M.S. Numerical Exam 2014

(Department of Mathematical Sciences, M.S.U.)

Instructions: Attempt 4 of the 6 questions. Show all work. Carefully Read and Follow Directions. Clearly label your work and attach it to this sheet.

1. Given a positive definite matrix $A \in \mathbb{R}^{n\times n}$, define the A-norm on $\mathbb{R}^n$ by

$$\|x\|_A = \sqrt{x^T A x}$$

   Show that this is indeed a norm on $\mathbb{R}^n$.

2. Using the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

   (a) Compute a reduced SVD factorization.

   (b) Compute a full SVD factorization.

3. Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (a) Compute a reduced QR factorization $A = \hat{Q}\hat{R}$.

   (b) Compute a full QR factorization $A = QR$.

   (c) Let $B \in \mathbb{R}^{n\times n}$ be a matrix with the property that columns $1, 3, 5, 7, \ldots$ are orthogonal to columns $2, 4, 6, 8, \ldots$. You also may assume that B has full rank. In the QR factorization of $B = QR$, describe the special structure that R possesses.

4. For the following IVP

$$y' = f(t, y)$$
$$y(t_0) = y_0,$$

   (a) Compute the Local Truncation Error of the implicit multistep method given by

$$y_{n+2} = \frac{4}{3}y_{n+1} - \frac{1}{3}y_n + \frac{2}{3}kf(t_{n+2}, y_{n+2})$$

   (b) Is the method Consistent?

   (c) Give the characteristic polynomial $\rho(\zeta)$, and assess the Zero-Stability property of this method.

   (d) Comment on the convergence properties of this method.

5. (a) Determine the general solution to the linear difference equation

$$2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = 0$$

Hint: One root of the characteristic polynomial is at $\zeta = 1$.

(b) Determine the solution to this difference equation with the starting values $U^0 = 11$, $U^1 = 5$, and $U^2 = 1$. What is U^{10}?

(c) For the following IVP

$$y' = f(t, y)$$
$$y(t_0) = y_0,$$

approximate $y(t)$ by applying the LMM described by

$$2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = k(\beta_0 f(U^n) + \beta_1 f(U^{n+1})).$$

For what values of β_0 and β_1 is local truncation error $\mathcal{O}(k^2)$?

(d) Suppose you use the values of β_0 and β_1 just determined in this LMM. Is this a convergent method?

6. For the two-point BVP described by

$$u'' - u = f(x), \quad x \in (0, 1)$$
$$u(0) = 0, \quad u(1) = 0,$$

give the general form of the linear system of equations that you would solve in order to approximate $u(x)$ using the finite difference technique of a 2nd order centered difference for the derivative approximation with a step size of h. Show that the system is guaranteed to be uniquely solvable.