PhD Dynamics Exam (Aug 2016)

NAME:

Pick, Circle, and Solve 5 problems. Good luck!

1. Consider a continuous map \( f : X \to X \) of a compact metric space. Show that
   a) for any \( x_0 \in X \), the omega limit set \( \omega(x_0) \) (which consist of all limits \( \lim_{k \to \infty} f^{n_k}(x_0) \)
   where \( n_k \to \infty \) is contained in \( Y := \bigcap_{n \geq 0} f^n(X) \);
   b) there exists \( x_0 \in X \) such that \( x_0 \in \omega(x_0) \).

2. Prove that, for any triple \( d_0, d_1, d_2 \) of decimal digits, there is \( n \in \mathbb{N} \) such that \( 3^n \)
   has decimal expansion that starts with the sequence, that is \( 3^n = d_0d_1d_2 \ldots \). (You can use without proof that \( \log_{10} 3 \) is irrational.)

3. For a continuous map \( f : X \to X \) of a compact metric space, show that \( f \) is
topologically transitive (i.e., has a point with dense orbit) if \( \bigcup_{n \in \mathbb{N}} f^n(U) \) is dense
in \( X \) for any non-empty open \( U \subset X \).

4. Let \( f : \mathbb{T}^2 \to \mathbb{T}^2 \) be induced by some integer matrix \( A \), i.e., \( f(x) = Ax \), with
   the arithmetic modulo one. (\( \mathbb{T}^2 \) is the usual flat torus \( \mathbb{R}^2/\mathbb{Z}^2 \).) Prove that every
   point \( x \in \mathbb{T}^2 \) with rational coordinates is pre-periodic (i.e., \( f^n(x) \) is periodic for
   some \( n \geq 0 \)). Additionally, argue that if all such points are actually periodic then
   \( A \) is invertible over the integers (i.e., \( \det(A) = \pm 1 \)).

5. Consider continuous maps of compact metric spaces \( f : X \to X \) and \( g : Y \to Y \)
   where \( g \) is a factor of \( f \) (i.e., \( \phi \circ f = g \circ \phi \) for some continuous surjective \( \phi : X \to Y \)).
   Prove that \( h_{\text{top}}(g) \leq h_{\text{top}}(f) \). Give an example of \( f, g \), and the factor map \( \phi \)
   where \( h_{\text{top}}(f) = h_{\text{top}}(g) \) even though \( f \) and \( g \) are not topologically conjugate.

6. Give an example of a subshift \( X \) that is not a subshift of finite type (SFT). (This
   includes showing that your \( X \) is not an SFT, i.e., there is not a finite forbidden set
   defining \( X \).)

7. Let \( \phi \) be the substitution \( 1 \mapsto 121, \ 2 \mapsto 1 \) and \( w = w_1w_2w_3 \ldots \) be an infinite
   fixed word, i.e., \( \phi(w) = w \). Compute the frequency of the symbol 1 in \( w \), as given
   by
   \[ \nu(1) := \lim_{n \to \infty} \frac{1}{n} \# \{ k \in \{ 1, \ldots, n \} : w_k = 1 \} \].
   You can take for granted that the limit exists.