} z_i(J))$$

Denote this map for $n < n+1$ by

$$\iota : \mathcal{D}(n) \to \mathcal{D}(n+1).$$

Define $\widehat{Div} = colim\ \mathcal{D}$. The projection $\widehat{Div} \to \mathcal{J}_g$ is a fibration in an obvious way. One can think of a point in the fiber $\widehat{Div}(J)$ over $J \in \mathcal{J}_g$ as a pair of disjoint 'infinite" positive divisors on the Riemann surface $(F_g \setminus x_0, J)$ whose difference with the pair $(\sum y_d, \sum z_d)$ is a pair of finite divisors.

Denote the subspace

$$\mathcal{D}_d(n) := (\mathcal{J}_g \times Div_{d+n}) \cap \mathcal{D}(n) \subset \mathcal{D}(n).$$

Clearly, $\mathcal{D}_d(n) \hookrightarrow \mathcal{J}_g \times Div_{d+n}$ is a homotopy equivalence. Moreover, for each $J \in \mathcal{J}_g$, the embedding $\iota : \mathcal{D}_d(n-1) \hookrightarrow \mathcal{D}_{d+1}(n)$ given by $(J, \eta, \zeta) \mapsto (\eta + y_n(J), \zeta + z_n(J))$ has trivial normal bundle meaning that it extends to an open embedding over $\mathcal{J}_g$ of $\mathcal{D}_d(n-1)\widetilde{\times}(V_{y_d(-)}\times V_{z_d(-)})$ where V_p is a sufficiently small open ball around $p \in F_g$. Denote by $\mathcal{D} \supset \mathcal{D}_d : \mathbb{N} \to \mathsf{Top}$ the subfunctor given by $n \mapsto \mathcal{D}_d(n)$. Write

$$\widehat{Div}_d \subset \widehat{Div}$$

for the component which is the colimit of $\mathcal{D}_d$.

Remark 2.0.1. The idea for introducing $\widehat{Div}_d =: \widehat{Div}_d(F_g)$ rather than being satisfied with $\mathcal{J}_g \times Div_d$ is that $\widehat{Div}(-)$ behaves better on non-compact arguments. Namely, restriction maps among $\widehat{Div}(-)$ which one expects to be quasifibrations from Dold-Thom theory are indeed quasifibrations.

3 The homology fibration

3.1 Phrasing Abel's Theorem

Let V_J be the space of holomorphic 1-forms on the Riemann surface (F_g, J). There is the natural inclusion $H_1(F_g;\mathbb{Z}) \hookrightarrow {V_J}^*$ as a non-degenerate lattice yielding the g-torus

$$T_J := {V_J}^*/H_1(F_g;\mathbb{Z})$$

known as the *Jacobian variety* of the Riemann surface (F_g, J).

Consider the space of pairs

$$\mathcal{J}_g\tilde{\times}T_J := \{(J, v) \mid v \in T_J\}.$$

Projection onto the first coordinate makes $\mathcal{J}_g\tilde{\times}T_J \to \mathcal{J}_g$ into a fiber bundle whose fiber over any $J \in \mathcal{J}_g$ is the Jacobian variety T_J.

For each $p \in F_g$, choose a piecewise smooth path $\gamma : [0,1] \to F_g$ from p to x_0. Integration along such γ gives a map $I : Div_d \to T_J$. Explicitly, given a positive divisor $\eta = \sum m_i p_i$ of F_g and paths γ_i from p_i to x_0, write $\int_{\gamma_\eta}$ for $\sum m_i \int_{\gamma_i}$. Define

$$I(\eta,\zeta) = (\alpha \mapsto (\int_{\gamma_\eta} - \int_{\gamma_\zeta})\alpha). \tag{3.1}$$

Note that the same formula extends I to a map $I : Sp(F_g) \times Sp(F_g) \to T_J$.

Interpreting Abel's theorem ([GH94], p. 231, for a general reference), the fiber of I over $0 \in T_J$ is the space of degree d meromorphic functions on (F_g, J), that is, $Hol_*^d((F_g, J), \mathbb{C}P^1)$. Here, the $*$ denotes based maps.

Definition 3.1.1. A map $p : E \to B$ is a *homology fibration up to degree* k if for each $b \in B$ the inclusion $fiber_b(p) \to hofiber_b(P)$ induces an isomorphism in $H_q(-)$ for $q < k$.

Remark 3.1.2. This definition is similar to that in [MS76] where they introduce the notion of a *homology fibration.*

Through out the paper, we will implicitly make use of the following

Proposition 3.1.3. *Let $E \xrightarrow{p} B$ be a continuous map with B paracompact and locally contractible. Then p is a homology fibration up to degree k if and only if for each $b \in B$ there is a contractible neighborhood $b \in U \subset B$ such that $p^{-1}(b) \hookrightarrow p^{-1}(U)$ induces an isomorphism in $H_q(-)$ for $q < k$.*

Proof. The proof is nearly identical to that of Proposition 5 in [MS76]. □

Segal shows that $I : Div_d \to T_J$ is a *homology fibration up to degree* $d-2g$. In this paper, we are interested in a similar statement as the complex structure J is allowed to vary. This is accomplished as follows.

