

Problem 1. Suppose that E is a bounded subset of \mathbb{R} and that $f : E \rightarrow \mathbb{R}$ is uniformly continuous. Prove that f is bounded.

Problem 2. Suppose that $f : [0, \frac{1}{2}] \rightarrow \mathbb{R}$ is continuous. Calculate (with justification) $\lim_{n \rightarrow \infty} \int_0^{\frac{1}{2}} f(x^n) dx$.

Problem 3. Prove that

$$\lim_{x \rightarrow 0} \frac{\int_x^{2x} \sum_{n=1}^{\infty} \frac{\sin(nt)}{n^3 t} dt}{x} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Problem 4. Suppose that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, that $\frac{\partial f}{\partial y}(x, y)$ exists for all $(x, y) \in \mathbb{R}^2$, and that $\left| \frac{\partial f}{\partial y}(x, y) \right| \leq \frac{1}{2}$ for all $(x, y) \in \mathbb{R}^2$. Prove that there is a unique continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $g(x) = f(x, g(x))$ for all $x \in \mathbb{R}$.
