Instructions: Attempt five of the six questions and show all work. Good luck!

1. For part (a) and (b) recall that for a sequence of real numbers \( (a_n) \),

\[
\limsup_{n \to \infty} a_n := \limsup_{n \to \infty} \{a_k : k \geq n\}.
\]

(a) Let \( (x_n) \) and \( (y_n) \) be two bounded sequences of real numbers. Prove that

\[
\limsup_{n \to \infty} (x_n + y_n) \leq \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n.
\]

(b) True or False:

\[
\limsup_{n \to \infty} (x_n y_n) = (\limsup_{n \to \infty} x_n)(\limsup_{n \to \infty} y_n).
\]

If true give a proof, if false provide a counter example.

(c) Suppose that \( 0 < r < 1 \) and \( |x_{n+1} - x_n| < r^n \) for all \( n \in \mathbb{N} \), show that \( (x_n) \) is a Cauchy sequence.

2. (a) State the Riemann–Lebesgue Theorem on Riemann integration.

(b) Give an example of a function that is not Riemann integrable.

(c) Consider the function

\[
f(x) = \begin{cases} 
1 & \text{if } x = 0 \\
1/q & \text{if } x \text{ is rational, } x = p/q \text{ in lowest terms and } q > 0 \\
0 & \text{if } x \text{ is irrational.}
\end{cases}
\]

Is this function Riemann integrable?

3. Let \( E \) be a closed, bounded, and nonempty subset of \( \mathbb{R}^m \) and let \( f : E \to E \) be a function satisfying \( |f(x) - f(y)| < |x - y| \) for all \( x, y \in E, x \neq y \). Prove that there is one and only one point \( x_0 \in E \) such that \( f(x_0) = x_0 \).

4. Let \( f : [0, \infty) \to [0, \infty) \) be a monotonically decreasing function with

\[
\int_0^\infty f(x)dx < \infty.
\]

Prove that \( \lim_{x \to \infty} x f(x) = 0 \).

5. Suppose that \( f : [0, 1] \to \mathbb{R} \) is continuous. Calculate the following limits (with proof)

(a)

\[
\lim_{n \to \infty} \int_0^1 x^n f(x)dx.
\]

(b)

\[
\lim_{n \to \infty} \int_0^{1/2} f(x^n)dx.
\]
6. Let \((f_n)\) be a sequence of uniformly bounded real valued continuous functions on \([0, 1]\) such that for every natural \(k \in \mathbb{N},\)

\[
\lim_{n \to \infty} \int_0^1 f_n(x) x^k \, dx = 0.
\]

Show that for every continuous \(\phi : [0, 1] \to \mathbb{R}\) we have

\[
\lim_{n \to \infty} \int_0^1 f_n(x) \phi(x) \, dx = 0.
\]