ANALYSIS: Master’s Comprehensive Exam

Instructions: Attempt all of the problems, showing work. Place at most one problem solution on a side for each sheet of paper turned in. Do not submit your scratch work.

1. Define the function \( f(x) = \sum_{n=0}^{\infty} \left( \frac{\sin x}{n+1} \right)^2 \). For what values of \( x \in \mathbb{R} \) is \( f \) continuous? Justify your answer.

2. Suppose that \( f : K \rightarrow \mathbb{R}^m \) is continuous on the compact set \( K \subset \mathbb{R}^n \). Define uniform continuity. Prove that \( f \) is uniformly continuous.

3. Suppose that \( f : [0,1] \rightarrow \mathbb{R} \) is a continuous function.

   (a) Evaluate \( \lim_{n \to \infty} \int_0^1 x^n f(x) \, dx \). Be sure to prove your answer is correct.

   (b) Evaluate \( \lim_{n \to \infty} n \int_0^1 x^n f(x) \, dx \). Be sure to prove your answer is correct.