Masters Comprehensive Exam - Topology
January 5, 2008

CHOOSE THREE

(1) Suppose that \( f : X \to Y \) is continuous and \( Y \) is Hausdorff. Prove that \( \{ (x_1, x_2) : f(x_1) = f(x_2) \} \) is a closed subset of \( X \times X \) (with the product topology).

(2) Suppose that \( f : X \to Y \) is a continuous surjection. Prove:
   a: If \( X \) is compact then \( Y \) is compact.
   b: If \( X \) is connected then \( Y \) is connected.

(3) Let \( X \) be the subspace of \( \mathbb{R}^3 \), \( X = \{ (x, y, z) : x^2 + y^2 \leq 1, z = -1 \} \cup \{ (x, y, z) : x^2 + y^2 \leq 1, z = 1 \} \), and let \( \sim \) be the equivalence relation defined on \( X \) by \( (x_1, y_1, z_1) \sim (x_2, y_2, z_2) \) if and only if \( (x_1, y_1, z_1) = (x_2, y_2, z_2) \) or \( x_1 = x_2, y_1 = y_2 \), and \( x_1^2 + y_1^2 = 1 \). Prove that the quotient space \( X/\sim \) is homeomorphic with the sphere \( S^2 = \{ (x, y, z) : x^2 + y^2 + z^2 = 1 \} \).

(4) Suppose that \( \gamma : [0, 1] \to X \) is a path from \( x_0 \) to \( x_1 \). Define \( \Gamma : \Omega(X, x_0) \to \Omega(X, x_1) \) (\( \Omega(X, x_i) \) is the set of loops in \( X \) based at \( x_i \)) by

\[
\Gamma(\alpha)(t) = \begin{cases} 
\gamma(1 - 3t), & 0 \leq t \leq 1/3, \\
\alpha(3t - 1), & 1/3 \leq t \leq 2/3, \\
\gamma(3t - 2), & 2/3 \leq t \leq 1.
\end{cases}
\]

Prove that:

a: If \( \alpha \) is path homotopic to \( \beta \) then \( \Gamma(\alpha) \) is path homotopic to \( \Gamma(\beta) \).

b: \( \Gamma(\alpha \star \beta) \) is path homotopic to \( \Gamma(\alpha) \star \Gamma(\beta) \).