Real Analysis Master Comprehensive Exam

(Jan 2011)

Name:

Pick and <u>circle</u> four out of the five problems below, then solve them.
If you rely on a theorem please state it carefully!
Good Luck!

1. Let f, g be two differentiable real functions on $\mathbb{R}$ such that, for some $a < b < c$,

$$f(a) > g(a) \quad \text{and} \quad f(b) < g(b) \quad \text{and} \quad f(c) > g(c).$$

a) Show that $f(x) = g(x)$ for some $x \in \mathbb{R}$.
b) Show that $f'(x) = g'(x)$ for some $x \in \mathbb{R}$.

2. Consider $(a_n), (b_n) \subset \mathbb{R}$ satisfying $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} b_n = 0$.
a) Show that, for any $m \in \mathbb{N}$, $\lim_{n\to\infty} a_n + b_n + \ldots + b_{n-m} = \infty$.
b) Give an example with $\lim_{n\to\infty} a_n + b_n + \ldots + b_1 \neq \infty$.

3. We are interested in subsets $A, B \subset \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$\sup(A \cap (-\infty, x]) = \sup(B \cap (-\infty, x]).$$

a) Show that the closures $\overline{A}$ and $\overline{B}$ are equal, $\overline{A} = \overline{B}$.
b) Give an example of such A and B with $A \neq B$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given as the sum of an infinite series

$$f(x) = \sum_{k=0}^{\infty} a_k \sin^2(kx)$$

where $(a_k) \subset \mathbb{R}$ is such that $\lim_{k\to\infty} a_k k^3 = 0$.
a) Prove that f is uniformly continuous on $\mathbb{R}$.
b) Prove that f is differentiable and f' is continuous.

5. Let (f_n) be a sequence of real valued continuous functions on $[0, 1]$ such that, for every natural $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_0^1 f_n(x) x^k dx = 0.$$

a) Argue that, for every continuous $\phi : [0, 1] \to \mathbb{R}$, it must be that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\phi(x) dx = 0.$$

b) Must it be that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$? Justify your answer.