Linear Algebra Masters Comprehensive Exam August 2015
Assume that all Vector Spaces are Finite Dimensional
Do Four of the following Six Problems and show all work.

1. Define \( T \in \mathcal{L}(P_2(\mathbb{R})) \) by \( T(f) = x \frac{df}{dx} \), so for example \( T(x^2) = x(2x) = 2x^2 \).
   
   (a) Find the matrix of \( T \) with respect to the basis \((1, x, x^2)\).
   
   (b) Find \( T^* \).
   
   (c) Find \( N_{T^*} \).
   
   (d) Now make \( P_2(\mathbb{R}) \) into an inner-product space by defining

   \[
   \langle p, q \rangle = \int_0^1 p(x)q(x) \, dx.
   \]

   Is \( T \) self-adjoint?

2. Let

   \[
   A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 6 \end{bmatrix}.
   \]

   (a) Find \( N_{AT} \).
   
   (b) Find conditions on \( b \) the guarantee that \( Ax = b \) has a solution.

3. Prove that if \( T \in \mathcal{L}(V) \) is normal, then

   \[ \text{range } T = \text{range } T^* \]

4. Prove that any linear map on a subspace of \( V \) can be extended to a linear map on \( V \). In other words, show that if \( U \) is a subspace of \( V \) and \( S \in \mathcal{L}(U, W) \), then there exists \( T \in \mathcal{L}(V, W) \) such that \( Tu = Su \) for all \( u \in U \).

5. Suppose that \( T \in \mathcal{L}(V, W) \) is injective (1-to-1) and \((v_1, \ldots, v_n)\) is linearly independent in \( V \). Prove that \((Tv_1, \ldots, Tv_n)\) is linearly independent in \( W \).

6. Prove that if \( T \in \mathcal{L}(V) \) is self-adjoint, then the singular values of \( T \) equal the absolute values of the eigenvalues of \( T \) (repeated appropriately).