Masters Comprehensive Exam - Topology

January 5, 2008

CHOOSE THREE

(1) Suppose that $f : \mathbf{X} \to \mathbf{Y}$ is continuous and $\mathbf{Y}$ is Hausdorff. Prove that $\{(x_1, x_2) : f(x_1) = f(x_2)\}$ is a closed subset of $\mathbf{X} \times \mathbf{X}$ (with the product topology).

(2) Suppose that $f : \mathbf{X} \to \mathbf{Y}$ is a continuous surjection. Prove:
a: If $\mathbf{X}$ is compact then $\mathbf{Y}$ is compact.
b: If $\mathbf{X}$ is connected then $\mathbf{Y}$ is connected.

(3) Let $\mathbf{X}$ be the subspace of $\mathbb{R}^3$, $\mathbf{X} = \{(x, y, z) : x^2 + y^2 \le 1, z = -1\} \cup \{(x, y, z) : x^2 + y^2 \le 1, z = 1\}$, and let $\sim$ be the equivalence relation defined on $\mathbf{X}$ by $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if $(x_1, y_1, z_1) = (x_2, y_2, z_2)$ or $x_1 = x_2, y_1 = y_2$, and $x_1^2 + y_1^2 = 1$. Prove that the quotient space $\mathbf{X}/\sim$ is homeomorphic with the sphere $\mathbb{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.

(4) Suppose that $\gamma : [0, 1] \to \mathbf{X}$ is a path from x_0 to x_1. Define $\Gamma : \Omega(\mathbf{X}, x_0) \to \Omega(\mathbf{X}, x_1)$ ($\Omega(\mathbf{X}, x_i)$ is the set of loops in $\mathbf{X}$ based at x_i) by

$$\Gamma(\alpha)(t) = \begin{cases} \gamma(1 - 3t), & 0 \le t \le 1/3, \\ \alpha(3t - 1), & 1/3 \le t \le 2/3, \\ \gamma(3t - 2), & 2/3 \le t \le 1. \end{cases}$$

Prove that:

a: If α is path homotopic to β then $\Gamma(\alpha)$ is path homotopic to $\Gamma(\beta)$.
b: $\Gamma(\alpha \star \beta)$ is path homotopic to $\Gamma(\alpha) \star \Gamma(\beta)$.