M.S. Real Analysis Exam 2016

Instructions: Attempt **Five** of the Seven questions and Show all work. Good Luck!

Problem 1.

(a) Let (x_n) and (y_n) be two sequences of real numbers. Prove that

$$\limsup_{n\to\infty} (x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

(b) True or False:

$$\limsup_{n\to\infty} x_n y_n = (\limsup_{n\to\infty} x_n) \cdot (\limsup_{n\to\infty} y_n),$$

if True give a proof, if false provide a counter example.

Problem 2. Let (f_n) be a sequence of real valued continuous functions on $[0,1]$ which are uniformly bounded on $[0,1]$ and such that for each k, $k = 0, 1, 2, 3, \ldots$,

$$\lim_{n\to\infty} \int_0^1 f_n(x) x^k dx = 0.$$

Show that for any continuous $\phi : [0,1] \to \mathbb{R}$, it must be that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\phi(x) dx = 0.$$

Problem 3. Let X be a compact metric space, Y another metric space (possibly non-compact), let $p : X \times Y \to Y$ be the map $p(x,y) = y$. Show that if Z is a closed subset of $X \times Y$ then $p(Z)$ is closed in Y.

Problem 4. Let (f_n) be a sequence of continuous maps $[0,1] \to \mathbb{R}$ such that

$$\int_0^1 (f_n(y))^2 dy \leq 4$$

for all n. Define $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x) = \int_0^1 (x+y) f_n(y) dy.$$

a. Find a constant $K > 0$ such that $|g_n(x)| \leq K$ for all n,

b. Prove that a subsequence of the sequence (g_n) converges uniformly on $[0,1]$.

For this problem you may assume the Schwarz inequality holds for two integrable functions:

$$\left| \int_a^b f(x)g(x) dx \right| \leq \left(\int_a^b f^2(x) dx \right)^{1/2} \left(\int_a^b g^2(x) dx \right)^{1/2}$$

Problem 5. Let $\sum a_n$ be a convergent series of positive real numbers and (f_n) a sequence of real-values functions defined on $S \subset \mathbb{R}$ such that

$$|f_{n+1}(x) - f_n(x)| < a_n, \text{for all } n \in \mathbb{N} \text{ and all } x \in S$$

Prove that (f_n) is uniformly Cauchy on S and hence it is uniformly convergent on S.

Problem 6. State the Riemann-Lebesgue Theorem on Integration and use it to prove that if $f_n \to f$ uniformly on $[a, b]$ and each f_n is Riemann integrable then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Problem 7. Let X and Y be metric spaces, with metrics d_X and d_Y, respectively. Let $f, f_1, f_2, \ldots$ be bijective functions from X to Y, with inverses $g, g_1, g_2, \ldots$, respectively. Assume that

a. g is uniformly continuous; and

b. $f_n \to f$ uniformly as $n \to \infty$.

Prove that $g_n \to g$ uniformly as $n \to \infty$.