# STAT 501/502 Comprehensive Exam
## Thursday, January 5, 2017

**Note: Start each problem on a new page. In the interest of time, for any problem requiring derivation of an MLE you do not have to confirm that you have found a global maximizer using a second derivative test.**

1. (5pts) An experiment has a sample space with a countably infinite number of outcomes.

   True or False: All points in the sample space can have a positive probability of occurring. Justify your answer.

2. (15 pts) Let $X$ and $Y$ be 2 continuous random variables with joint pdf

$$f(x, y) = C(1-x)y^2$$

   where $-1 < y < 1$, $0 < x < 1$ , $0 < y^2 < x$, and $C = 105/8$. Find the marginal distribution for $X$ and then find the distribution of $V = -\log(X)$.

3. (5 pts) Let $X \sim Geom(p)$. Let $Y = X - 1$. Use the moment generating function of $X$, $M_X$ to find $M_Y$ and, based on that result, deduce the probability distribution of $Y$.

4. Let $X_1, \cdots, X_n$ be a sample from a normal distribution with mean $\mu$, and variance $\sigma^2$. Let $Cov(X_i, X_j) = \sigma^2\rho$ for $i \neq j$ and $\rho > 0$. We wish to use $\overline{X}$ to estimate $\mu$.

   (a) (3 pts) Is $\overline{X}$ still an unbiased estimator of $\mu$? Justify your answer.

   (b) (5 pts) Show that

$$Var\left(\overline{X}\right) = \frac{\sigma^2}{n}\left(1 + (n-1)\rho\right)$$

   and compare this to the variance under an assumption of independence.

   (c) (5 pts) Is $\overline{X}$ a consistent estimator of $\mu$ when the data have the covariance structure given above? Justify your answswer.

5. Let $X_1, X_2, \cdots, X_n \overset{iid}{\sim} Expon(\theta)$.

   (a) (3 pts) Give the MLE of $\theta$. You can state it without proof if you remember it).

   (b) (5 pts) Find the MLE of $\tau(\theta) = P(X > 1)$ where $X$ follows the exponential distribution.

   (c) (5 pts) Find the CRLB for unbiased estimators of $\tau(\theta)$.

6. (10 pts) Let $X_1, X_2, \cdots, X_n \overset{iid}{\sim} N(\mu, \theta)$. The sample variance, an estimator of $\theta$, is

$$S^2 = \frac{1}{n-1}\sum\left(X_i - \overline{X}\right)^2$$

   (a) Show that

$$\sqrt{n}\left(S^2 - \theta\right) \overset{d}{\to} N(0, 2\theta^2)$$

   The following two results (which you do not need to prove) may be useful:

   i. $\frac{(n-1)S^2}{\theta} \sim \chi^2_{n-1}$

ii. If $U_\nu \sim \chi^2_\nu$ then

$$Z_\nu = \frac{U_\nu - \nu}{\sqrt{2\nu}} \xrightarrow{d} N(0,1)$$

(Also: Remember Slutsky's Theorem).

(b) (5 pts) Use the preceding result to construct an approximate $100(1-\alpha)\%$ CI for $\theta$. A Wald interval is fine.

7. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with pdf

$$f_X(x;\theta) = \frac{3x^2}{\theta}\exp(-x^3/\theta) I_{(0,\infty)}(x)$$

for $\theta > 0$.

(a) (5 pts) Show that $T = \sum X_i^3$ is a complete sufficient statistic for $\theta$.

(b) (10 pts) Find a UMVUE for $\theta$. Hint: $Y = X^3 \sim Expon(\theta)$.

8. (15pts) Let $X_1, \cdots, X_n \overset{iid}{\sim} Geom(p)$ Find the form of the test statistic for a likelihood ratio test of $H_0 : p = 1/2$ versus the alternative that $p \neq 1/2$. You do not need to simplify the test statistic but describe how you could use it to conduct an approximate large sample level $\alpha$ test of the null hypothesis based on a chi-squared distribution.