There is a natural inclusion $V_J^* \hookrightarrow H_1(F_g;\mathbb{C})$. This along with the projection $\mathbb{C} \to \mathbb{R}$ yields the canonical homeomorphism

$$V_J^* \cong H_1(F_g;\mathbb{R})$$

which is $H_1(F_g;\mathbb{Z})$-eqivariant. This results in a canonical homeomorphism of g-tori

$$T_J \cong H_1(F_g;\mathbb{R})/H_1(F_g;\mathbb{Z}).$$

Denote this standard g-torus $H_1(F_g;\mathbb{R})/H_1(F_g;\mathbb{Z})$ by T_0. There results an isomorphism of fiber bundles

$$\begin{array}{ccc} \mathcal{J}_g \tilde{\times} T_J & \xrightarrow{\cong} & \mathcal{J}_g \times T_0 \\ \downarrow & & \downarrow \\ \mathcal{J}_g & \xrightarrow{id} & \mathcal{J}_g. \end{array}$$

Define $\mathcal{J}_g^d(\mathbb{C}P^1) \subset \mathcal{J}_g \times Map(F_g, \mathbb{C}P^n)$ to be the subspace consisting of those pairs (J,h) for which h is J-holomorphic. Denote subspace $\mathcal{J}_g^d(\mathbb{C}P^1)_* \subset \mathcal{J}_g^d(\mathbb{C}P^1)$ consisting of those pairs (J,h) for which h is a based map. Using Abel's theorem, the fiber of the composite

$$\mathcal{I} : \mathcal{J}_g \times Div_d \xrightarrow{id \times I} \mathcal{J}_g \tilde{\times} T_J \xrightarrow{\cong} \mathcal{J}_g \times T_0 \xrightarrow{pr} T_0$$

is the space $\mathcal{J}_g^d(\mathbb{C}P^1)_*$. Extend notation and write $\mathcal{I} : \mathcal{J}_g \times Sp \times Sp \to T_0$ for the map given by the same formula (3.1).

3.2 The Homology Fibration

The following theorem is the crux of the paper and is the most technical argument presented.

Theorem 3.2.1. *The map $\mathcal{I} : \mathcal{J}_g \times Div_d \to T_0$ is a homology fibration up to degree $d-2g$.*

Proof. Through out this proof, we will suppress the subscript g wherever is should appear.

The product $P_d := Sp_d \times Sp_d$ is stratified by the spaces $P_{d,k} := \{(\eta,\zeta) \mid deg(\eta \cap \zeta) \geq k\}$:

$$Sp_d \subset P_{d,d-1} \subset ... \subset P_{d,1} \subset P_d = Sp_d \times Sp_d.$$

Notice that the inclusion $Div_{d-k} \times Sp_k \hookrightarrow P_{d,k}$, given by $((\eta,\zeta),\xi) \mapsto (\eta+\xi, \zeta+\xi)$, is a homeomorphism onto the open stratum $P_{d,k} \setminus P_{d,k+1}$.

Extend the path-integral map $\mathcal{I}$ from above to each $\mathcal{J} \times P_{d,k}$ in the apparent way. Let $S \subset T_0$ be a subset. Denote $(\mathcal{J} \times P_{d,k})_S := \mathcal{I}^{-1}(S) \subset \mathcal{J} \times P_{d,k}$. There is a sequence of inclusions

$$(\mathcal{J} \times Sp_d)_S \subset (\mathcal{J} \times P_{d,d-1})_S \subset \cdots \subset (\mathcal{J} \times P_{d,1})_S \subset (\mathcal{J} \times P_d)_S.$$

with $(\mathcal{J} \times P_{d,k})_S \setminus (\mathcal{J} \times P_{d,k+1})_S \cong (\mathcal{J} \times (Div_{d-k} \times Sp_k))_S$. For $v \in W \subset T_0$ a contractible neighborhood of v, let j_- denote any such inclusion $(-)_v \hookrightarrow (-)_W$. Unwinding definitions, the proof of the lemma amounts to showing $H_r(j_{\mathcal{J}\times Div_d}) = 0$ when $r < d - 2g$.

The inclusion $Div_{d-k} \times Sp_k \hookrightarrow P_{n,k}$ is an open embedding. The inclusion $P_{d,k} \hookrightarrow P_{d,k-1}$ has normal bundle denoted $\nu_{P_{d,k}}$. It is thus possible to form the well-behaved normal bundle $\nu_{\mathcal{J}\times P_{d,k}}$ of the embedding $\mathcal{J} \times P_{d,k} \hookrightarrow \mathcal{J} \times P_{d,k-1}$, namely,

$$\nu_{\mathcal{J}\times P_{d,k}} := pr^* \nu_{P_{d,k}}$$

where $pr : \mathcal{J} \times P_{d,k} \to P_{d,k}$ is projection onto the second factor. There is an analogous normal bundle for the two embeddings $(\mathcal{J} \times P_{d,k})_{v,W} \hookrightarrow (\mathcal{J} \times P_{d,k-1})_{v,W}$.

