Real Analysis Master Comprehensive Exam
(Dec 2015)

Name:

Pick and circle four out of the five problems below, then solve them. If you rely on a theorem please state it carefully! Good Luck!

1. Construct a series $\sum_{k=0}^{\infty} a_k$ such that $(-1)^k a_k > 0$ for all $k$, $a_k \to 0$ as $k \to \infty$, but the series diverges.

2. Let $(x_n)$ be a sequence in $\mathbb{R}$. Prove that $(x_n)$ has a monotone subsequence.

3. Let $M$ be a compact metric space and $A$ be a dense subset of $M$. Prove that for any $\delta > 0$, there exists a finite subset $\{a_1, \ldots, a_k\} \subset A$ which is $\delta$-dense in $M$ in the sense that each $x \in M$ lies within distance $\delta$ of at least one of the points $a_j, j = 1, \ldots, k$.

4. Let $(f_n)$ be a sequence of differentiable functions defined on $[0,1]$, and assume that for all $n$, $f_n(0) = f'_n(0)$. Suppose also that for all $n$ and $x \in [0,1]$, $|f'_n(x)| \leq 1$. Prove that there is a subsequence of $(f_n)$ converges uniformly on $[0,1]$.

5. Let $f(x) : [0,1] \to \mathbb{R}$, be a continuous function with continuous derivative $f'(x)$, and there exists a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in [0,1]$. Prove that for any $n \in \mathbb{N}$,

$$\left| \frac{1}{n} \sum_{k=1}^{n-1} f \left( \frac{k}{n} \right) - \int_{0}^{1} f(x) \, dx \right| \leq \frac{M}{2n}$$