1. Which of these analytic functions has a primitive (i.e., can be written as the derivative of an analytic function) in the indicated domain $D$? Justify your answers.
   (a) $f(z) = e^{-z^2}$ in $D = \mathbb{C}$.
   (b) $g(z) = e^{1/z}$ in $D = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
   (c) $h(z) = e^{1/z^2}$ in $D = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
   (Hint: Explicit integration is not the way to go in either of these.)

2. Let $f$ be a conformal map from the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ onto some domain $D \subset \mathbb{C}$. Show that $D = \mathbb{C} \setminus \{w_0\}$ for some $w_0 \in \mathbb{C}$. (Hint: What type of isolated singularities can $f$ have at 0 and at $\infty$?)

3. Let us say that a domain $D \subset \mathbb{C}$ is a circle domain if it is a bounded domain whose boundary components are disjoint circles. I.e., there exist points $z_0, \ldots, z_n \in \mathbb{C}$ and radii $r_0, \ldots, r_n > 0$ such that the closed disks $\overline{D_k} = \overline{D_{z_k}(r_k)}$ for $k = 1, \ldots, n$ are mutually disjoint subsets of $D_0 = D_{z_0}(r_0)$ with $D = D_0 \setminus \bigcup_{k=1}^n \overline{D_k}$.
   (a) Show that every analytic function $f : \overline{D} \to \mathbb{C}$ has a decomposition $f = f_0 + \ldots + f_n$ where $f_0$ is analytic in $D_0$, and each $f_k$ for $k = 1, \ldots, n$ is analytic in $\Delta_k = \mathbb{C} \setminus \overline{D_k}$ (where $\mathbb{C} = \mathbb{C} \cup \{\infty\}$, i.e., $f_1, \ldots, f_n$ are analytic at $\infty$, too.)
   (b) Show that this decomposition is unique up to constants.
   (Hint: Mimic the proof of the Laurent decomposition. Note that it is assumed for simplicity that $f$ is analytic in the closure of $D$, i.e., in a domain containing $\overline{D} = D \cup \partial D$.)

4. Let $n$ be a positive integer and consider the equation $\cos z = 3z^n$.
   (a) Find the number of solutions (counted with multiplicity) in the unit disk.
   (b) Show that these are all solutions in the strip $\{z \in \mathbb{C} : |\text{Im } z| < 1\}$.

5. Let $f : \mathbb{D} \to \mathbb{D}$ be analytic with $f(1/2) = f(-1/2) = 0$. Show that
   $$|f(z)| \leq \frac{|4z^2 - 1|}{|4 - z^2|}$$
   (Hint: Show first that the map $g(z) = (4z^2 - 1)/(4 - z^2)$ maps the unit disk to itself and fixes the unit circle. Then consider $h(z) = f(z)/g(z)$.)