N = natural numbers      Z = integers      Q = rationals      R = reals

1. A subgroup \( H \) of a group \( G \) is said to be characteristic in \( G \) if \( \varphi(H) \subseteq H \) for every (surjective) isomorphism \( \varphi : G \to G \)

   (a) Prove that if \( H \) is characteristic in \( G \) then \( H \) is a normal subgroup of \( G \).

   (b) Prove that the center \( Z(G) := \{ a \in G : ab = ba \text{ for all } b \in G \} \) is characteristic in \( G \).

2. Suppose \( G \) is a finite abelian group and that \( n \in \mathbb{N} \) is relatively prime to the order of \( G \). Prove that for each \( y \in G \) there is an \( x \in G \) so that \( nx = y \).

3. Prove that:

   (a) \( \mathbb{Q}[x] \) is a principal ideal domain.

   (b) \( \mathbb{Z}[x] \) is not a principal ideal domain.

   (c) The kernel of the ring homomorphism \( \varphi : \mathbb{Z}[x] \to \mathbb{R} \) that takes \( x \) to \( 1 + \sqrt{2} \) is a principal ideal.

4. Let \( f(x) = x^5 - 4x + 2 \in \mathbb{Q}[x] \) and let \( G \) be the Galois group of \( f(x) \). Prove that:

   (a) \( 5 \mid |G| \).

   (b) \( 2 \mid |G| \).