1. Find a two term expansion of the positive singular root $X$ of

$$e x^3 - x + \epsilon = 0$$

2. Let $y(x, \epsilon)$ be the solution of the boundary value problem:

$$\epsilon y'' + y' + y^2 = 0, \quad x \in (0, 1)$$

$$y(0) = \frac{1}{4}, \quad y(1) = \frac{1}{2}$$

Find the outer, inner and uniformly valid approximations of $y$ in the limit $\epsilon \to 0$ assuming a layer exists at $x = 0$.

3. A functional $J : \mathcal{A} \to \mathbb{R}$ where

$$J(y) \equiv \int_0^1 x y(x) + \frac{1}{2} y'(x)^2 \, dx$$

and

$$\mathcal{A} = \{y \in C^2[0, 1] : y(0) = 1\}$$

a) Derive the natural boundary condition for $y(x)$.

b) Find the unique extrema $\bar{y}(x)$ of $J$ over $\mathcal{A}$.

4. Define the operator $L$ and domain $D$ by:

$$Lu \equiv u''$$

$$D \equiv \{u \in C^2[0, \pi] : u(0) = u'(\pi) = 0\}$$

a) Find all the eigenvalues $\lambda_n > 0$ and normalized eigenfunctions $\phi_n(x)$ of $L$

b) Find the series representation $g(x, \zeta)$ of the Green’s function solving:

$$Lu = f(x), \quad u(x) = \int_0^\pi g(x, \zeta) d\zeta$$

5. Find the solution of the integral equation:

$$\int_0^1 e^{x+y} u(y) \, dy + u(x) = e^{-x}$$

6. Use Laplace convolution Theorem to find the bounded general solution of

$$u_{tt} = u_{xx} + g(t), \quad x > 0, \quad t > 0$$

$$u(x, 0) = u_t(x, 0) = u(0, t) = 0$$