ANALYSIS: Master’s Comprehensive Exam January ’06

Instructions: Attempt all of the problems, showing work. Place at most one problem solution on a side for each sheet of paper turned in. Do not submit your scratch work.

(1) If $0 < x < \frac{\pi}{2}$, prove that $\frac{2}{\pi} < \frac{\sin x}{x} < 1$.

(2) Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous and $\lim_{x \to \infty} f(x) = 1$.
   
   (a) For $0 < R < \infty$, evaluate $\lim_{n \to \infty} \frac{1}{n} \int_0^R e^{-\frac{x}{n}} f(x) \, dx$

   (b) Show $\lim_{n \to \infty} \frac{1}{n} \int_0^\infty e^{-\frac{x}{n}} f(x) \, dx = 1$

(3) Let $(f_n)$ denote a sequence of continuous functions on a domain $D \subset \mathbb{R}$ such that $f_n$ converges uniformly to the function $f$ on $D$.

   Show that $\lim_{n \to \infty} f_n(x_n) = f(x)$, if $\lim_{n \to \infty} x_n = x$ and $x \in D$. 