1. Given a positive definite matrix \( A \in \mathbb{R}^{n \times n} \), define the \( A \)-norm on \( \mathbb{R}^n \) by

\[
\| x \|_A = \sqrt{x^T A x}
\]

Show that this is indeed a norm on \( \mathbb{R}^n \).

2. Using the matrix \( A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \)

(a) Compute a reduced SVD factorization.
(b) Compute a full SVD factorization.

3. Consider the matrix \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \)

(a) Compute a reduced QR factorization \( A = \hat{Q} \hat{R} \).
(b) Compute a full QR factorization \( A = QR \).
(c) Let \( B \in \mathbb{R}^{n \times n} \) be a matrix with the property that columns 1, 3, 5, 7, \ldots are orthogonal to columns 2, 4, 6, 8, \ldots. You also may assume that \( B \) has full rank. In the QR factorization of \( B = QR \), describe the special structure that \( R \) possesses.

4. For the following IVP

\[
y' = f(t, y) \\
y(t_0) = y_0,
\]

(a) Compute the Local Truncation Error of the implicit multistep method given by

\[
y_{n+2} = \frac{4}{3}y_{n+1} - \frac{1}{3}y_n + \frac{2}{3}k f(t_{n+2}; y_{n+2})
\]

(b) Is the method Consistent?
(c) Give the characteristic polynomial \( \rho(\zeta) \), and assess the Zero-Stability property of this method.
(d) Comment on the convergence properties of this method.
5. (a) Determine the general solution to the linear difference equation

\[ 2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = 0 \]

**Hint:** One root of the characteristic polynomial is at \( \zeta = 1 \).

(b) Determine the solution to this difference equation with the starting values \( U^0 = 11, U^1 = 5, \) and \( U^2 = 1 \). What is \( U^{10} \)?

(c) For the following IVP

\[ y' = f(t, y) \]
\[ y(t_0) = y_0, \]

approximate \( y(t) \) by applying the LMM described by

\[ 2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = k(\beta_0 f(U^n) + \beta_1 f(U^{n+1})). \]

For what values of \( \beta_0 \) and \( \beta_1 \) is local truncation error \( O(k^2) \)?

(d) Suppose you use the values of \( \beta_0 \) and \( \beta_1 \) just determined in this LMM. Is this a convergent method?

6. For the two-point BVP described by

\[ u'' - u = f(x), \quad x \in (0, 1) \]
\[ u(0) = 0, \quad u(1) = 0, \]

give the general form of the linear system of equations that you would solve in order to approximate \( u(x) \) using the finite difference technique of a 2nd order centered difference for the derivative approximation with a step size of \( h \). Show that the system is guaranteed to be uniquely solvable.