Real Analysis Comprehensive PhD Exam  
(*Aug 2017*)

**NAME:**

*Pick, Circle, and Solve 4 problems. Good luck!*

1. Suppose that \( \mu \) and \( \nu \) are finite measures (on the same \( \sigma \)-algebra \( \mathcal{X} \)) and that \( \nu \ll \mu \) (i.e. \( \mu(E) = 0 \implies \nu(E) = 0 \) for all \( E \in \mathcal{X} \)). Show that, for any \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( \mu(E) < \delta \implies \nu(E) < \varepsilon \) for all \( E \in \mathcal{X} \).

2. Suppose that \( \mathcal{A} \) is a collection of subsets of a set \( X \) that is closed under countable unions and finite intersections (i.e. \( A_n \in \mathcal{A} \implies \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A} \) and \( A_1, A_2 \in \mathcal{A} \implies A_1 \cap A_2 \in \mathcal{A} \)). A function \( f : X \to \mathbb{R} \) is \( \mathcal{A} \)-**measurable** iff the sets \( \{ x \in X : f(x) > \alpha \} \) belong to \( \mathcal{A} \) for all \( \alpha \in \mathbb{R} \). Prove that, if \( f \) and \( g \) are \( \mathcal{A} \)-measurable, then \( f + g \) is \( \mathcal{A} \)-measurable. (Please, make sure to carefully justify all steps.)

3. Show that any bounded linear functional \( \xi : L^1(\mathbb{R}) \to \mathbb{R} \) can be decomposed, \( \xi = \xi_1 + \xi_2 \), into two bounded linear functionals \( \xi_1, \xi_2 : L^1(\mathbb{R}) \to \mathbb{R} \) where \( \xi_1 \) is supported on \( [0, \infty) \) and \( \xi_2 \) is supported on \( (-\infty, 0] \). (A functional \( \xi : L^1(\mathbb{R}) \to \mathbb{R} \) is supported on a segment \( J \) iff \( \xi(f) = 0 \) for any \( f \in L^1(\mathbb{R}) \) that a.e. vanishes on \( J \).)

4. Without leaning on Lebesgue Density Theorem, prove that if the Lebesgue measure \( \lambda(A) \) of a subset \( A \subset [0, 1] \) is positive then there is a segment \( J \subset \mathbb{R} \) such that \( \lambda(A \cap J) > \frac{1}{2} \lambda(J) \).

5. Let \( f : \mathbb{R} \to \mathbb{R} \) be integrable and essentially bounded. Define

\[
F(x) := \int_{-\infty}^{\infty} f(x - t)f(x + t) \, dt.
\]

(i) Show that \( F \) is a well defined element of \( L^1(\mathbb{R}) \) and that \( \| F \|_1 \leq \frac{1}{2} \| f \|_1^2 \).

(ii) Show that \( F \) is continuous.