**M.S. Numerical Exam 2002**  
(DEPARTMENT OF MATHEMATICAL SCIENCES, M.S.U.)

**Instructions:** Attempt all questions. Show all work.

1. Consider the system

\[
\begin{align*}
    x^2 + \frac{1}{3}y^3 &= 9, \\
    y^2 - x &= 9.
\end{align*}
\]

Let \( x^{(n)} = (x_n, y_n)^T, n \geq 0 \), be the \( n \)-th iterate of a Newton’s Method approximation of a root of (1)-(2). Compute \( x^{(1)} \) for the initial guess \( x^{(0)} = (0, 1)^T \).

2. Let \( U \in \mathbb{R}^{n \times n} \) be a nonsingular upper triangular matrix and \( e_j \) be the \( j \)-th column of the \( n \times n \) identity matrix. Solutions \( x \) of \( Ux = e_j \) have the form \( x = (x_1, x_2, \ldots, x_j, 0, \ldots, 0)^T, 1 \leq j \leq n \). Given this special solution structure, how many flops are required to solve

\[ Ux = e_j + e_{j-1} \]

using backward substitution? Assume that all upper off diagonal elements of the matrix \( U \) are nonzero.

3. Let \( Q(f) \) be the 2-point quadrature approximation

\[ Q(f) = f \left( -\sqrt{3} \right) + f \left( \sqrt{3} \right) \]

of the integral \( I(f) \equiv \int_{-1}^{1} f(x)dx \).

   a. Define the term **degree of accuracy** for the quadrature rule \( Q \).

   b. Through explicit calculation, compute the degree of accuracy for the quadrature rule \( Q \).

4. Consider the scalar initial value problem

\[
\begin{align*}
    \frac{dy}{dt} &= f(y, t) \\
    y(0) &= 0
\end{align*}
\]

The Runge-Kutta method of order 2 known as Heun’s Method is defined by the following

\[ w_{i+1} = w_i + \frac{h}{2} \left[ f(w_i, t_i) + f(w_i + hf(w_i, t_i), t_{i+1}) \right], \quad (3) \]

for \( i = 0, 1, 2, \ldots \) where \( t_i = ih, h > 0 \) is the step size for the method and \( w_i \) is the approximation for \( y(t_i) \). Show that if \( f(0, t) = 0 \) the global error at \( t_i \) for the method is \( |y(t_i)| \).