TOPOLOGY EXAM

AUGUST 2016

Instructions. Do as many parts of each problem as you can. Show all of your work; partial credit will be given. Justify your answers and assertions.

Conventions. For each non-negative integer n, we denote the *closed (unit) n-disk* as the subspace

$$\mathbb{D}^n \ := \ \{p \in \mathbb{R}^n \mid \|p\| \leq 1\} \ \subset \ \mathbb{R}^n$$

of Euclidean n-space, which is endowed with its standard (metric) topology; we denote the *(unit) $(n-1)$-sphere* as the subspace

$$S^{n-1} \ := \ \{p \in \mathbb{R}^n \mid \|p\| = 1\} \ \subset \ \mathbb{D}^n \ \subset \ \mathbb{R}^n \ .$$

Problems

(1) Let $f : S^1 \to \mathbb{R}$ be a continuous map. Prove that there is a point $p \in S^1$ so that $f(p) = f(-p)$.

(2) Consider the subspace

$$\mathsf{Conf}_2(\mathbb{D}^2) := \{(p, q) \in \mathbb{D}^2 \times \mathbb{D}^2 \mid p \neq q\} \subset \mathbb{D}^2 \times \mathbb{D}^2 \ .$$

Prove or disprove each of the following assertions concerning the topological space $\mathsf{Conf}_2(\mathbb{D}^2)$:

(a) Every continuous real-valued function

$$f \colon \mathsf{Conf}_2(\mathbb{D}^2) \longrightarrow \mathbb{R}$$

attains its minimum.

(b) Every continuous map from the circle $\gamma \colon S^1 \to \mathsf{Conf}_2(\mathbb{D}^2)$ extends to a continuous map from the 2-disk:

$$\begin{array}{ccc} S^1 & \xrightarrow{\ \gamma\ } & \mathsf{Conf}_2(\mathbb{D}^2) \\ \downarrow & \nearrow_{\overline{\gamma}} & \\ \mathbb{D}^2 & & \end{array}$$

Hint: Notice the continuous maps

$$S^1 \longrightarrow \mathsf{Conf}_2(\mathbb{D}^2) \ , \qquad z \mapsto (z, -z) \ ,$$

and

$$\mathsf{Conf}_2(\mathbb{D}^2) \longrightarrow S^1 \ , \qquad (p,q) \mapsto \frac{p-q}{\|p-q\|} \ .$$

(c) Every continuous map from the 2-sphere $\alpha \colon S^2 \to \mathsf{Conf}_2(\mathbb{D}^2)$ extends to a continuous map from the 3-disk:

$$\begin{array}{ccc} S^2 & \xrightarrow{\ \alpha\ } & \mathsf{Conf}_2(\mathbb{D}^2) \\ \downarrow & \nearrow_{\overline{\alpha}} & \\ \mathbb{D}^3 & & \end{array} \ .$$

Hint: Elaborate on the hint from the previous part by considering the assignment

$$\mathsf{Conf}_2(\mathbb{D}^2) \times [0,1] \longrightarrow \mathsf{Conf}_2(\mathbb{D}^2) \ , \qquad ((p,q),t) \mapsto \Big(\ (1-t)p + t\frac{p-q}{\|p-q\|} \ , \ (1-t)q + t\frac{q-p}{\|p-q\|} \ \Big) \ .$$

(3) (a) For $m \neq n$ distinct non-negative integers, prove that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.
Hint: Consider the homology of $\mathbb{R}^i \setminus \{0\}$.
(b) Prove that there are no non-negative integers m, n for which S^m is homeomorphic to $\mathbb{R}^n$.

(4) Prove that there is a unique continuous function $f \colon \mathbb{R}^2 \to \mathbb{R}$ for which $f(x,y) = x^2 + \frac{y^2}{4}$ whenever the pair of real numbers (x,y) has the following property:
Either $x = 0$, or, if $x \neq 0$, the ratio $\frac{y}{x}$ is a rational number.

(5) A *rotation (of $\mathbb{R}^3$)* is a 3×3 matrix A with the following properties:
- the set of columns of A are an orthonormal basis for $\mathbb{R}^3$;
- the determinant $\mathsf{det}(A) = 1$.

Let $A_1, \ A_2, \ A_3, \ \dots$ be a sequence of rotations of $\mathbb{R}^3$. Prove that there is a rotation R of $\mathbb{R}^3$ with the following property.
For each $\epsilon > 0$, there are infinitely many i for which $\|A_i(v) - R(v)\| < \epsilon \cdot \|v\|$ for every non-zero vector $0 \neq v \in \mathbb{R}^3$.

Hint: You might want to establish, and make use of, the following assertions:
- The collection of rotations of $\mathbb{R}^3$ is a closed and bounded subset of $\mathbb{R}^9 \cong \mathsf{Mat}_{3\times 3}$, which is the set of 3×3 matrices;
- The evaluation map

$$\mathbb{R}^3 \times \mathsf{Mat}_{3\times 3} \longrightarrow \mathbb{R}^3 \ , \qquad (v, A) \mapsto A(v) \ ,$$

is continuous.

(6) Denote by M the topological space which is the (closed) Mobius band. Consider the topological space

$$W := (S^1 \times S^1) \coprod_{\partial M} M ,$$

which is obtained by gluing the torus and the Mobius band along a homeomorphism $\partial M \cong S^1 \times \{x_0\}$, for some point $x_0 \in S^1$. Compute the homology of W.