Real Analysis Comprehensive Exam

9:00 am - 1:00 pm, Monday, August 17th, 2015

Solve 4 of the following 5 problems. Clearly mark the solutions to be graded.

1. Let \( m \) denote the Lebesgue measure on \( \mathbb{R} \), and let \( 0 < \epsilon < 1 \). Show that there is a subset \( U \) of \([0,1]\) with the following properties:
   
   - \( U \) is open,
   - \( U \) is dense in \([0,1]\), and
   - \( m(U) < \epsilon \).

2. Let \((X, \mathcal{S}, \mu)\) be a measure space, and let \( 0 < p < r < q < \infty \). Show that
   
   \[ L^p(\mu) \cap L^q(\mu) \subseteq L^r(\mu). \]

3. Let \( m \) denote the Lebesgue measure on \( \mathbb{R} \). Let \( A \) and \( B \) be measurable subsets of \([0,1]\) with
   
   \[ m(A) = 1 = m(B). \]

   Show that for every \( 0 < \lambda < 1 \), there are points \( a \in A \) and \( b \in B \) with \( a - b = \lambda \).

4. Let \((X, \mathcal{S}, \mu)\) be a measure space, and let \( f \) be a non-negative measurable function in \( L^1(\mu) \).

   Show that for every \( \epsilon > 0 \), there is a number \( \delta > 0 \) such that if \( E \in \mathcal{S} \) is a measurable set satisfying \( \mu(E) < \delta \), then
   
   \[ \int_E f \ d\mu < \epsilon. \]

5. Let \((X, \mathcal{S}, \mu)\) be a measure space with \( \mu(X) < \infty \), and let \( \{f_n\}_{n\in\mathbb{N}} \) be a sequence of measurable functions on \( X \). Show that \( \{f_n\}_{n\in\mathbb{N}} \) converges to 0 in measure if and only if
   
   \[ \lim_{n \to \infty} \int_X \frac{|f_n|}{1 + |f_n|} \ d\mu = 0. \]