Real Analysis Master Comprehensive Exam

(Jan 2010)

Name:

Pick and circle four out of the five problems below, then solve them. If you rely on a theorem please state it carefully! *Good Luck!*

1. Let $(x_n) \subset \mathbb{R}$ be such that $\liminf_{n\to\infty} x_n = -\infty$ and $\limsup_{n\to\infty} x_n = +\infty$. Show that if $\sup_{n\in\mathbb{N}} |x_{n+1} - x_n| \leq 1$, then (x_n) has a subsequence convergent to some $x \in \mathbb{R}$.

2. Fix $f : [0,1] \to \mathbb{R}$. Consider the sets

$$G_\epsilon := \{x \in [0,1] : \exists_{\delta>0} \forall_{t\in[0,1]} |t - x| < \delta \implies |f(t) - f(x)| < \epsilon\}.$$

For each $\epsilon > 0$, construct an open set U_ϵ such that

$$G_\epsilon \subset U_\epsilon \subset G_{2\epsilon}$$

and carefully argue that $\bigcap_{\epsilon>0} G_\epsilon = \bigcap_{\epsilon>0} U_\epsilon$.

3. Assuming that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$ and is C^2 smooth (i.e., f'' exists and is continuous), show that the following series converges

$$\sum_{n=1}^{\infty} (-1)^n f(1/n).$$

4. Suppose that, for each $n \in \mathbb{N}$, P_n is a quadratic polynomial. Argue that if $\lim_{n\to\infty} P_n(x)$ exists for the three arguments $x \in \{-1, 0, 1\}$, then (P_n) converges uniformly on the segment $[-1, 1]$. Give a counter example when P_n are allowed to be cubic.

5. Compute the limit

$$\lim_{\mu\to 0} \int_0^1 \frac{\mu^2 x^2}{x^4 + \mu^4}\, dx.$$

Give a rigorous argument supporting your answer.

The gist of the solutions:

1. Infinitely many x_n must reside in $[-1, 1]$, which is sequentially compact.

2. For $x \in G_\epsilon$, let $\delta(x) := \delta > 0$ be as stipulated by the definition of G_ϵ. Set $U_\epsilon := \bigcup_{x \in G_\epsilon} (x - \delta(x), x + \delta(x))$. BTW: the intersection $\bigcap_{\epsilon > 0} G_\epsilon$ is the set of continuity if f.

3. Use Taylor expansion to get

$$(-1)^n f(1/n) = f'(0)(-1)^n 1/n + [\text{something bounded}] \cdot 1/n^2$$

and lean on convergence of $\sum (-1)^n 1/n$ and absolute convergence of $\sum 1/n^2$.

4. For quadratic P_n, determine the three coefficients of P_n in terms of the values $P_n(-1), P_n(0), P_n(1)$ to see that they converge. Uniform convergence of the P_n follows.

5. Fix $\epsilon > 0$. Over $[\epsilon, 1]$, the integrand converges uniformly to zero and so does the integral $\int_\epsilon^1$. Over $[0, \epsilon]$ the integrand is bounded by some constant C (in fact $C = 1/2$) and so the integral $\int_0^\epsilon$ does not exceed $C\epsilon$. (Graph the integrand if this seems unnatural.)