M.S. Dynamical Systems Exam 2006

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Instructions: Attempt all questions. Show all work.

1. Define the system:

$$\begin{aligned} \dot{x} &= y^3 - \mu x \quad , \quad \mu \in \mathbb{R} \\ \dot{y} &= -\frac{1}{2}\, x - y \end{aligned}$$

a) Find the Hamiltonian for this system when $\mu = -1$.

b) Show that for $\mu > 0$ and an appropriate value of m, $V = x^2 + y^m$ is a Liapunov function for the system.

c) Does the system have any periodic orbits for $\mu > 0$? Explain.

d) Draw a bifurcation diagram of the system's equilibria in the (μ, y)-plane.

2. In polar coordinates (r, θ) planar flow is described by:

$$\frac{dr}{dt} = f(r, \mu) = r(\mu - r^2) \quad , \quad \mu \in \mathbb{R} \tag{1}$$

$$\frac{d\theta}{dt} = g(\theta, \mu) = \mu - \sin\theta \tag{2}$$

a) Draw qualitatively accurate phase portraits in the (x, y)-plane for $\mu = \frac{1}{2}$ labelling the stable and unstable manifolds $W^s(p)$ and $W^u(p)$ of any saddles p as they occur.

b) Summarize the stability of all periodic orbits and fixed points (as they exist) for all μ.

c) For what μ (if any) does the system have a homoclinic orbit?

3. Consider the Tent Map $x \mapsto f(x)$ where

$$f(x) = \begin{cases} 2x & , \ x \in [0, 1/2] \\ 2 - 2x & , \ x \in [1/2, 1] \end{cases}$$

a) Draw an accurate graph of the second iterate map $f^2(x)$.

b) The map has a period 2-orbit $\gamma(a) = \{a, b, a, b, \ldots\}$. Determine a and b

c) Determine all those initial conditions x_0 such that the orbit $\gamma(x_0)$ is eventually periodic to the periodic orbit $\gamma(a)$ after the second iterate. That is, the orbit $\gamma(x_0)$ should look like one of the following:

$$\gamma(x_0) = \{x_0, a, b, a, b\}$$

or

$$\gamma(x_0) = \{x_0, b, a, b, a\}$$