Complex Analysis Ph. D. Comprehensive Exam
August 2015

Solve 4 of the following 5 problems.

NOTATION: $\mathbb{C}$ is the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.

1. Let f be an analytic function which has a zero of order $N \geq 1$ at $z_0 \in \mathbb{C}$. Find the residues of

$$g(z) = \frac{f'(z)}{f(z)} \quad \text{and} \quad h(z) = zg(z)$$

at z_0.

2. Find a conformal map from the slit unit disk $D = \mathbb{D} \setminus [0, 1)$ onto the parallel strip $S = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$.

3. (a) Let $D \subseteq \mathbb{C}$ be a simply connected domain and let $f : D \to \mathbb{C}$ be an analytic function without zeros. Show that there exists an analytic function $g : D \to \mathbb{C}$ with $f = e^g$.

(b) Find a domain $D \subseteq \mathbb{C}$ and an analytic function without zeros $f : D \to \mathbb{C}$ which can not be written in the form $f = e^g$ with $g : D \to \mathbb{C}$ analytic. (Note that by the first part, D will necessarily not be simply connected.)

4. Let $S = \{z \in \mathbb{C} : |\operatorname{Re} z| < \pi/4\}$ and let $f : S \to \mathbb{D}$ be analytic with $f(0) = 0$. Show that $|f(z)| \leq |\tan z|$ for all $z \in S$. What can you say if equality holds for some $z \neq 0$?

5. Let f be an entire function with the property that for every $x \in \mathbb{R}$ there exists $n \geq 0$ with $f^{(n)}(x) = 0$. Show that f is a polynomial.