Proceeding by (downward) induction on d, assume for $0 < k \leq d$ that $H_r(j_{\mathcal{J}\times Div_{d-k}\times Sp_k}) = 0$ when $r+k \leq d-k-2g$. That is to say $j_{\mathcal{J}\times Div_{d-k}\times Sp_k}$ induces an isomorphism in H_r for $r < d-k-2g$ and a surjection in H_{d-k-2g}. Refer to this inductive hypothesis as the *primary* inductive hypothesis. We wish to prove the case $k = 0$.

Consider the diagram (3.2) of exact sequences of homology groups associated to the pair

$$(\mathcal{J} \times P_{d,k}\ ,\ \mathcal{J} \times (P_{d,k} \setminus P_{d,k+1})) \cong (\mathcal{J} \times P_{d,k}\ ,\ \mathcal{J} \times Div_{d-k} \times Sp_k).$$

From the above discussion, the relative term is homotopy equivalent to the Thom space $Th(\nu_{\mathcal{J}\times P_{d,k+1}})$ of the normal bundle.

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_{r-1}Th(\nu_{(\mathcal{J}\times P_{d,k+1})_v}) & \longrightarrow & H_r(\mathcal{J} \times Div_{d-k} \times Sp_k)_v & \longrightarrow & H_r(\mathcal{J} \times P_{d,k})_v & \longrightarrow \cdots \\ & \downarrow j & & \downarrow j & & \downarrow j & \\ \cdots \longrightarrow & H_{r-1}Th(\nu_{(\mathcal{J}\times P_{d,k+1})_W}) & \longrightarrow & H_r(\mathcal{J} \times Div_{d-k} \times Sp_k)_W & \longrightarrow & H_r(\mathcal{J} \times P_{d,k})_W & \longrightarrow \cdots \end{array} \tag{3.2}$$

Invoke a nested *secondary* (downward) induction argument on $0 < k \leq d$ to assume $H_r(j_{\mathcal{J}\times P_{d,l}}) = 0$ for $r \leq d - l$ with $k < l \leq d$. For the moment, assume the base case $k = d$ of this secondary induction hypothesis. Using the primary inductive hypothesis, the Thom isomorphism, and the 5-lemma on (3.2), it follows that $H_r(j_{\mathcal{J}\times P_{d,k}}) = 0$ for $r \leq d - k$ $(k > 0)$. The case $k = d$ is easy enough as outlined by the following two facts.

Firstly, the Riemann-Roch formula and Abel's theorem tells us that, for $d > 2g$, $I : \{J\} \times Sp_d \to T_0$ is a fiber bundle having fiber the $d - g$ dimensional complex vector space of degree d (based) meromorphic functions on (F_g, J). Secondly, as we allow variation in $J \in \mathcal{J} \simeq *$, the map $\mathcal{I} : \mathcal{J} \times Sp_d \to T_0$ is a fiber bundle with fiber identified with the product $\mathcal{J} \times \mathbb{C}^{d-g}$. These same two facts, imply that $H_r(j_{\mathcal{J}\times P_d}) = 0$ since $P_d = Sp_d \times Sp_d$.

Considering diagram (3.2) for $k = 0$, the 5-lemma tells us that $H_r(j_{\mathcal{J}\times Div_d}) = 0$ provided $d \geq 2g$ and $r \leq d - 2g$. This completes the inductive step.

□

Let $(\eta,\zeta) \in Sp(F_g \setminus U_n) \times Sp(F_g \setminus U_n)$. Note that $\mathcal{I}(J,\eta,\zeta) = \mathcal{I}(J, \eta + y_{n+1}(J), \zeta + z_{n+1}(J)) \in T_0$ exactly because the pair $(y_{n+1}(J), z_{n+1}(J))$ occurs as the zero and pole of a meromorphic function on (F_g, J). In particular, the diagram

$$\begin{array}{ccc} \mathcal{D}_d(n) & \xrightarrow{\iota} & \mathcal{D}_d(n+1) \\ \downarrow{\scriptstyle \mathcal{I}} & & \downarrow{\scriptstyle \mathcal{I}} \\ T_0 & \xrightarrow{=} & T_0 \end{array} \tag{3.3}$$

commutes. In any event, there results a universal map from the colimit

$$\mathcal{I} : \widehat{Div} \to T_0.$$

The next task is to prove the following

Lemma 3.2.2. *Let $d, n \in \mathbb{N}$. The induced map on fibers in the above diagram (3.3) induces an isomorphism in $H_q(-)$ for $q < d - 2g$.*

Proof. To prove Lemma 3.2.2 it will suffice to show $\iota : \mathcal{D}_d(n) \to \mathcal{D}_d(n+1)$ induces an isomorphism in $H_q(-)$ for $q < d-2g$. This is indeed sufficient using the Zeeman comparison theorem ([Zee57]) for the apparent Leray-Serre spectral sequences (see [McC01], §5). Recall that the canonical inclusion $\mathcal{D}_d(n) \hookrightarrow \mathcal{J}_g \times Div_{d+n}$ is a homotopy equivalence. Choose once and for all a homotopy inverse $\mathcal{J}_d \times Div_{d+1} \to \mathcal{D}_d(n)$ for each $n \in \mathbb{N}$ and denote the resulting map again as

$$\iota : \mathcal{J}_g \times Div_d \to \mathcal{J}_g \times Div_{d+1} \tag{3.4}$$

