APPLIED MATH MASTER'S EXAM

January 2006

Directions: Attempt each question, and justify your answers to obtain full credit.

1. (a) Use standard techniques (i.e., not the Poincare-Lindstedt method) to find a two-term regular perturbation approximation to the solution of the ODE IVP

$$\frac{d^2y}{dt^2} + \epsilon\frac{dy}{dt} + y = 0, \qquad t > 0$$
$$y(0) = 1, \ \ \frac{dy}{dt}(0) = 0, \ \ 0 < \epsilon \ll 1.$$

Call the approximation $y_\epsilon(t)$.

   (b) Is it possible for $y_\epsilon(t)$ to converge uniformly to the true solution to the ODE IVP on $0 \le t < \infty$ as $\epsilon \to 0$? Explain why or why not.

2. Find the extremal for the functional

$$J(y) = \frac{1}{2}\int_0^1 [y'(x)^2 + y(x)^2]\, dx - 2y(1)$$

subject to the constraints $y \in C^2[0,1]$ and $y(0) = 0$.

3. (a) Compute the eigenvalues and corresponding eigenfunctions for the differential operator

$$Lu(x) = \frac{d^2u}{dx^2}, \quad 0 < x < 1,$$

with associated homogeneous Dirichlet boundary conditions

$$u(0) = 0, \ u(1) = 0.$$

   (b) Use part (a) to obtain a series representation for the solution to the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0$$
$$u(x,0) = f(x), \qquad 0 < x < 1.$$

4. Use Fourier transform methods to obtain an expression for the solution to the following evolution equation in terms of the initial data $f(x)$.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \ t > 0,$$
$$u(x,t) \to 0, \qquad x \to \pm\infty, \ t > 0,$$
$$u(x,0) = f(x), \qquad -\infty < x < \infty.$$