Problem 1.

(a) Let \((x_n)\) and \((y_n)\) be two sequences of real numbers. Prove that
\[
\limsup_{n \to \infty} (x_n + y_n) \leq \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n.
\]

(b) True or False:
\[
\limsup_{n \to \infty} x_n y_n = (\limsup_{n \to \infty} x_n) \cdot (\limsup_{n \to \infty} y_n),
\]
if True give a proof, if false provide a counter example.

Problem 2. Let \((f_n)\) be a sequence of real valued continuous functions on \([0, 1]\) which are uniformly bounded on \([0, 1]\) and such that for each \(k, k = 0, 1, 2, 3, \ldots,\)
\[
\lim_{n \to \infty} \int_0^1 f_n(x)x^k dx = 0.
\]
Show that for any continuous \(\phi : [0, 1] \to \mathbb{R},\) it must be that
\[
\lim_{n \to \infty} \int_0^1 f_n(x)\phi(x)dx = 0.
\]

Problem 3. Let \(X\) be a compact metric space, \(Y\) another metric space (possibly non-compact), let \(p : X \times Y \to Y\) be the map \(p(x, y) = y.\) Show that if \(Z\) is a closed subset of \(X \times Y\) then \(p(Z)\) is closed in \(Y.\)

Problem 4. Let \((f_n)\) be a sequence of continuous maps \([0, 1] \to \mathbb{R}\) such that
\[
\int_0^1 (f_n(y))^2 dy \leq 4
\]
for all \(n.\) Define \(g_n : [0, 1] \to \mathbb{R}\) by
\[
g_n(x) = \int_0^1 (x + y)f_n(y)dy.
\]

a. Find a constant \(K > 0\) such that \(|g_n(x)| \leq K\) for all \(n,\)

b. Prove that a subsequence of the sequence \((g_n)\) converges uniformly on \([0, 1].\)

For this problem you may assume the Schwarz inequality holds for two integrable functions:
\[
\left| \int_a^b f(x)g(x)dx \right| \leq \left( \int_a^b f^2(x)dx \right)^{1/2} \left( \int_a^b g^2(x)dx \right)^{1/2}
\]
Problem 5. Let $\sum a_n$ be a convergent series of positive real numbers and $(f_n)$ a sequence of real-values functions defined on $S \subset \mathbb{R}$ such that

$$|f_{n+1}(x) - f_n(x)| < a_n, \text{ for all } n \in \mathbb{N} \text{ and all } x \in S$$

Prove that $(f_n)$ is uniformly Cauchy on $S$ and hence it is uniformly convergent on $S$.

Problem 6. State the Riemann-Lebesgue Theorem on Integration and use it to prove that if $f_n \to f$ uniformly on $[a,b]$ and each $f_n$ is Riemann integrable then

$$\lim_{n \to \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx.$$

Problem 7. Let $X$ and $Y$ be metric spaces, with metrics $d_X$ and $d_Y$, respectively. Let $f, f_1, f_2, \ldots$ be bijective functions from $X$ to $Y$, with inverses $g, g_1, g_2, \ldots$, respectively. Assume that

a. $g$ is uniformly continuous; and

b. $f_n \to f$ uniformly as $n \to \infty$.

Prove that $g_n \to g$ uniformly as $n \to \infty$. 