Departments of Mathematics
Montana State University

Fall 2015

Real Analysis Comprehensive Exam

9:00 am - 1:00 pm, Monday, August 17th, 2015

Solve 4 of the following 5 problems. Clearly mark the solutions to be graded.

1. Let m denote the Lebesgue measure on $\mathbb{R}$, and let $0 < \epsilon < 1$. Show that there is a subset U of $[0, 1]$ with the following properties:
   - U is open,
   - U is dense in $[0, 1]$, and
   - $m(U) < \epsilon$.

2. Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $0 < p < r < q < \infty$. Show that

$$\mathrm{L}^p(\mu) \cap \mathrm{L}^q(\mu) \subseteq \mathrm{L}^r(\mu).$$

3. Let m denote the Lebesgue measure on $\mathbb{R}$. Let A and B be measurable subsets of $[0, 1]$ with

$$m(A) = 1 = m(B).$$

   Show that for every $0 < \lambda < 1$, there are points $a \in A$ and $b \in B$ with $a - b = \lambda$.

4. Let $(X, \mathcal{S}, \mu)$ be a measure space, and let f be a non-negative measurable function in $\mathrm{L}^1(\mu)$. Show that for every $\epsilon > 0$, there is a number $\delta > 0$ such that if $E \in \mathcal{S}$ is a measurable set satisfying $\mu(E) < \delta$, then

$$\int_E f \, d\mu < \epsilon.$$

5. Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions on X. Show that $\{f_n\}_{n \in \mathbb{N}}$ converges to 0 in measure if and only if

$$\lim_{n \to \infty} \int_X \frac{|f_n|}{1 + |f_n|} \, d\mu = 0.$$