Applied Mathematics Comprehensive Exam
Fall 2009

Instructions: Answer 3 of the problems from **Part A**, and answer 3 of the problems from **Part B**. Indicate clearly which questions you wish to be graded.

Part A

A.1 (a) Find the Singular Value Decomposition, $A = U\Sigma V^T$, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

(b) Find conditions on $\vec{y}$ for which the system

$$A\vec{x} = \vec{y}$$

has a solution (where A is as given above).

(c) Find the least squares solution of $A\vec{x} = \vec{b}$ with A given above and

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

A.2 Let $A \in \mathbb{R}^{N\times N}$ be a Symmetric Positive Definite matrix. Recall the quadratic form $q(x) = \langle Ax, x\rangle$. Show that the level surface $q(x) = 1$ is an ellipsoid.

A.3 (a) State in detail the Fredholm Alternative Theorem for

$$u(x) = f(x) + \lambda \int_a^b k(x,y)u(y)dy$$

(b) Solve the equation

$$u(x) = \cos^2 x + \lambda \int_0^{2\pi} \sum_{k=0}^{2} \cos(kx)\cos(ky)u(y)dy$$

for the function $u(x)$, $0 \le x \le 2\pi$, and for any choice of the constant λ.

A.4 Determine conditions on $f(x)$, α and β for which there are solutions of

$$u'' + u = f(x), \quad u(0) - u(2\pi) = \alpha, \quad u'(0) - u'(2\pi) = -\beta.$$

Part B

B.1 Consider the variational problem

$$\min \int_a^b F(x, y(x), \frac{dy}{dx}(x))\, dx$$

over the space of functions y that are piecewise continuously differentiable on $[a, b]$ with $y(a) = 0$ and $y(b)$ unspecified. Assume that $F(x, y, y')$ is a smooth function. Derive the Euler-Lagrange equation for this problem, and clearly identify the appropriate boundary conditions. Be sure to describe the appropriate space of admissible variations.

B.2 Use the appropriate eigenfunction expansion to represent the solution of the given problem.

$$-u'' = f(x), \quad 0 < x < \pi$$

$$u(0) = \alpha, \quad u(\pi) = \beta$$

B.3 Use a Green's function to solve

$$u'' = f(x), \quad 0 < x < 1$$

$$u(0) = 0, \quad \int_0^1 u(x) = 0.$$

B.4 Let $f(x)$ be a continuously differentiable function except at a discrete set of points $x_1, x_2 \ldots, x_n$, where f has finite jump discontinuities $\Delta f_1, \Delta f_2, \ldots, \Delta f_n$. Show that the derivative of $f(x)$ in the sense of distributions is given by

$$f' = \frac{df}{dx} + \sum_{j=1}^{n} \Delta f_j \delta(x - x_j),$$

where df/dx is the classical derivative of f wherever it exists.