It turns out that most of the work has already been done. As in the proof of Theorem 3.2.1, consider the sequence of closed inclusions

$$(\mathcal{J}_g \times Sp_d) \subset (\mathcal{J}_g \times P_{d,d-1}) \subset ... \subset (\mathcal{J}_g \times P_{d,1}) \subset (\mathcal{J}_g \times P_d) = (\mathcal{J}_g \times Sp_d \times Sp_d). \tag{3.5}$$

In the natural way, extend the map ι in (3.4) to a map to a map of sequences (3.5). In the proof of Theorem 3.2.1, we compared the H_*-long exact sequences for the pairs $((P_{d,k})_v, (Div_{d-k} \times Sp_k)_v)$ and $((P_{d,k})_W, (Div_{d-k} \times Sp_k)_W)$. Here we compare the H_*-long exact sequences of the pairs $((P_{d,k})_v, (Div_{d-k} \times Sp_k)_v)$ and $((P_{d+1,k})_v, (Div_{d+1-k} \times Sp_k)_v)$. The appropriate nested induction argument here is nearly identical to that of Theorem 3.2.1. Again, the essential fact comes from Abel's theorem which is that both $\mathcal{J}_g \times Sp_d \to T_0$ and $\mathcal{J}_g \times Sp_d \times Sp_d \to T_0$ are fiber bundles for $d \geq 2g$. Details are left to the interested reader. □

Corollary 3.2.3. *The integration map*

$$\mathcal{I} : \widehat{Div}_d \to T_0$$

is a homology fibration up to degree $d - 2g$.

Proof. It follows from Theorem 3.2.1 and Proposition 3 of [MS76] that the resulting map of telescopes

$$\mathcal{I} : (hocolim\ \mathcal{D}_d) \to hocolim(T_0 \xrightarrow{=} T_0 \xrightarrow{=} \dots)$$

is a homology fibration up to degree $d - 2g$. The result follows from the observation that $\mathcal{D}(n) \hookrightarrow \mathcal{D}(n+1)$ extends to an open inclusion of a trivial bundle over $\mathcal{D}(n)$, and is thus a cofibration. □

4 Comparing Fibration Sequences

4.1 The comparison fibration

Theorem 3.2.1 thus gives the sequence

$$\mathcal{J}_g^d(\mathbb{C}P^n)_* \to \mathcal{J}_g \times Div_d \xrightarrow{\mathcal{I}} T_0 \tag{4.1}$$

as a homology fibration up to degree $d - 2g$. The idea now is to compare this sequence with a homotopy theoretic sequence whose fiber is $\mathcal{J}_g \times Map_*^d(F_g, S^2)$. For this, consider the (homotopy) fibration sequence

$$S^2 \to \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \to \mathbb{C}P^\infty \tag{4.2}$$

which is defined as follows.

The topological group S^1 acts on S^2 by rotating S^2 fixing the north and south poles. Associated to this S^1-action there is a fibration sequence

$$S^2 \to ES^1 \times_{S^1} S^2 \to BS^1. \tag{4.3}$$

Regard the total space of this fiber bundle as a union of two disk bundles, one corresponding to the northern hemisphere of S^2, the other to the southern; the two disk bundles are glued together along their fiber-wise boundary S^1-bundle. The total space of each disk-bundle is homotopy equivalent to BS^1. The equatorial S^1-bundle is a model for ES^1 and is thus contractible. The fibration seuqnce (4.3) then becomes the (homotopy) fibration sequence

$$S^2 \to BS^1 \vee BS^1 \to BS^1. \tag{4.4}$$

Lastly, recall that $\mathbb{C}P^\infty$ is a model for BS^1.

4.2 The Scanning Map

Our goal now is to define a 'scanning map'

$$S : \widehat{Div} \to \mathcal{J} \times Map_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty).$$

Regard $\mathbb{C}P^\infty$ as the projectivization of the vector space $\mathbb{C}[z]$. There is a map $\mathbb{C}[z] \to Sp(S^2, \infty)$ which sends a polynomial to its roots. This map descends to a homeomorphism $\mathbb{C}P^\infty \cong Sp(S^2, \infty)$.

Write $B_\epsilon(0) \subset \mathbb{C}$ for the ϵ-neighborhood of $0 \in \mathbb{C}$. Once and for all, choose a continuous family of diffeomorphisms $\phi_\epsilon : B_\epsilon(0) \xrightarrow{\cong} \mathbb{C}$ which fix a neighborhood of the origin. This induces a continuous family of diffeomorphisms $\phi_\epsilon^* : (B_\epsilon(0)^*, *) \xrightarrow{\cong} (S^2, \infty)$ from the one-point compactifications.

