Instructions: Attempt all questions. Show all work.

1. Consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by
   
   $$x \to f(x) = \frac{x}{\mu + x}$$

   where $\mu$ is a real parameter.

   a) Find all the fixed points of this map and determine the $\mu$ for which they are stable.

   b) Does this map have a minimal period 2 orbit for any $\mu$? If so, what $\mu$?

2. In polar coordinates $(r, \theta)$ planar flow is described by:

   $$\frac{dr}{dt} = r(\mu + r^2 - r^4)$$  \hspace{1cm} (1)

   $$\frac{d\theta}{dt} = 1 + r^2 \sin^2 \theta$$  \hspace{1cm} (2)

   where $\mu$ is a real parameter.

   a) Draw a bifurcation diagram for the system (1)-(2) in the $(\mu, r)$-plane labelling the stability of all periodic orbits and fixed points (Use a solid line for stable and a dashed line for unstable).

   b) Does the system have a Hopf bifurcation? Explain.

   c) Draw a qualitatively accurate phase portrait in the $(x, y)$-plane for $\mu = -\frac{1}{8}$.

3. The following questions apply to the planar system:

   $$\frac{dx}{dt} = x(4 - y - x^2)$$  \hspace{1cm} (3)

   $$\frac{dy}{dt} = y(1 - x)$$  \hspace{1cm} (4)

   a) Determine the coordinates and stability of all fixed points. Label the location of these fixed points in the $xy$-plane along with the $x$ and $y$ nullclines.

   b) Does this system have any periodic orbits? Explain.