Masters Test

Linear Algebra - August, 2016

Do three of the following four problems.

1. Let $\mathcal{M}_{2\times 2}$ be the set of all 2×2 matrices and let E_{ij} be the 2×2 matrix with a 1 in the ij position and zeros elsewhere. That is,

$$\begin{aligned}\mathcal{M}_{2\times 2} &= \text{span}\left\{E_{ij} : \left\{\begin{array}{c} i=1,2 \\ j=1,2\end{array}\right\}\right\} = \text{span}\left\{B_M\right\} \\ &= \text{span}\left\{E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix},\right. \\ &\qquad\left. E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}\right\}.\end{aligned}$$

The set $B_M = \{E_{ij}\}_{i=1,1}^{2,2}$ is a basis for $\mathcal{M}_{2\times 2}$. Let $B_P = \{1, x, x^2, x^3\}$ be a basis for $\mathcal{P}_3$. Define the linear transformation $T : \mathcal{P}_3 \longrightarrow \mathcal{M}_{2\times 2}$ by

$$T(p(x)) = \begin{bmatrix} p(0) & p'(1) - 2p(1) \\ p^{iv}(17) & p''(-1)\end{bmatrix}, \quad p^{iv}(x) = \frac{d^4 p(x)}{dx^4}.$$

(a) Find the null space of T.

(b) Find a basis for the range of T.

(c) Find the matrix of T with respect to the given basis.

2. Let $\mathcal{M}_{2\times 2}$ denote the set of all 2×2 matrices. If $M_a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$ and $M_b = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22}\end{bmatrix}$ then

$$\langle M_a, M_b\rangle = \sum_{i=1}^{2}\sum_{i=1}^{2} a_{ij}b_{ij}$$

defines an inner product on $\mathcal{M}_{2\times 2}$. Let $\mathcal{S} \subset \mathcal{M}_{2\times 2}$ be defined by

$$\mathcal{S} = \text{span}\left\{\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{bmatrix} : a_{11}, a_{12}, a_{22} \text{ are arbitrary real numbers}\right\}.$$

(a) Write out a basis for $\mathcal{S}$?

(b) Find $\mathcal{S}^{\perp} = \{M \in \mathcal{M}_{2\times 2} : \langle M, \mathcal{L}\rangle = 0 \text{ for all } \mathcal{L} \in \mathcal{S}\}$.

3. An $n \times n$ matrix is called a permutation matrix if its columns are the same as the rows of the identity matrix in an arbitrary order. For example if $n = 3$ there are six permutation matrices

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Is the set $\mathcal{P} = \{P_j\}_{j=1}^{6}$ a linearly independent set. If yes why? If no, find a dependence relation among the P_j. Hint: Using the check for linear independence (or dependence), finish writing out the matrix (find the $*$'s)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \sum_{j=1}^{6} a_j P_j = \begin{bmatrix} a_1 + a_2 & a_3 + a_4 & * \\ a_3 + a_5 & * & * \\ * & * & * \end{bmatrix} \equiv A.$$

For problem # 4, you are given the following information: Assume λ is an eigenvalue for the linear operator, $T \in \mathcal{L}(V)$. If $v \in \mathcal{N}(T - \lambda I)$ so that $(T - \lambda I)v = 0$ and if $w \in \mathcal{N}((T - \lambda I)^2)$ but $w \notin \mathcal{N}(T - \lambda I)$ then $(T - \lambda I)w = v$.

4. Assume $\dim \mathcal{V} = 3$, $T \in \mathcal{L}(\mathcal{V})$, $Tv_1 = \lambda_1 v_1$, $Tv_2 = \lambda_2 v_2$, $\lambda_1 \neq \lambda_2$ and $\dim \mathcal{N}(T - \lambda_2 I) = 1$. One can show (you are not required to show this) that there is a basis for $\mathcal{V}$, say $\mathcal{U} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, so that the matrix for T with respect to the basis $\mathcal{U}$ is upper triangular $[\,T\,]_{\mathcal{U}} = \begin{bmatrix} \lambda_1 & 0 & r_{13} \\ 0 & \lambda_2 & r_{23} \\ 0 & 0 & \lambda_2 \end{bmatrix}$.

   (a) Show that the vector v_3 in $\mathcal{U}$ can be selected so that the matrix representation of T in the new basis, call it $\mathcal{W}$, is $[\,T\,]_{\mathcal{W}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$.
   Hint: Compare $\mathcal{N}([\,T\,]_{\mathcal{U}} - \lambda_2 I)$ with $\mathcal{N}(([\,T\,]_{\mathcal{U}} - \lambda_2 I)^2)$.

   (b) Carry out (a) for $T : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 + x_3 \\ 2x_1 + 3x_2 + 3x_3 \\ x_1 + 4x_3 \end{bmatrix}$. Hint: You may assume $\lambda_1 = 5$ and $\lambda_2 = 3$