Once and for all, fix a parallelization of $F_g \setminus x_0$ and a continuous family of Riemannian metric on $F_g \setminus x_0$ parametrized by $\mathcal{J}_g$. Assume for each $J \in \mathcal{J}_g$ that the metric is such that the injectivity radius of $F_g \setminus x_0$ is bounded away from 0. That is, there is an $\epsilon_J > 0$ such that for each $p \in F_g \setminus x_0$, an ϵ_J-neighborhood of p is a convex ball. For $\epsilon > 0$ as so, it follows that the exponential map

$$exp_p : B_\epsilon(0) \to F_g \setminus x_0$$

is an embedding where $B_\epsilon(0) \subset \mathbb{C}$ is an ϵ-neighborhood of $0 \in \mathbb{C}$. Assume further that this family of metrics is chosen so that for each $J \in \mathcal{J}_g$ the distances are bounded below

$$\epsilon_J < \inf_{i,j \in \mathbb{N}} \{dist(y_i(J), z_j(J))\}. \tag{4.5}$$

Fix a continuous family $\epsilon : \mathcal{J}_g \to (0, \infty)$ as in the above paragraph. Denote by $\mathcal{D}^\epsilon \subset \mathcal{D}$ the subfunctor with $\mathcal{D}^\epsilon(n) \subset \mathcal{D}(n)$ the subspace consisting of those triples (J, η, ζ) for which

$$\emptyset = (\bigcup_{u \in \eta} B_{\epsilon_J}(u)) \cap (\bigcup_{v \in \zeta} B_{\epsilon_J}(v)) \subset F_g \setminus x_0.$$

Because of condition (4.5), this condition is indeed preserved under the structure maps of the diagram $\mathcal{D}$. Each of the inclusions $\mathcal{D}^\epsilon(n) \hookrightarrow \mathcal{D}(n)$ in addition to the inclusion $\widehat{Div}^\epsilon \hookrightarrow \widehat{Div}$ is a homotopy equivalence.

Let $n \in \mathbb{N}$. There is a 'scanning map'

$$S_n^\epsilon : \mathcal{D}^\epsilon(n) \to Map_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)$$

whose adjoint is given as

$$((J, \eta, \zeta), p) \mapsto (exp_p^{-1}(\eta) - exp_p^{-1}(\sum_0^n y_i(J)) \vee (exp_p^{-1}(\zeta) - exp_p^{-1}(\sum_0^n z_i(J)),$$

for $p \neq x_0$, where $-$ denotes subtraction in the twice delooped $B^2\mathbb{Z} \cong \mathbb{C}P^\infty \cong_{\phi_\epsilon} Sp(B_\epsilon(0)^*, *)$; and for $p = x_0$ by the assignment $((J, \eta, \zeta), x_0) \mapsto \infty \in \mathbb{C}P^\infty \vee \mathbb{C}P^\infty$ to the base point. By construction the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}^\epsilon(n) & \longrightarrow & \mathcal{D}^\epsilon(n+1) \\ & {\scriptstyle S_n^\epsilon}\searrow \quad \swarrow{\scriptstyle S_{n+1}^\epsilon} & \\ & Map_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty). & \end{array} \tag{4.6}$$

This gives the data of a map from the colimit

$$S^\epsilon : \widehat{Div}^\epsilon \to Map_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty).$$

Refer to this map also as a 'scanning map'. Upon the (contractible) choice of a homotopy inverse, from the zig-zag

$$\widehat{Div} \xleftarrow{\simeq} \widehat{Div}^\epsilon \xrightarrow{S^\epsilon} Map_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)$$

there results a map $\widehat{Div} \xrightarrow{S} Map_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)$, referred to as the *scanning map*. It is straightforward to verify that the homotopy class of S is independent of the choice of the family $\epsilon_J > 0$. Think of S as scanning F_g for either η or ζ with a very zoomed microscope through which, via the parallelization, F_g looks like $\mathbb{C}$.

For $J \in \mathcal{J}_g$ a complex structure, denote by $\widehat{Div}(J) \subset \widehat{Div}$ the fiber over J of the projection map $\widehat{Div} \to \mathcal{J}_g$.

Theorem 4.2.1 (Segal ([Seg79] §4))**.** *The the scanning map restricts to a homotopy equivalence* $S : \widehat{Div}(J) \to Map_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)$.

Corollary 4.2.2. *The scanning map*

$$S : \widehat{Div} \to \mathcal{J} \times Map_*(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)$$

homotopy equivalence.

Note that S restricts on the component $S : \widehat{Div}_d \to Map_*^d(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)$ to a map whose target is the space of maps with bi-degree (d, d). By Corollary 4.2.2 this restriction is a homotopy equivalence.

