Ph.D. Comprehensive Exam - Topology

August 2015

Every answer to a question in this exam needs a proof.

(1) Let X be a compact space, Y a Hausdorff space, $f : X \to Y$ is continuous, one-to-one and onto. Prove that f is a homeomorphism.

(2) Let X be a compact Hausdorff space and let $C_1 \supset C_2 \supset \ldots$ be a decreasing sequence of closed connected subsets of X. Prove that their intersection $C = \bigcap_i C_i$ is connected. Give an example to show that this may fail if we do not assume that X is compact.

(3) **a:** Compute the fundamental group of the space obtained from the disjoint union of two spaces, each homeomorphic to the torus $S^1 \times S^1$, by identifying a circle $S^1 \times 1$ in one torus with the corresponding circle $S^1 \times 1$ in the other torus.

b: Let $X \subset R^m$ be the union of convex open sets $X_1, \ldots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all $i, j, k = 1, \ldots, n$. Show that X is connected and simply-connected.

(4) Let S^n be an n-sphere.

a: Show that S^n admits a continuous field of non-zero tangent vectors if, and only if, n is odd.

b: Suppose $f : S^n \to S^n$ is a continuous map with degree different from $(-1)^{n+1}$. Show that f has a fixed point.