APPLIED MATH MASTER'S EXAM

January 2011

Instructions: Attempt 4 of the following 6 questions. Show all work. Clearly label your work and attach it to this sheet.

1. Find a two term expansion of the positive singular root X of
$$\epsilon x^3 - x + \epsilon = 0$$

2. Let $y(x,\epsilon)$ be the solution of the boundary value problem:
$$\begin{aligned} \epsilon y'' + y' + y^2 &= 0 \quad , \quad x \in (0,1) \\ y(0) &= \frac{1}{4} \quad , \quad y(1) = \frac{1}{2} \end{aligned}$$
Find the outer, inner and uniformly valid approximations of y in the limit $\epsilon \to 0$ assuming a layer exists at $x = 0$.

3. A functional $J : \mathcal{A} \to \mathbb{R}$ where
$$J(y) \equiv \int_0^1 xy(x) + \frac{1}{2}y'(x)^2 \, dx$$
and
$$\mathcal{A} = \left\{ y \in C^2[0,1] \ : y(0) = 1 \right\}$$
   a) Derive the natural boundary condition for $y(x)$.
   b) Find the unique extrema $\bar{y}(x)$ of J over $\mathcal{A}$.

4. Define the operator L and domain D by:
$$\begin{aligned} Lu &\equiv u'' \\ D &\equiv \{u \in C^2[0,\pi] : u(0) = u'(\pi) = 0\} \end{aligned}$$
   a) Find all the eigenvalues $\lambda_n > 0$ and normalized eigenfunctions $\phi_n(x)$ of L
   b) Find the series representation $g(x,\zeta)$ of the Green's function solving:
$$Lu = f(x) \qquad , \qquad u(x) = \int_0^\pi g(x,\zeta) d\zeta$$

5. Find the solution of the integral equation:
$$\int_0^1 e^{x+y} \, u(y) \, dy + u(x) = e^{-x}$$

6. Use Laplace convolution Theorem to find the bounded general solution of
$$\begin{aligned} u_{tt} &= u_{xx} + g(t) \quad , \quad x > 0 \quad , \quad t > 0 \\ u(x,0) &= u_t(x,0) = u(0,t) = 0 \end{aligned}$$