4.3 Comparing Sequences

Observe that the connected component of the constant map $Map_*^0(F_g, \mathbb{C}P^\infty)$ as well as the g-torus T_0 are both models for $K(\mathbb{Z}^g, 1)$. So, abstractly, there is a homotopy equivalences $D : T_0 \to Map_*^0(F_g, \mathbb{C}P^\infty)$. In fact, as in ([Seg79], §4), it is possible to choose the homotopy equivalence D via Poincare' duality such that there results a homotopy commutative diagram

$$\begin{array}{ccc}
\mathcal{J}_g^d(\mathbb{C}P^1)_* & \longrightarrow & \mathcal{J}_g \times Map_*^d(F_g, S^2) \\
\downarrow & & \downarrow \\
\widehat{Div}_d & \xrightarrow{S} & \mathcal{J}_g \times Map_*^d(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty) \\
\downarrow{\scriptstyle \mathcal{I}} & & \downarrow{\scriptstyle sub} \\
T_0 & \xrightarrow{D} & Map_*^0(F_g, \mathbb{C}P^\infty \vee \mathbb{C}P^\infty)
\end{array}$$

where the indicated map is induced by subtraction in $\mathbb{C}P^\infty \cong B^2\mathbb{Z}$ so that the right vertical sequence is the fibration induced from the sequence in (4.2).

Moreover, imitating Segal's lines ([Seg79], Lemma 4.7), one can verify that the resulting map on fibers is the standard inclusion

$$\mathcal{J}_g^d(\mathbb{C}P^1)_* \hookrightarrow \mathcal{J}_g \times Map_*^d(F_g, S^2).$$

Because S and D are equivalences (Theorem 4.2.1), we conclude the following lemma.

Lemma 4.3.1. *The inclusion*

$$\mathcal{J}_g^d(\mathbb{C}P^1)_* \hookrightarrow \mathcal{J}_g \times Map_*^d(F_g, S^2)$$

induces an isomorphism in $H_q(-)$ *for* $q < d - 2g$.

5 Proof of Theorem 0.1.2

5.1 Unbased Mapping spaces

We are interested in the unbased mapping spaces $\mathcal{J}_g^d(\mathbb{C}P^1)$ and $Map^d(F_g, S^2)$.

Lemma 5.1.1. *The inclusion* $\mathcal{J}_g^d(\mathbb{C}P^1) \hookrightarrow Map^d(F_g, S^2)$ *induces an isomorphism in* $H_q(-)$ *for* $q < d - 2g$.

Proof. Consider the base-point evaluation fibrations

$$\begin{array}{ccc} \mathcal{J}_g^d(\mathbb{C}P^1)_* & \longrightarrow & \mathcal{J}_g \times Map_*^d(F_g, S^2) \\ \downarrow & & \downarrow \\ \mathcal{J}_g^d(\mathbb{C}P^1) & \longrightarrow & \mathcal{J}_g \times Map^d(F_g, S^2) \\ \downarrow & & \downarrow \\ \mathbb{C}P^1 & \longrightarrow & S^2. \end{array} \tag{5.1}$$

There results a morphism of Leray-Serre H_*-spectral sequences. Lemma 4.3.1 shows that this functor induces an isomorphism on the $E_{p,q}^2$ page for $q < d - 2g$. We thus have an isomorphism on the $E_{p,q}^r$ pages for all r when $p + q < d - 2g$. In particular, there is an isomorphism on the $E_{p,q}^\infty$ page for $p + q < d - 2g$.

We are now confronted with the common issue of concluding the middle horizontal map in diagram 5.1 induces an isomorphism in H_*, for $* = p + q < d - 2g$, knowing it is an isomorphism on the filtration quotients $E_{p,q}^\infty$. For this one uses induction on the filtration degree; the inductive step is clear in light of the 5-lemma.

□

5.2 The action of $Diff_g^+$

Our ultimate goal being to understand moduli space, we now want a version of Lemma 5.1.1 which is quotiented by the action of $Diff_g^+$.

Theorem 5.2.1 (Main theorem ($n = 1$))**.** *The map* $\mathcal{M}_g^d(\mathbb{C}P^1) \to \mathcal{MT}_g^d(\mathbb{C}P^1)$ *induced by the natural inclusion induces an isomorphism in* $H_q(-)$ *for* $q < d - 2g$.

Proof. There is the following morphism of fibration sequences

$$\begin{array}{ccc} \mathcal{J}_g^d(\mathbb{C}P^1) & \xrightarrow{\cong_{H_{<d-2g}}} & \mathcal{J}_g \times Map^d(F, S^2) \\ \downarrow & & \downarrow \\ EDiff_g^+ \times_{Diff_g^+} \mathcal{J}_g^d(\mathbb{C}P^1) & \longrightarrow & EDiff_g^+ \times_{Diff_g^+} (\mathcal{J}_g \times Map^d(F, S^2)) \\ \downarrow & & \downarrow \\ BDiff_g^+ & \xrightarrow{id} & BDiff_g^+ \end{array}$$

Lemma 5.1.1 shows that the top horizontal map is an isomorphism in $H_q(-)$ for $q < d - 2g$. Recognizing the total spaces in the above diagram as the *homotopy* moduli spaces, the same spectral sequence argument proving Lemma 5.1.1 finishes the proof.

□

5.3 Maps to $\mathbb{C}P^n$ for $n \geq 1$

In this subsection we will sketch the idea for how to deal with maps to $\mathbb{C}P^n$ for $n > 1$.

