1. A small pond is stocked with game fish, both trout and bass. Let \( x(t) \) denote the population of the trout in the pond at any time \( t \) and let \( y(t) \) denote the population of the bass in the pond at any time \( t \). Denote the fundamental dimensions as \([x] = [y] = P\), and \([t] = T\). Assume that without the bass, the trout population could grow indefinitely (growth rate proportional to the population \( x \) with proportionality constant \( k_1 \)). However, the bass population decreases the growth rate of the trout population in a manner proportional to the number of possible interactions between the two species, say proportional to the product of \( x \) and \( y \) with proportionality constant \( k_2 \). A similar situation is assumed for the growth of the bass population, where the proportionality constant for the interaction term is \( k_2 \) again. If the trout population is initially \( x_0 \) and the bass population is initially \( y_0 \), a mathematical model that describes \( x(t) \), the trout population, and \( y(t) \), the bass population, at time \( t \geq 0 \) is given by

\[
\frac{dx}{dt} = k_1 x - k_2 xy, \quad x(0) = x_0 \\
\frac{dy}{dt} = k_3 y - k_2 xy, \quad y(0) = y_0
\]

(a) Use dimensional analysis to determine the fundamental dimensions of all the variables and parameters. Identify the fundamental dimensions of \( x, y, t, k_1, k_2, k_3, x_0, \) and \( y_0 \).

(b) Determine how many independent dimensionless variables and parameters are necessary to describe this physical process.

(c) Scale the problem to arrive at a dimensionless model involving dimensionless populations \( X \) and \( Y \) and dimensionless time \( s \). Carefully identify the dimensionless parameters.

2. Consider the nonlinear initial-value problem

\[
y''(x) + y'(x) + \varepsilon y^2 = 0 \\
y(0) = 1 \\
y'(0) = 0
\]

where \( \varepsilon \) is a small positive parameter.

(a) Carefully identify the leading order and first-order initial-value problems to be solved for a perturbation solution.

(b) Solve the two initial-value problems above.

(c) Give the two-term perturbation solution which involves the leading order approximation and the first-order correction.
3. Consider the isoperimetric problem

\[ J(y) = \int_0^1 xy(x)dx \]

subject to the constraint

\[ W(y) = \int_0^1 (y(x))^2 dx = \frac{1}{12} . \]

Define the functional \( J^*(y) = J(y) + \lambda W(y) \) and the class of admissible functions \( A = \{ y \in C[1, 2] : W(y) = \frac{1}{12} \} \).

(a) Write down the Euler equation for this problem and show that the extremals in \( A \) are \( y_0(x) = \pm x/2 \).

(b) Show that \( J(y_0 + y) = J(y_0) + J(y) \).

(c) Show that if \( y_0 + y \) is in the admissible class \( A \) then \( W(y_0 + y) = W(y_0) + W(y) \pm J(y) \) for either extremal \( y_0 \).

(d) Beginning with \( y_0 = \pm x/2 \in A \), assume that \( y \) is such that \( y_0 + y \) is in \( A \). Then \( W(y_0) = W(y_0 + y) = \frac{1}{12} \). Use this and part (c) to show that \( W(y) = \mp J(y) \).