Topology

Master's Exam

January 10, 2007

Instructions: Work at most one problem per side of the furnished paper.

(1) Suppose A, X, and Y are topological spaces. Give $X \times Y$ the product topology. Suppose $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are given by $(x, y) \overset{\pi_X}{\mapsto} x$ and $(x, y) \overset{\pi_Y}{\mapsto} y$. Prove that a function $f : A \to X \times Y$ is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.

(2) Suppose (X, d) is a metric space such that the set $A := \{d(x_1, x_2) : \ x_i \in X.\ i = 1, 2\}$ is the closed interval $[0, 1]$.

   (a) Is X necessarily compact?

   (b) Is X necessarily connected?

(3) Suppose $\{0, 1\}$ has the discrete topology, and the countably infinite product $X = \prod_{n \in \mathbb{N}} \{0, 1\}$ has the product topology . Find the closures of the following subsets of X:

   (a) $A = \{(0, 0, 0, \dots),\ (1, 0, 0, \dots),\ (1, 1, 0, 0, \dots),\ (1, 1, 1, 0, 0, \dots), \dots\}$.

   (b) $B = \{(x_n) : \ x_n = 0$ for infinitely many values of n $\}$

(4) Suppose $X := \mathbb{R}^2 - (y\text{-axis})$ has the subspace topology.

   (a) Show that X is locally compact.

   (b) Define $\mathbb{S}^2$ to be the one-point compactification of $\mathbb{R}^2$. Is the one-point compactification $\overline{X}$ of X homeomorphic to $\mathbb{S}^2$?