Theorem 5.3.1 (Main Theorem)**.** *The map $\mathcal{M}_g^d(\mathbb{C}P^n) \to \mathcal{MT}_g^d(\mathbb{C}P^n)$ induces an isomorphism in $H_q(-)$ for $q < (d - 2g)(2n - 1)$.*

Proof (sketch). The proof is analogous to the proof of Theorem 5.2.1. We will sketch the appropriate modifications. Details will be left to the interested reader. Some helpful details can be found in [Seg79].

A holomorphic map $h \in \mathcal{J}_g^d(\mathbb{C}P^1)_*$ is a based meromorphic function and is thus determined by its poles and zeros. Equivalently, we could regard h as a pair (r_0, r_1) of degree d polynomial functions on (F_g, J) whose zeros are disjoint. This is the essence of Abel's theorem that we used earlier to regard $\mathcal{J}_g^d(\mathbb{C}P^1)_*$ as a subspace of $\mathcal{J}_g \times Div_d$. Now, think of $h \in \mathcal{J}_g^d(\mathbb{C}P^n)_*$ similarly as an $(n+1)$-tuple $(r_0, ..., r_n)$ of degree d polynomials with $\bigcap_{i=0}^n \{r_i = 0\} = \emptyset$. One would thus generalize Div_d to $Div_d^{(n)}$ consisting of $(n+1)$-tuples of positive divisors which are $(n+1)$-wise disjoint.

A modification of Abel's theorem shows that such an $(n+1)$-tuple comes from a map into $\mathbb{C}P^n$ if and only if each divisor has the same image in the Jacobian variety and, moreover, the fiber of the similarly defined integration map $\mathcal{I} : \mathcal{J}_g \times Div_d^{(n)} \to T_0^n$ is the space of based holomorphic maps $\mathcal{J}_g^d(\mathbb{C}P^n)_*$. The surrounding analogous constructions are similar.

On the homotopy theoretic side, one constructs the space

$$W_{n+1}(\mathbb{C}P^\infty) \subset \prod^{n+1} \mathbb{C}P^\infty$$

consisting of $(n+1)$-tuples with at least one entry the base point of $\mathbb{C}P^\infty$. Indeed, $W_2(\mathbb{C}P^\infty) = \mathbb{C}P^\infty \vee \mathbb{C}P^\infty$. Similar to sequence 4.2 is the (homotopy) fibration sequence

$$\mathbb{C}P^n \to W_{n+1}(\mathbb{C}P^\infty) \to \prod^n \mathbb{C}P^\infty.$$

There is a similar scanning map used to compare $Div_d^{(n)}$ to $Map_*^d(F_g, W_{n+1}(\mathbb{C}P^\infty))$. Details are left to the interested reader.

□

6 Variants of the main theorem

In this final section we outline two simple variations on the main theorem 0.1.2.

6.1 Marked points

Choose k points $\{r_i\} \subset F_g$. Let $Diff^+_{g,k}$ denote the subgroup of $Diff^+_g$ consisting of those diffeomorphisms of F_g which fix each r_i for $1 \leq i \leq k$. Define the *moduli space of degree d marked holomorphic curves of genus g in $\mathbb{C}P^n$* as the orbit space

$$\mathcal{M}^d_{g,k}(\mathbb{C}P^n) := \mathcal{J}^d_g(\mathbb{C}P^n)//Diff^+_{g,k}.$$

Do similarly for the topological counterpart $\mathcal{MT}^d_{g,k}(\mathbb{C}P^n)$.

Theorem 6.1.1. *The standard map*

$$\mathcal{M}^d_{g,k}(\mathbb{C}P^n) \to \mathcal{MT}^d_{g,k}(\mathbb{C}P^n)$$

induces an isomorphism in $H_q(-)$ for $q < (d-2g)(2n-1)$.

Proof. The proof is nearly identical to that of Theorem 5.3.1. □

6.2 Moduli of singular Riemann surfaces

We define a notion of a singular surface specific to our purposes. Let F be a compact Hausdorff space. A structure of a singular surface on F is the data of a finite subset $P \subset F$ and the structure of an oriented smooth surface on $F \setminus P$. Such an F endowed with this structure will be referred to as a *singular surface*. Such singular surfaces are characterized as quotient spaces

$$F = (\amalg^n_1 F_i)/\sim .$$

where each F_i is a smooth oriented surface with a finite collection of marked points $\{p_{i_k}\}^{n_i}_1 \subset F_i$, which are identified $p_{i_k} \sim p_{j_l}$ in some way. Refer to the data $(F_i, \{p_{i_k}\}^{n_i}_1)$ as a *normalization* of F; note that this data is unique. Refer to the subset $P \subset F$ as the *nodes* and denote the cardinality $n(F) := |P|$. Define the *genus* of F as the number $g(F)$ given by

$$\chi(F) = 2 - 2g(F) + n(F)$$

Say a singular surface is *irreducible* if its normalization consists of a single connected surface.

A complex structure J on such a singular surface F is the data of a complex structure J_i on each component F_i of its normalization. Write $\mathcal{J}_F$ for the space of such complex structures with the apparent topology. Refer to a pair (F, J) as a *singular Riemann surface*. A holomorphic map from (F, J) to a complex manifold Y is the data of holomorphic maps $(F_i, J_i) \xrightarrow{h_i} Y$ such that $h_i(p_{i_k}) = h_j(p_{j_l})$ whenever $p_{i_k} \sim p_{j_l}$ in F. The degree of such a map is the sum $\Sigma_i deg(h_i) \in H_2(Y)$. Let $Hol^\alpha((F, J), Y)$ be the space of degree α holomorphic maps from (F, J) into Y, topologized in the obvious way.

Remark 6.2.1. In what follows, most of the statements about singular surfaces are stated only for *irreducible* singular surfaces. One could make statements about non-irreducible surfaces provided one specifies the degree of maps on each component of the normalization.

In [Seg79], Segal proves the following

Theorem 6.2.2 (Segal)**.** *Let F be an irreducible singular surface. The standard inclusion*

$$Hol^d((F,J), \mathbb{C}P^n) \hookrightarrow Map^d(F, \mathbb{C}P^n)$$

induces an isomorphism in $H_q(-)$ for $q < (d - 2(g(F) - n(F) + 1))(2n-1)$.

The idea of the proof is that a holomorphic map $(F,J) \to \mathbb{C}P^n$ from a singular surface *is* a holomorphic map $(F_1, J_1) \to \mathbb{C}P^n$ with conditions on the marked points $\{p_{1_k}\}$. The theorem follows from Segal's previous work when F is not singular (i.e., the set of nodes is empty, $P = \emptyset$).

For F a singular surface, write $Diff^+(F) \subset Diff^+(\coprod F_i)$ as the topological subgroup of those diffeomorphisms ϕ of $\coprod F_i$ for which $\phi(p_{i_k}) \sim p_{i_k}$. The group $Diff^+(F)$ acts on the space of pairs

$$\mathcal{J}^d_{[F]}(\mathbb{C}P^n) := \{(J,h) \mid J \in \mathcal{J}_F \text{ and } h \in Hol^d((F,J),Y)\}$$

by pulling back complex structures and precomposing with holomorphic maps.

Let F be a singular surface. Define the *moduli space of type* $[F]$ to be the homotopy orbit space

$$\mathcal{M}^d_{[F]}(\mathbb{C}P^n) := \mathcal{J}^d_{[F]}(\mathbb{C}P^n)//Diff^+(F). \tag{6.1}$$

Define similarly the *topological moduli space of type* $[F]$ as

$$\mathcal{MT}^d_{[F]}(\mathbb{C}P^n) := (\mathcal{J}_F \times Map(F, \mathbb{C}P^n))//Diff^+(F).$$

Theorem 6.2.3. *Let F be an irreducible singular surface. The standard map*

$$\mathcal{M}^d_{[F]}(\mathbb{C}P^n) \to \mathcal{MT}^d_{[F]}(\mathbb{C}P^n) \simeq Map(F, \mathbb{C}P^n)//Diff^+(F)$$

induces an isomorphism in $H_q(-)$ for $q < (d - 2(g(F) - n(F) + 1))(2n-1)$.

Proof. The proof will merely be outlined. As demonstrated with the proof of Theorem 0.1.2, it is possible to follow the lines of Segal for singular Riemann surfaces while keeping track of the complex structure on such Riemann surfaces as a variable. This leads to a statement analogous to Lemma 5.1.1 for singular surfaces. The same spectral sequence argument from the proof of Theorem 5.2.1 is in place to account for quotienting by the action of $Diff^+(F)$. □

Remark 6.2.4. This remark serves to report a curiosity of the author. Consider the compactified moduli space $\overline{\mathcal{M}}^d_g(\mathbb{C}P^n)$ of Gromov-Witten theory (see [MS04] for a general reference). An S-point of this space (stack) is in particular a family of nodal Riemann surfaces over S together with a degree d holomorphic maps into $\mathbb{C}P^n$. There is a *topological* counterpart $\overline{\mathcal{MT}}^d_g(\mathbb{C}P^n)$ which replaces holomorphic maps by continuous maps. Write

$$\overline{\mathcal{M}}^{irr}_g(\mathbb{C}P^n) \hookrightarrow \overline{\mathcal{M}}_g(\mathbb{C}P^n)$$

for the subspace (substack) consisting of irreducible nodal surfaces, and similarly for the topological counterpart. There is a natural stratification of $\overline{\mathcal{M}}_g^{irr}(\mathbb{C}P^n)$ in terms of the number of nodes of surfaces (see [KV99] for example). It is a naive observation that the open strata are moduli spaces of the form (6.1). One might expect then from Theorem 6.2.3 that the apparent comparison

$$\overline{\mathcal{M}}_g^{d,irr}(\mathbb{C}P^n) \hookrightarrow \overline{\mathcal{MT}}_g^{d,irr}(\mathbb{C}P^n)$$

is then an isomorphism in H_q for $q << g << d$. A needed ingredient is a result on the existence of regular neighborhoods of the closed strata of $\overline{\mathcal{M}}_g^{irr}(\mathbb{C}P^n)$.

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